11.3

Show explicitly that two successive Lorentz transformations in the same direction are equivalent to a single Lorentz transformation with a velocity

$$v = \frac{v_1 + v_2}{1 + \left(v_1 v_2 / c^2\right)}$$

This is an alternative way to derive the parallel-velocity addition law.

Sol:

Choose a coordinate system $K'$ moves with a velocity $\vec{v} = v_1 \hat{x}$ relative to $K$, and another coordinate system $K''$ moves with a velocity $\vec{v} = v_2 \hat{x}$ relative to $K'$.

Therefore,

$$A_1 = \begin{bmatrix} \gamma_1 & -\beta_1\gamma_1 & 0 & 0 \\ -\beta_1\gamma_1 & \gamma_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} \gamma_2 & -\beta_2\gamma_2 & 0 & 0 \\ -\beta_2\gamma_2 & \gamma_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where $\gamma_1 = \dfrac{1}{\sqrt{1 - \dfrac{v_1^2}{c^2}}}$, $\beta_1 = \dfrac{v_1}{c}$, $\gamma_2 = \dfrac{1}{\sqrt{1 - \dfrac{v_2^2}{c^2}}}$ and $\beta_2 = \dfrac{v_2}{c}$.

$$A_1 A_2 = \begin{bmatrix} \gamma_1 & -\beta_1\gamma_1 & 0 & 0 \\ -\beta_1\gamma_1 & \gamma_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma_2 & -\beta_2\gamma_2 & 0 & 0 \\ -\beta_2\gamma_2 & \gamma_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \gamma_1\gamma_2 + \beta_1\beta_2\gamma_1\gamma_2 & -\beta_2\gamma_1\gamma_2 - \beta_1\gamma_1\gamma_2 & 0 & 0 \\ -\beta_1\gamma_1\gamma_2 - \beta_2\gamma_1\gamma_2 & \gamma_1\gamma_2 + \beta_1\beta_2\gamma_1\gamma_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where

$$\gamma_1\gamma_2 + \beta_1\beta_2\gamma_1\gamma_2$$

$$= \frac{1}{\sqrt{1 - \frac{v_1^2}{c^2}}} \frac{1}{\sqrt{1 - \frac{v_2^2}{c^2}}} + \frac{v_1}{c}\frac{v_2}{c} \frac{1}{\sqrt{1 - \frac{v_1^2}{c^2}}} \frac{1}{\sqrt{1 - \frac{v_2^2}{c^2}}}$$

$$= \frac{1}{\sqrt{1 - \frac{v_1^2}{c^2} - \frac{v_2^2}{c^2} + \frac{v_1^2}{c^2}\frac{v_2^2}{c^2}}} + \frac{v_1}{c}\frac{v_2}{c} \frac{1}{\sqrt{1 - \frac{v_1^2}{c^2} - \frac{v_2^2}{c^2} + \frac{v_1^2}{c^2}\frac{v_2^2}{c^2}}}$$

$$= \frac{1 + \frac{v_1}{c}\frac{v_2}{c}}{\sqrt{1 - \frac{v_1^2}{c^2} - \frac{v_2^2}{c^2} + \frac{v_1^2}{c^2}\frac{v_2^2}{c^2}}} = \frac{1 + \frac{v_1}{c}\frac{v_2}{c}}{\sqrt{\left(1 + \frac{v_1}{c}\frac{v_2}{c}\right)^2 - 2\frac{v_1}{c}\frac{v_2}{c} - \frac{v_1^2}{c^2} - \frac{v_2^2}{c^2}}}$$

$$= \frac{1 + \frac{v_1}{c}\frac{v_2}{c}}{\sqrt{\left(1 + \frac{v_1}{c}\frac{v_2}{c}\right)^2 - \left(\frac{v_1}{c} + \frac{v_2}{c}\right)^2}}$$

$$= \frac{1}{\sqrt{1 - \frac{\left(\frac{v_1}{c} + \frac{v_2}{c}\right)^2}{\left(1 + \frac{v_1 v_2}{c^2}\right)^2}}} = \frac{1}{\sqrt{1 - \frac{\frac{(v_1 + v_2)^2}{\left(1 + \frac{v_1 v_2}{c^2}\right)^2}}{c^2}}} = \frac{1}{\sqrt{1 - \frac{\left(\frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}\right)^2}{c^2}}}$$

$$= \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \equiv \gamma$$

$$v \equiv \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}$$

$$\Rightarrow \gamma_1 \gamma_2 (1 + \beta_1 \beta_2) = \gamma$$

Then

$$\gamma_1 \gamma_2 (\beta_1 + \beta_2) = \gamma_1 \gamma_2 \frac{(1 + \beta_1 \beta_2)}{(1 + \beta_1 \beta_2)} (\beta_1 + \beta_2) = -\frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2} \gamma$$

$$= -\frac{\frac{v_1}{c} + \frac{v_2}{c}}{1 + \frac{v_1}{c}\frac{v_2}{c}} \gamma = -\frac{1}{c} \frac{v_1 + v_2}{1 + \frac{v_1}{c}\frac{v_2}{c}} \gamma$$

$$= -\frac{v}{c} \gamma = -\beta \gamma$$

$$\Rightarrow A_1 A_2 = \begin{bmatrix} \gamma_1 & -\beta_1 \gamma_1 & 0 & 0 \\ -\beta_1 \gamma_1 & \gamma_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma_2 & -\beta_2 \gamma_2 & 0 & 0 \\ -\beta_2 \gamma_2 & \gamma_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \gamma_1 \gamma_2 + \beta_1 \beta_2 \gamma_1 \gamma_2 & -\beta_2 \gamma_1 \gamma_2 - \beta_1 \gamma_1 \gamma_2 & 0 & 0 \\ -\beta_1 \gamma_1 \gamma_2 - \beta_2 \gamma_1 \gamma_2 & \gamma_1 \gamma_2 + \beta_1 \beta_2 \gamma_1 \gamma_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \gamma & -\beta \gamma & 0 & 0 \\ -\beta \gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $v = \dfrac{v_1 + v_2}{1 + \dfrac{v_1 v_2}{c^2}}$

11.4

A possible clock is shown in the figure. It consists of a flashtube *F* and a photocell *P* shielded so that each views only the mirror *M*, located a distance *d* away, and mounted rigidly with respect to the flashtube-photocell assembly. The electronic innards of the box are such that when the photocell responds to a light flash from the mirror, the flashtube is triggered with a negligible delay and emits a short flash toward the mirror. The clock thus "ticks" once every ( $2d/c$ ) seconds when at rest.

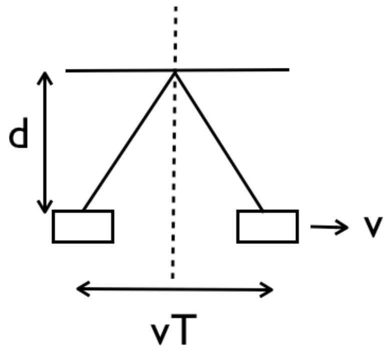
(a) Suppose that the clock moves with a uniform velocity $v$, perpendicular to the line from *PF* to *M*, relative to an observer. Using the second postulate of relativity, show by explicit geometrical or algebraic construction that the observer sees the relativistic time dilatation as the clock moves by.

(b) Suppose that the clock moves with a velocity *v* parallel to the line from *PF* to *M*. Verify that here, too, the clock is observed to tick more slowly, by the same time dilatation factor.

Sol:

(a)

The perpendicular motion is fairly easy to handle. If the box moves to the right at a uniform velocity v, we have the situation




Denoting the round-trip time T, the box moves a horizontal distance of $vT$ during one complete period. Using a bit of geometry, the distance D traveled by a beam of light from the box to the mirror and back is simply

$$D = 2\sqrt{d^2 + (vT/2)^2} = \sqrt{(2d)^2 + (vT)^2} \,.$$

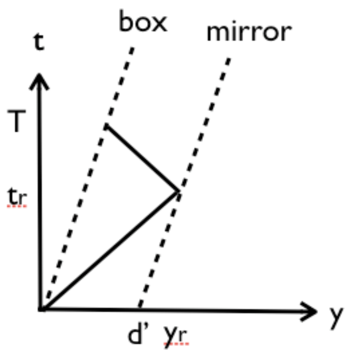
For a constant speed of light, this gives $T = D/c \Rightarrow cT = \sqrt{(2d)^2 + (vT)^2}$ .

Solving this for T gives the familiar time dilatation expression $T = \gamma T_0$, where

$$\gamma = \frac{1}{\sqrt{1-(v/c)^2}} \text{ and } T_0 = \frac{2d}{c}.$$

(b)

Here, consider a spacetime diagram




Here the box (and mirror) is moving in its parallel direction. We ought to know that this gives rise to a length contraction $d \to d/\gamma$. However, for now, we simply suppose that the box–mirror contraption appears to have length $d'$. The light beam reflects at position $y_r$ at time $t_r$, and is recaptured by the box at time T. We first work out $t_r$ algebraically. From the figure, the mirror's position is given by $y_r = d' + vt_r$, while the light ray travels according to $y_r = ct_r$. Solving this set of equations gives

$$y_r = \frac{d'}{1-v/c}, t_r = \frac{d'/c}{1-v/c}.$$

On the return, the light ray is captured by the box at time T and position $y = vT$.

Noting that the return path of the light ray is given by

$$y = y_r - c(t - t_r) = 2y_r - ct = \frac{2d'}{1-v/c} - ct.$$

We equate this to $vT$ to obtain $T = \frac{2d'/c}{1-v^2/c^2} = \gamma^2 \frac{2d'}{c}$.

Here, we realize that if lengths are contracted, $d' = d/\gamma$, then

$T = \gamma^2 \frac{2d'}{c} = \gamma \frac{2d}{c} = \gamma T_0$ gives the same time dilatation factor as part (a).

Alternatively, by demanding that the time dilatation factor is universal, we may obtain the length contraction relation $d' = d/\gamma$ as a result of this computation.

11.5

A coordinate system $K'$ moves with a velocity $\vec{v}$ relative to another system $K$. In $K'$ a particle has a velocity $\vec{u}'$ and an acceleration $\vec{a}'$. Find the Lorentz transformation law for accelerations, and show that in the system $K$ the components of acceleration parallel and perpendicular to $\vec{v}$ are

$$\vec{a}_{\parallel} = \frac{\left(1 - \frac{v^2}{c^2}\right)^{3/2}}{\left(1 + \frac{\vec{v}\cdot\vec{u}'}{c^2}\right)} \vec{a}'_{\parallel}$$

$$\vec{a}_{\perp} = \frac{\left(1 - \frac{v^2}{c^2}\right)^{3/2}}{\left(1 + \frac{\vec{v}\cdot\vec{u}'}{c^2}\right)} \left(\vec{a}'_{\perp} + \frac{\vec{v}}{c^2} \times \left(\vec{a}' \times \vec{u}'\right)\right)$$

Sol:

Instead of working directly with perpendicular and parallel components, we may start with a particular boost in the $x-t$ direction, and then generalize our results. We thus take a boost for the form

$$x^0 = \gamma\left(x^{0'} + \beta x'\right)$$

$$x = \gamma\left(x' + \beta x^{0'}\right)$$

$$y = y'$$

$$z = z'$$

In frame $K$, the path of a particle is specified by the vector function $\vec{x}\left(x^0\right)$, while in frame $K'$, this is instead $\vec{x}'\left(x^{0'}\right)$. 3-velocities and 3-accelerations are then defined in a frame dependent manner

Frame $K$:

$$\vec{u} = c\frac{\partial \vec{x}}{\partial x^0}$$

$$\vec{a} = c\frac{\partial \vec{u}}{\partial x^0}$$

Frame $K'$:

$$\vec{u}' = c\frac{\partial \vec{x}'}{\partial x^{0'}}$$

$$\vec{a} = c\frac{\partial \vec{u}'}{\partial x^{0'}}$$

To transform between the two frames, we need not just the transformation of the 3-vectors, but also the transformation relating times $x^0$ and $x^{0'}$. Noting from

$$x^0 = \gamma\left(x^{0'} + \beta x'\right)$$

$$x = \gamma\left(x' + \beta x^{0'}\right)$$

$$y = y'$$

$$z = z'$$

that a particle following a path $\vec{x}'\left(x^{0'}\right)$ yields a time relation

$$x^0 = \gamma\left(x^{0'} + \beta x'\left(x^{0'}\right)\right)$$

we may write

$$\frac{dx^0}{dx^{0'}} = \gamma\left(1 + \beta u'_x/c\right)$$

The inverse relation is simply

$$\frac{dx^{0'}}{dx^0} = \frac{1}{\gamma\left(1 + \beta u'_x/c\right)}$$

This useful expression is basically all we need. We start with velocities

$$u_x = c\frac{dx}{dx^0} = c\frac{dx^{0'}}{dx^0}\frac{dx}{dx^{0'}} = \frac{c}{\gamma\left(1 + \beta u'_x/c\right)}\frac{d}{dx^{0'}}\gamma\left(x' + \beta x^{0'}\right)$$

$$= \frac{u'_x + c\beta}{1 + \beta u'_x/c}$$

and

$$u_y = c\frac{dy}{dx^0} = c\frac{dx^{0'}}{dx^0}\frac{dy}{dx^{0'}} = \frac{c}{\gamma\left(1+\beta u'_x/c\right)}\left(\frac{u'_y}{c}\right)$$

$$= \frac{u'_y}{\gamma\left(1+\beta u'_x/c\right)}$$

writing $\beta u'_x = \vec{\beta}\cdot\vec{u}'$, and noting that the $x$ direction is the parallel direction while the $y$ direction is the perpendicular direction, it is easy to see that these velocity transformations may be written as

$$\vec{u}_\parallel = \frac{\vec{u}'_\parallel + c\vec{\beta}}{1+\vec{\beta}\cdot\frac{\vec{u}'}{c}}$$

$$\vec{u}_\perp = \frac{\vec{u}'_\perp}{\gamma\left(1+\vec{\beta}\cdot\frac{\vec{u}'}{c}\right)}$$

We now go on to accelerations, From $u_x$ and $u_y$, we have

$$a_x = c\frac{du_x}{dx^0} = \frac{c}{\gamma\left(1+\beta u'_x/c\right)}\frac{d}{dx^{0'}}\frac{u'_x + c\beta}{1+\beta\frac{u'_x}{c}}$$

$$= \frac{c}{\gamma\left(1+\beta u'_x/c\right)}\frac{\left(1+\beta\frac{u'_x}{c}\right)\left(\frac{a'_x}{c}\right) - \left(u'_x + c\beta\right)\left(\frac{\beta a'_x}{c^2}\right)}{\left(1+\beta\frac{u'_x}{c}\right)^2}$$

$$= \frac{\left(1-\beta^2\right)a'_x}{\gamma\left(1+\beta u'_x/c\right)^3}$$

$$= \frac{a'_x}{\gamma^3\left(1+\beta u'_x/c\right)^3}$$

$$a_y = c\frac{du_y}{dx^0} = \frac{c}{\gamma\left(1+\beta u_x'/c\right)}\frac{d}{dx^{0'}}\frac{u_y'}{\gamma\left(1+\beta\frac{u_x'}{c}\right)}$$

$$= \frac{c}{\gamma^2\left(1+\beta u_x'/c\right)}\frac{\left(1+\beta\frac{u_x'}{c}\right)\left(\frac{a_y'}{c}\right)-u_y'\left(\frac{\beta a_x'}{c^2}\right)}{\left(1+\beta\frac{u_x'}{c}\right)^2}$$

$$= \frac{a_y' + \frac{\beta\left(u_x'a_y' - u_y'a_x'\right)}{c}}{\gamma^2\left(1+\beta u_x'/c\right)^3}$$

It is straightforward to convert the expression for $a_x$ into one for $\vec{a}_\parallel$. The result is

$$\vec{a}_\parallel = \frac{\vec{a}_\parallel'}{\gamma\left(1+\vec{\beta}\cdot\vec{u}'/c\right)^3}$$

For the perpendicular direction, we have to be a bit more clever. Noting that $x$ components in $a_y$ are related to $\vec{\beta}\cdot(\cdots)$, while $y$ components are directly related to the $\perp$ direction, we have

$$\vec{a}_\perp = \frac{\vec{a}_\perp' + \vec{a}'\left(\vec{\beta}\cdot\vec{u}'\right) - \vec{u}'\frac{\left(\vec{\beta}\cdot\vec{a}'\right)}{c}}{\gamma^2\left(1+\vec{\beta}\cdot\vec{u}'/c\right)^3}$$

Use of the BAC-CAB rule finally gives

$$\vec{a}_\perp = \frac{\vec{a}_\perp' + \vec{\beta}\times\frac{\left(\vec{a}'\cdot\vec{u}'\right)}{c}}{\gamma^2\left(1+\vec{\beta}\cdot\vec{u}'/c\right)^3}$$

11.6

Assume that a rocket ship leaves the earth in the year 2100. One of a set of twins born in 2080 remains on earth; the other rides in the rocket. The rocket ship is so constructed that it has an acceleration $g$ in its own rest frame (this makes the occupants feel at home). It accelerates in a straight-line path for 5 years (by its own clocks), decelerates at the same rate for 5 more years, turns around, accelerates for 5 years, decelerates for 5 years, and lands on earth. The twin in the rocket is 40 years old. (a) What year is it on earth? (b) How far away from the earth did the rocket ship travel?

Sol:

(a)

To calculate the time interval in the earth frame *K* along the first acceleration leg,

$$T_1 = \int_0^{5a} \gamma(\tau)\, d\tau = \int_0^{5a} \frac{1}{\sqrt{1 - \frac{u^2(\tau)}{c^2}}} d\tau \text{ -----(1)}$$

we require $u(\tau)$. To find $u(\tau)$, we use the parallel-component result of Problem 11.5 for the case that $K'$ is the instantaneously co-moving frame of the rocket and *K* is the earth frame. Then, $dt' = d\tau$ and $u_{\parallel}' = 0$, and

$$\frac{du}{dt} = \frac{1}{\gamma_u{}^3}\frac{du'}{dt} = \frac{1}{\gamma_u{}^3}\frac{du'}{d\tau} = \frac{g}{\gamma_u{}^3} \text{ -----(2)}$$

Also, due to time dilation between the earth and the instantaneously co-moving rocket frame it is $dt = \gamma_u d\tau$, and therefore

$$\frac{du}{dt} = \frac{du}{\gamma_u d\tau} = \frac{g}{\gamma_u^3}$$

$$\gamma_u^2 du = g d\tau$$

$$\int_{u=0}^{u(\tau)} \frac{1}{1 - u^2 / c^2} du = \int_{\tau=0}^{\tau} g\, d\tau = g\tau$$

$$\frac{u(\tau)}{c} = \tanh(\frac{g\tau}{c})$$

Insertion into (1) allows us to calculate the travel time of the first leg observed in *K*,

$$\int_{\tau=0}^{5a} \frac{1}{\sqrt{1 - \tanh^2\left(\frac{g\tau}{c}\right)}} d\tau = \int_{\tau=0}^{5a} \cosh\left(\frac{g\tau}{c}\right) d\tau = \frac{c}{g}\sinh\left(\frac{g5a}{c}\right) = 84a$$

By symmetry, all other legs give the same result, yielding a travel time observed in *K* of $T = 4T_1 = 336a$ . Thus, the year of return to earth is 2436.

Note that in the analysis the instantaneously co-moving frame (ICMF) of the rocket $K'$ is an inertial frame; for that reason the presented analysis is valid. As the rocket moves along, it marks the origins of an infinite sequence of different ICMFs. The rocket itself is not an inertial frame, of course, but the rocket frame never enters in our analysis.

(b)

The travel distance in *K* along the first leg

$$L_1 = \int_{t=0}^{T_1} u(t)\,dt = \int_{t=0}^{84a} u(t)\,dt$$

requires knowledge of the rocket velocity *u*(*t*) observed in *K*. From (2) it follows

$$\gamma^3(u)du = gdt$$

$$\int_{u=0}^{u(t)} \frac{1}{\sqrt{1-\frac{u^2}{c^2}}^{3}}\,du = \int_{o}^{t} g\,dt$$

$$u(t)\frac{1}{\sqrt{1-\frac{u(t)^2}{c^2}}} = gt$$

$$u(t) = \frac{gt}{\sqrt{1+\frac{g^2t^2}{c^2}}}$$

Insertion into the previous equation yields

$$L_1 = \frac{c^2}{g}\int_{t=0}^{T_1\times g/c} \frac{gt/c}{\sqrt{1+\frac{g^2t^2}{c^2}}}d(dt/c) = \frac{c^2}{g}\left[\sqrt{1+\frac{g^2T_1^2}{c^2}}-1\right]$$

Since $\frac{gT_1}{c} \gg 1$, the square-root can be developed, yielding with $T_1 = 84a$.

$$L_1 \approx cT_1 + \frac{c^2}{g}\left(\frac{c}{2gT_1} - 1\right) \approx cT - \frac{c^2}{g} = (84 - 0.969) lightyears$$

Since all legs have, for symmetry, the same length, we find a total travel distance of $L = 2L_1 = 166$ *lightyears*, which is almost as far as a beam of light would travel (which would be 168 *lightyears*).

11.9

An infinitesimal Lorentz transformation and its inverse can be written as

$$x'^{\alpha} = \left(g^{\alpha\beta} + \varepsilon^{\alpha\beta}\right)x_{\beta}$$
$$x^{\alpha} = \left(g^{\alpha\beta} + \varepsilon'^{\alpha\beta}\right)x'_{\beta}$$

where $\varepsilon^{\alpha\beta}$ and $\varepsilon'^{\alpha\beta}$ are infinitesimal.

(a) Show from the definition of the inverse that $\varepsilon'^{\alpha\beta} = -\varepsilon^{\alpha\beta}$.

(b) Show from the preservation of the norm that $\varepsilon^{\alpha\beta} = -\varepsilon^{\beta\alpha}$.

(c) By writing the transformation in terms of contravariant components on both sides of the equation, show that $\varepsilon^{\alpha\beta}$ is equivalent to the matrix $L$ (11.93).

Sol:

(a) We consider the effect of the consecutive application of the transformation given in the problem and its inverse,

$$\begin{aligned}
x''^{\alpha} &= \left(g^{\alpha\beta} + \varepsilon'^{\alpha\beta}\right)x'_{\beta} \\
&= \left(g^{\alpha\beta} + \varepsilon'^{\alpha\beta}\right)g_{\beta\gamma}x'^{\gamma} \\
&= \left(g^{\alpha\beta} + \varepsilon'^{\alpha\beta}\right)g_{\beta\gamma}\left(g^{\gamma\delta} + \varepsilon^{\gamma\delta}\right)x_{\delta} \\
&= \left(g^{\alpha\beta} + \varepsilon'^{\alpha\beta}\right)g_{\beta\gamma}\left(g^{\gamma\delta} + \varepsilon^{\gamma\delta}\right)g_{\delta\eta}x^{\eta}
\end{aligned}$$

Since the two consecutive transformations are the inverse of each other, it also is $x''^{\alpha} = \delta^{\alpha}{}_{\eta}x^{\eta}$ for all $x$. By comparison with the previous equation, we can write

$$\begin{aligned}
\delta^{\alpha}{}_{\eta} &= \left(g^{\alpha\beta} + \varepsilon'^{\alpha\beta}\right)g_{\beta\gamma}\left(g^{\gamma\delta} + \varepsilon^{\gamma\delta}\right)g_{\delta\eta} \\
&= \left[\delta^{\alpha}{}_{\gamma}\left(g^{\gamma\delta} + \varepsilon^{\gamma\delta}\right) + \varepsilon'^{\alpha\beta}g_{\beta\gamma}\left(g^{\gamma\delta} + \varepsilon^{\gamma\delta}\right)\right]g_{\delta\eta} \\
&= \left[g^{\alpha\delta} + \varepsilon^{\alpha\delta} + \varepsilon'^{\alpha\beta}\left(\delta_{\beta}{}^{\delta} + g_{\beta\gamma}\varepsilon^{\gamma\delta}\right)\right]g_{\delta\eta} \\
&= \left[g^{\alpha\delta} + \varepsilon^{\alpha\delta} + \varepsilon'^{\alpha\delta} + \varepsilon'^{\alpha\beta}g_{\beta\gamma}\varepsilon^{\gamma\delta}\right]g_{\delta\eta} \\
&= \left[g^{\alpha\delta} + \varepsilon^{\alpha\delta} + \varepsilon'^{\alpha\delta}\right]g_{\delta\eta} \\
&= \delta^{\alpha}{}_{\eta} + g_{\delta\eta}\left(\varepsilon^{\alpha\delta} + \varepsilon'^{\alpha\delta}\right) = \delta^{\alpha}{}_{\eta} + \left(\varepsilon^{\alpha}{}_{\eta} + \varepsilon'^{\alpha}{}_{\eta}\right)
\end{aligned}$$

Note that due to the infinitesimal character of the elements of the $\varepsilon$-tensors, we were allowed to drop terms quadratic in them. From the last line it follows that $\varepsilon^{\alpha\delta} = -\varepsilon'^{\alpha\delta}$, q.e.d.

(b) Beginning with norm conservation, we find by application of the infinitesimal transformation law specified in the problem,

$$
\begin{aligned}
x^\alpha x_\alpha &= x'^\alpha x'_\alpha \\
&= \left(g^{\alpha\beta} + \varepsilon^{\alpha\beta}\right)x_\beta x'_\alpha = \left(g^{\alpha\beta} + \varepsilon^{\alpha\beta}\right)x_\beta g_{\alpha\gamma} x'^\gamma \\
&= \left(g^{\alpha\beta} + \varepsilon^{\alpha\beta}\right)x_\beta g_{\alpha\gamma}\left(g^{\gamma\delta} + \varepsilon^{\gamma\delta}\right)x_\delta \\
&= \left(g^{\alpha\beta} + \varepsilon^{\alpha\beta}\right)\left[g_{\alpha\gamma}\left(g^{\gamma\delta} + \varepsilon^{\gamma\delta}\right)\right]g_{\delta\eta} x_\beta x^\eta \\
&= \left(g^{\alpha\beta} + \varepsilon^{\alpha\beta}\right)\left(\delta_\alpha{}^\delta + g_{\alpha\gamma}\varepsilon^{\gamma\delta}\right)g_{\delta\eta} x_\beta x^\eta \\
&= \left(g^{\alpha\beta} + \varepsilon^{\alpha\beta}\right)\left[\left(\delta_\alpha{}^\delta + g_{\alpha\gamma}\varepsilon^{\gamma\delta}\right)g_{\delta\eta}\right]x_\beta x^\eta \\
&= \left(g^{\alpha\beta} + \varepsilon^{\alpha\beta}\right)\left(g_{\alpha\eta} + g_{\alpha\gamma}\varepsilon^{\gamma\delta} g_{\delta\eta}\right)x_\beta x^\eta \\
&= \left(\delta^\beta{}_\eta + \delta^\beta{}_\gamma \varepsilon^{\gamma\delta} g_{\delta\eta} + \varepsilon^{\alpha\beta} g_{\alpha\eta}\right)x_\beta x^\eta \\
&= x_\beta x^\beta + \left(\varepsilon^{\beta\delta} g_{\delta\eta} + \varepsilon^{\alpha\beta} g_{\alpha\eta}\right)x_\beta x^\eta \\
&= x_\alpha x^\alpha + \left(\varepsilon^{\beta\alpha} g_{\alpha\eta} + \varepsilon^{\alpha\beta} g_{\alpha\eta}\right)x_\beta x^\eta \\
&= x_\alpha x^\alpha + \left(\varepsilon^{\beta\alpha} + \varepsilon^{\alpha\beta}\right)g_{\alpha\eta} x_\beta x^\eta \\
&= x^\alpha x_\alpha + \left(\varepsilon^{\beta\alpha} + \varepsilon^{\alpha\beta}\right)x_\alpha x_\beta
\end{aligned}
$$

Again, due to the infinitesimal character of the elements of the $\varepsilon$-tensors we were allowed to drop terms quadratic in them. Since the result must be valid for all $x$, it follows that $\varepsilon^{\alpha\beta} = -\varepsilon'^{\beta\alpha}$, q.e.d.

(c) It is

$$x'^\alpha = \left(g^{\alpha\beta} + \varepsilon^{\alpha\beta}\right)x_\beta = \left(g^{\alpha\beta} + \varepsilon^{\alpha\beta}\right)g_{\beta\gamma} x^\gamma = \left(\delta^\alpha{}_\gamma + \varepsilon^{\alpha\beta} g_{\beta\gamma}\right)x^\gamma$$

Also, an infinitesimal Lorentz transformation matrix with generator $L$ is of the form $A = \exp(L) = 1 + L$. In index notation, the effect of such a transformation is
$x'^\alpha = A^\alpha{}_\gamma x^\gamma = \left(\delta^\alpha{}_\gamma + L^\alpha{}_\gamma\right)x^\gamma$

Comparison of the last two equations shows

$$L^\alpha{}_\gamma = \varepsilon^{\alpha\beta} g_{\beta\gamma}$$

which is equivalent to $L^\alpha{}_\gamma g^{\gamma\delta} = \varepsilon^{\alpha\beta} g_{\beta\gamma} g^{\gamma\delta} = \varepsilon^{\alpha\beta}\delta_\beta{}^\delta = \varepsilon^{\alpha\delta}$.

11.16

In the rest frame of a conducting medium the current density satisfies Ohm's law $\vec{J}' = \sigma\vec{E}'$, where $\sigma$ is the conductivity and primes denote quantities in the rest frame.

(a) Taking into account the possibility of convection current as well as conduction current, show that the covariant generalization of Ohm's law is

$$J^{\alpha} - \frac{1}{c^2}(U_{\beta}J^{\beta})U^{\alpha} = \frac{\sigma}{c}F^{\alpha\beta}U_{\beta},$$

where $U^{\alpha}$ is the 4-velocity of the medium.

(b) Show that if the medium has a velocity $\vec{v} = c\vec{\beta}$ with respect to some inertial frame that the 3-vector current in that frame is

$$\vec{J} = \gamma\sigma[\vec{E} + \vec{\beta}\times\vec{B} - \vec{\beta}(\vec{\beta}\cdot\vec{E})] + \rho\vec{v} \ ,$$

where $\rho$ is the charge density observed in that frame.

(c) If the medium is uncharged in its rest frame ( $\rho' = 0$), what is the charge density and the expression for $\vec{J}$ in the frame of part b? This is the relativistic generalization of the equation $\vec{J} = \sigma(\vec{E} + \vec{v}\times\vec{B})$ (see p. 320).

Sol:

(a)

There are several methods for obtaining the covariant generalization of Ohm's law. We start by taking the rest frame result $\vec{J}' = \sigma\vec{E}'$ and converting 3-vectors into 4-vectors. To do so, we note that the rest frame may be defined as the frame where the 4-velocity has the form

$$U'^{\mu} = (c, 0)$$

This allows us to realize the electric field as a contraction of the Maxwell field strength tensor with the 4-velocity

$$F'^{\mu\nu}U'_{\nu} = (0, c\vec{E}')$$

Thus the right hand side of Ohm's law can be written as

$$(0, \sigma\vec{E}') = \frac{\sigma}{c}F'^{\mu\nu}U'_{\nu} \quad \text{------(1)}$$

The left hand side of Ohm's law ought to be the 3-vector current density. It is natural to take $\vec{J}' \rightarrow J'^{\mu}$. However, this contains more than the 3-current, since it includes the charge density $\rho'$ as well. Because $\rho'$ is not involved in Ohm's law, we need to subtract it out. This can be done by noting that $J'^{\nu}U'_{\nu} = c^2 U'^{\mu}$. Hence

$$(0, \vec{J}') = J'^{\mu} - \frac{1}{c^2}\left(J'^{\nu}U'_{\nu}\right)U'^{\mu} \text{ ------(2)}$$

Combining (1) and (2) gives the covariant form of Ohm's law

$$J^{\mu} - \frac{1}{c^2}\left(J^{\nu}U_{\nu}\right)U^{\mu} = \frac{\sigma}{c}F^{\mu\nu}U_{\nu} \text{ ------(3)}$$

where we have now dropped all primes, since the equation is in covariant form (ie frame independent). Note that this can equivalently be written as

$$J^{\mu} = \frac{\sigma}{c}F^{\mu\nu}U_{\nu} + \frac{1}{c^2}\left(J^{\nu}U_{\nu}\right)U^{\mu},$$

where the first term on the right corresponds to the conduction current and the second to the convection current (charge times velocity, where charge is given by the Lorentz invariant $J^{\nu}U_{\nu}$ and the velocity is given by $U^{\mu}$).

(b)
Working in the (inertial) lab frame, we take

$$U^{\mu} = (\gamma c, \gamma \vec{v}), J^{\mu} = (c\rho, \vec{J})$$

Substituting this into (3) and noting that

$$\frac{1}{c^2}J^{\nu}U_{\nu} = \gamma\left(\rho - \frac{1}{c^2}\vec{v}\cdot\vec{J}\right)$$

we obtain the time component equation

$$-v^2\gamma^2\rho + \gamma^2\vec{v}\cdot\vec{J} = \sigma\gamma\vec{v}\cdot\vec{E} \text{ ------(4)}$$

as well as the space components equation

$$\vec{J} - \gamma^2\rho\vec{v} + \frac{\gamma^2}{c^2}\vec{v}(\vec{v}\cdot\vec{J}) = \sigma\gamma(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}) \text{ -----(5)}$$

Solving (4) for $\vec{v}\cdot\vec{J}$ and substituting this into (5) gives

$$\vec{J} = \sigma\gamma[\vec{E} + \vec{\beta}\times\vec{B} - \vec{\beta}(\vec{\beta}\cdot\vec{E})] + \rho\vec{v} \text{ ----(6)}$$

This gives an explicit realization of the conduction and convection currents (the latter being the $\rho\vec{v}$ term in the above).

(c)

Note that if the medium is uncharged in its rest frame, the 4-current must satisfy the relation $J^\nu U_\nu = 0$. In this case, the covariant Ohm's law (3) reduces to

$$J^\mu = \frac{\sigma}{c}F^{\mu\nu}U_\nu$$

Taking $J^\mu = (c\rho, \vec{J})$ and $U^\mu = (c\gamma, \vec{v}\gamma)$ in the above then gives directly

$$\rho = \frac{\sigma\gamma}{c}(\vec{\beta}\cdot\vec{E}), \vec{J} = \sigma\gamma(\vec{E} + \vec{\beta}\times\vec{B})$$

It is straightforward to check that this is consistent with (6) obtained above.

11.19

A particle of mass $M$ and 4-momentun $P$ decays into two particles of masses $m_1$ and $m_2$.

(a) Use the conservation of energy and momentum in the form, $p_2 = P - p_1$, and the invariance of scalar products of 4-vectors to show that the total energy of the first particle in the rest frame of the decaying particle is

$$E_1 = \frac{M^2 + m_1^2 + m_2^2}{2M}$$

and that $E_2$ is obtained by interchanging $m_1$ and $m_2$.

(b) Show that the kinetic energy $T_i$ of the i-th particle in the same frames is

$$T_i = \Delta M\left(1 - \frac{m_i}{M} - \frac{\Delta M}{2M}\right)$$

where $\Delta M = M - m_1 - m_2$ is the mass excess or Q value of the process.

(c) The charged pi-meson ( $M = 139.6\ \mathrm{MeV}$ ) decays into a mu-meson ( $m_1 = 105.7\ \mathrm{MeV}$ ) and a neutrino ( $m_2 = 0\ \mathrm{MeV}$ ). Calculate the kinetic energies of the mu-meson and the neutrino in the pi-meson's rest frame. The unique kinetic energy of the muon is the signature of a two-body decay. It entered importantly in the discovery of the pi-meson in photographic emulsions by Powell and coworkers in 1947.

Sol:

(a) From Jackson 11.152

$$\vec{E}(ct,x,y,z) = -\hat{z}\frac{q\gamma(vt-z)}{\sqrt{r_\perp^2+\gamma^2(vt-z)^2}^3} + \vec{r}_\perp\frac{q\gamma}{\sqrt{r_\perp^2+\gamma^2(vt-z)^2}^3}$$

$$\vec{B}(ct,x,y,z) = \hat{z}\times\vec{r}_\perp\frac{q\gamma}{\sqrt{r_\perp^2+\gamma^2(vt-z)^2}^3}$$

Where $\vec{r}_\perp = (x,y,0)$. To see the equivalence, perform a suitable translation and a rotation about the $z$-axis to get back to eq. 11.152.
To obtain the limit $\gamma\to\infty$, we first consider the electric field. Considering the denominator, we see that the field generally only is appreciable if $|vt-z|$ is of order $r_\perp/\gamma$ of less. Thus, in the limit $\gamma\to\infty$ non-zero fields only exist if $|vt-z| << r_\perp$. Thus, in the limit $\gamma\to\infty$ $z$-component of the electric field is negligible. Next, we observe that

$$\frac{\gamma}{\sqrt{r_\perp^2+\gamma^2(vt-z)^2}^3} = \begin{cases} \dfrac{\gamma}{r_\perp^3}\to\infty, vt-z=0 \\ \dfrac{1}{\gamma^2(vt-z)^3}\to 0, vt-z\neq 0 \end{cases} \text{ in the limit } \gamma\to\infty$$

Further, at fixed time the integral over $z$ is

$$\int_{z=-\infty}^{\infty} \frac{\gamma}{\sqrt{r_\perp{}^2 + \gamma^2 (vt - z)^2}^3} dz = \int_{z=-\infty}^{\infty} \frac{\gamma}{\sqrt{r_\perp{}^2 + \gamma^2 z^2}^3} dz = \gamma \frac{1}{r_\perp{}^2} \left[ \frac{\gamma}{\sqrt{r_\perp{}^2 + \gamma^2 z^2}} \right]_{-\infty}^{\infty} = \frac{2}{r_\perp{}^2}$$

This result can, of course, also obtained by considering a fixed position and integrating over $ct$. Thus, in the limit $\gamma \to \infty$

$$\frac{\gamma}{\sqrt{r_\perp{}^2 + \gamma^2 (vt - z)^2}^3} = \frac{2}{r_\perp{}^2} \delta(ct - z)$$

and therefore

$$\vec{E}(ct, x, y, z) = \vec{r}_\perp \frac{2q}{r_\perp{}^2} \delta(ct - z)$$

$$\vec{B}(ct, x, y, z) = \hat{z} \times \vec{r}_\perp \frac{2q}{r_\perp{}^2} \delta(ct - z)$$

(b) $\nabla \cdot \vec{E} = 4\pi\rho$, For the above $\vec{E}$, it is

$$\nabla \cdot \vec{E} = \nabla \cdot \left[ \vec{r}_\perp \frac{2q}{r_\perp{}^2} \delta(ct - z) \right] = 2q\delta(ct - z) \left[ \frac{\partial}{\partial x} \left( \frac{x}{x^2 + y^2} \right) + \frac{\partial}{\partial y} \left( \frac{x}{x^2 + y^2} \right) \right] = 0$$

unless $r_\perp = 0$ and $z = ct$. Thus, $\nabla \cdot \vec{E}$ is of the form

$$\nabla \cdot \vec{E} = 4\pi f \delta^2(\vec{r}_\perp) \delta(ct - z)$$

with a constant $f$ that we can determine by integrating this equation over an infinitesimal spherical volume centered around the particle location $(0,0,ct)$:

$$\int \nabla \cdot \vec{E} dxdyd(ct - z) = \int 4\pi f \delta^2(\vec{r}_\perp) \delta(ct - z) dxdyd(ct - z)$$

$$= \oint \vec{E} \cdot d\vec{a} = 4\pi f$$

Since the field is localized to the plane $ct = z$, the area integral only yields contributions from a thin azimuthal band in the $ct = z$ plane. We can therefore write the area integral in the form

$$\int_{ct-z=-\varepsilon}^{\varepsilon} \int_{\phi=0}^{2\pi} \vec{r}_\perp \frac{2q}{r_\perp{}^2} \delta(ct - z) \cdot \vec{r}_\perp d(ct - z) d\phi = 4\pi f$$

$$q = f$$

There, $\varepsilon$ is an infinitesimal length. Thus, from the given field alone we have derived that

$$\nabla \cdot \vec{E} = 4\pi f \delta^2(\vec{r}_\perp) \delta(ct - z)$$

By Gauss's law, it must also be $\nabla \cdot \vec{E} = 4\pi\rho$. Thus, the charge density for the given field is $\rho(\vec{x}) = q\delta^2(\vec{r}_\perp)\delta(ct - z)$. The zero-th component of the four-current producing the field in part a) must therefore be

$$J^0 = cq\delta^2(\vec{r}_\perp) \delta(ct - z)$$

This is in agreement with the 0-component of the current specified in the problem.

$\nabla \cdot \vec{B} = 0$ : The validity can be verified explicitly for locations $x \neq (0,0,ct)$. It is then concluded that $\nabla \cdot \vec{B} = 4\pi g \delta^2(\vec{r}_\perp)\delta(ct - z)$. The constant $g$ is determined via a small volume integral,

$$\int \nabla \cdot \vec{B} dx dy d(ct - z) = \int 4\pi g \delta^2(\vec{r}_\perp)\delta(ct - z) dx dy d(ct - z)$$

$$= \oint \vec{B} \cdot d\vec{a} = 4\pi g$$

Since the $\vec{B}$-field is also localized to the plane $ct = z$, the area integral is, with an infinitesimal $\varepsilon$,

$$\oint \vec{B} \cdot d\vec{a} = \int_{ct-z=-\varepsilon}^{\varepsilon} \int_{\phi=0}^{2\pi} (r_\perp \hat{\phi}) \frac{2q}{r_\perp^2} \delta(ct - z) \cdot \vec{r}_\perp d(ct - z) d\phi = 0$$

Thus, it is $g = 0$, and it is, as required, $\nabla \cdot \vec{B} = 0$ everywhere. We conclude that the $\vec{B}$-field given in a) is consistent with Gauss's law for $\vec{B}$.

$\nabla \times \vec{B} - \frac{\partial}{\partial ct} \vec{E} = \frac{4\pi}{c} \vec{J}$ :

By direct calculation using the given fields, it is found that

$$\nabla \times \vec{B} = \hat{x} \tfrac{x}{r_\perp^2} \delta'(ct - z) + \hat{y} \tfrac{y}{r_\perp^2} \delta'(ct - z) + \hat{z} \delta(ct - z) \cdot 0$$

where

$$\delta'(ct - z) = \frac{d}{dx} \delta(x) \bigg|_{x = ct - z}$$

Also, it is found that

$$\frac{\partial}{\partial ct} \vec{E} = \hat{x} \frac{x}{r_\perp^2} \delta'(ct - z) + \hat{y} \frac{y}{r_\perp^2} \delta'(ct - z)$$

so that $\nabla \times \vec{B} - \frac{\partial}{\partial ct} \vec{E} = 0$, unless $r_\perp = 0$ and $z = ct$. Thus, $\nabla \times \vec{B} - \frac{\partial}{\partial ct} \vec{E}$ must be of the form

$$\nabla \times \vec{B} - \frac{\partial}{\partial ct} \vec{E} = \vec{h} \delta^2(\vec{r}_\perp)\delta(ct - z)$$

with a vector constant $\vec{h}$ to be determined. We note that due to the cylindrical symmetry of the fields on the left side of the equation, the right side must have cylindrical symmetry as well. We conclude that $\vec{h}$ can only point in the $z$ -direction, and thus

$$\nabla \times \vec{B} - \frac{\partial}{\partial ct} \vec{E} = \hat{z} h \delta^2(\vec{r}_\perp)\delta(ct - z)$$

with a scalar constant $h$ to be determined. To find $h$, we consider the area integral of that over a small disk centered around the location $(0,0,ct)$ with area vector in the $+z$ -direction. Using Stokes's theorem, the left side yields, with the given electric and magnetic fields,

$$\int\left(\nabla\times\vec{B}-\frac{\partial}{\partial ct}\vec{E}\right)\cdot d\vec{a}=\oint\vec{B}\cdot d\vec{l}-\int\frac{\partial}{\partial ct}\vec{E}\cdot(\hat{z}da)=4\pi q\delta(ct-z)$$

Then area integral of the right side,

$$\int\hat{z}h\delta^2(\vec{r}_\perp)\delta(ct-z)\cdot\hat{z}da=h\delta(ct-z)$$

Comparing the last two equations, we see $h=4\pi q$, and therefore

$$\nabla\times\vec{B}-\frac{\partial}{\partial ct}\vec{E}=\hat{z}4\pi q\delta^2(\vec{r}_\perp)\delta(ct-z)$$

Note that this result is obtained solely from the given fields. By Maxwell-Ampere's law, it must in addition be $\nabla\times\vec{B}-\frac{\partial}{\partial ct}\vec{E}=\frac{4\pi}{c}\vec{J}$. By comparison we see that the current density must be

$$\vec{J}=\hat{z}qc\delta^2(\vec{r}_\perp)\delta(ct-z)$$

This is in agreement with the spatial components of the current specified in the problem.

$\nabla\times\vec{E}-\frac{\partial}{\partial ct}\vec{B}=0$:

For locations $x\neq(0,0,ct)$, validity of Faraday's law can be shown by direct calculation. To verify consistency at the particle location, consider the area integral of the field-side of Faraday's law over a small disk centered around the location $(0,0,ct)$ with area vector in the $+z$ -direction. Using Stokes's theorem, from the given electric and magnetic fields it is, finally and thankfully, found that

$$\oint\vec{E}\cdot d\vec{l}+\int\frac{\partial}{\partial ct}\vec{B}\cdot(\hat{z}da)=0$$

Combining the above results, the four-current $J^\alpha$ that is consistent with the given fields and with Maxwell's equations is

$$J^\alpha(\rho c,\vec{J})=(J^0,\vec{J})=zqc\delta^2(\vec{r}_\perp)\delta(ct-z)(1,\hat{v})$$

(c) To derive the fields from the potentials, use

$\vec{B}=\nabla\times\vec{A}$ and $\vec{E}=-\frac{\partial}{\partial ct}\vec{A}-\nabla\cdot A^0$

or equivalently,

$$F^{\alpha\beta}=\partial^\alpha A^\beta-\partial^\beta A^\alpha=\left(\frac{\partial}{\partial ct},-\nabla\right)\cdot\left(A^0,\vec{A}\right)$$

and

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

For $A^\alpha = -2q\delta(ct-z)\ln(\lambda r_\perp)(1,0,0,1) = -2q\delta(ct-z)\ln\left(\lambda\sqrt{x^2+y^2}\right)(1,0,0,1)$, we find

$$E_x = \partial^1 A^0 - \partial^0 A^1 = 2q\delta(ct-z)\frac{\partial}{\partial x}\ln\left(\lambda\sqrt{x^2+y^2}\right) = 2q\delta(ct-z)\frac{x}{r_\perp^{\ 2}}$$

$$E_y = \partial^2 A^0 - \partial^0 A^2 = 2q\delta(ct-z)\frac{\partial}{\partial y}\ln\left(\lambda\sqrt{x^2+y^2}\right) = 2q\delta(ct-z)\frac{y}{r_\perp^{\ 2}}$$

$$E_z = \partial^3 A^0 - \partial^0 A^3 = 2q\ln\left(\lambda\sqrt{x^2+y^2}\right)\left[\frac{\partial}{\partial z} + \frac{\partial}{\partial ct}\right]\delta(ct-z) = 0$$

$$B_x = \partial^3 A^2 - \partial^2 A^3 = -2q\delta(ct-z)\frac{\partial}{\partial y}\ln\left(\lambda\sqrt{x^2+y^2}\right) = -2q\delta(ct-z)\frac{y}{r_\perp^{\ 2}}$$

$$B_y = \partial^1 A^3 - \partial^3 A^1 = 2q\delta(ct-z)\frac{\partial}{\partial x}\ln\left(\lambda\sqrt{x^2+y^2}\right) = 2q\delta(ct-z)\frac{x}{r_\perp^{\ 2}}$$

$$B_y = \partial^2 A^1 - \partial^1 A^2 = 0$$

which agrees with the fields specified in part a).
For

$$A^\alpha = -2q\Theta(ct-z)(0,\nabla_\perp \ln(\lambda r_\perp)) = -2q\Theta(ct-z)\ln(\lambda r_\perp)\left(0,\frac{x}{r_\perp^{\ 2}},\frac{y}{r_\perp^{\ 2}},0\right)$$

$$= -2q\Theta(ct-z)\ln(\lambda r_\perp)\left(0,\frac{x}{x^2+y^2},\frac{y}{x^2+y^2},0\right)$$

we find

$$E_x = \partial^1 A^0 - \partial^0 A^1 = 2q\frac{x}{r_\perp^{\ 2}}\frac{\partial}{\partial ct}\Theta(ct-z) = 2q\delta(ct-z)\frac{x}{r_\perp^{\ 2}}$$

$$E_y = \partial^2 A^0 - \partial^0 A^2 = 2q\frac{y}{r_\perp^{\ 2}}\frac{\partial}{\partial ct}\Theta(ct-z) = 2q\delta(ct-z)\frac{y}{r_\perp^{\ 2}}$$

$$E_z = \partial^3 A^0 - \partial^0 A^3 = 0$$

$$B_x = \partial^3 A^2 - \partial^2 A^3 = 2q\frac{x}{r_\perp^{\ 2}}\frac{\partial}{\partial ct}\Theta(ct-z) = 2q\delta(ct-z)\frac{y}{r_\perp^{\ 2}}$$

$$B_y = \partial^1 A^3 - \partial^3 A^1 = -2q\frac{y}{r_\perp^{\ 2}}\frac{\partial}{\partial ct}\Theta(ct-z) = 2q\delta(ct-z)\frac{x}{r_\perp^{\ 2}}$$

$$B_z = \partial^2 A^1 - \partial^1 A^2 = 2q\Theta(ct-z)\left[\frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right)\right] = 0$$

which also agrees with the fields specified in part a).

11.23

In a collision process a particle of mass $m_2$, at rest in the laboratory, is struck by a particle of mass $m_1$, momentum $\vec{p}_{LAB}$ and total energy $E_{LAB}$. In the collision the two initial particles are transformed into two others of mass $m_3$ and $m_4$. The configurations of the momentum vectors in the center of momentum (cm) frame (traditionally called the center-of-mass frame) and the laboratory frame are shown in the figure.

(a) Use invariant scalar products to show that the total energy W in the cm frame has its square given by $W^2 = m_1^2 + m_2^2 + 2m_2 E_{LAB}$ and that the cms 3-momentum $\vec{p}'$ is ,

$$\vec{p}' = \frac{m_2 \vec{p}_{LAB}}{W}.$$

(b) Show that the Lorentz transformation parameters $\vec{\beta}_{cm}$ and $\gamma_{cm}$ describing the velocity of the cm frame in the laboratory are

$$\vec{\beta}_{cm} = \frac{\vec{p}_{LAB}}{m_2 + E_{LAB}}, \gamma_{cm} = \frac{m_2 + E_{LAB}}{W}.$$

(c) Show that the results of parts a and b reduce in the nonrelativistic limit to the familiar expressions,

$$W \approx m_1 + m_2 + \left( \frac{m_2}{m_1 + m_2} \right) \frac{p_{LAB}{}^2}{2m_1}$$

$$\vec{p}' \approx \left( \frac{m_2}{m_1 + m_2} \right) \vec{p}_{LAB}, \vec{\beta}_{cm} \approx \frac{\vec{p}_{LAB}}{m_1 + m_2}.$$

Sol:

(a)

Let $P$ and $P'$ be 4-vectors in lab and CM frame respectively, then we have

$$P_1 = (E_1, \vec{p}_{LAB}), P_2 = (m_2, \vec{0}); P_1' = (E_1', \vec{p}'), P_2' = (E_2', -\vec{p}')$$

From the energy and momentum conservation in the lab frame, we have

$$P_1 + P_2 = P_3 + P_4$$

The total center-of-mass energy W:

$$W^2 = \left(E_1' + E_2'\right)^2 = \left(E_1' + E_2'\right)^2 - \left(\vec{p}_1' + \vec{p}_2'\right)^2 = \left(P_1' + P_2'\right)^2$$

Now note $(P_1' + P_2')^2$ is Lorentz invariant, we have

$$W^2 = (P_1' + P_2')^2 = (P_1 + P_2)^2 = P_1^2 + P_2^2 + 2P_1 \cdot P_2 = m_1^2 + m_2^2 + 2m_2 E_1$$

To find $\vec{p}'$, we consider $(P_1 \cdot P_2)^2$ and $(P_1' \cdot P_2')^2$:

$$(P_1 \cdot P_2)^2 = (m_2 E_1)^2 = m_2^2 (p_1^2 + m_1^2) = m_2^2 p_1^2 + m_1^2 m_2^2$$

$$\begin{aligned}
(P_1' \cdot P_2')^2 &= (E_1' E_2' + p'^2)^2 = E_1'^2 E_2'^2 + 2E_1' E_2' p'^2 + p'^4 \\
&= (p'^2 + m_1^2)(p'^2 + m_2^2) + 2E_1' E_2' p'^2 + p'^4 \\
&= 2p'^4 + (m_1^2 + m_2^2) p'^2 + 2E_1' E_2' p'^2 + m_1^2 m_2^2 \\
&= p'^2 (2p'^2 + m_1^2 + m_2^2 + 2E_1' E_2') + m_1^2 m_2^2 \\
&= p'^2 (E_1'^2 + 2E_1' E_2' + E_2'^2) + m_1^2 m_2^2 = p'^2 W^2 + m_1^2 m_2^2
\end{aligned}$$

From Lorentz invariance, we have

$$(P_1 \cdot P_2)^2 = (P_1' \cdot P_2')^2 \Rightarrow m_2^2 p_1^2 = p'^2 W^2 \Rightarrow p' = \frac{m_2}{W} p_1$$

Since $\vec{p}_1$ and $\vec{p}'$ are in the same direction (the Lorentz boost is along $\vec{p}_1$), therefore we have $\vec{p}' = \frac{m_2}{W}\vec{p}_1$.

(b)

We can also obtain $\vec{p}'$ from Lorentz transformation of $\vec{p}_1$ ( and $-\vec{p}'$ from $\vec{p}_2$)

$$p' = \gamma_{cm}(p_1 - \beta_{cm}E_1); (-p') = \gamma_{cm}(-\beta_{cm}m_2)$$

Thus,

$$\beta_{cm} = \frac{p_1}{m_2 + E_1} \Rightarrow \vec{\beta}_{cm} = \frac{\vec{p}_1}{m_2 + E_1}$$

$$\gamma_{cm} = \frac{1}{\sqrt{1-\beta_{cm}^2}} = \frac{m_2 + E_1}{\sqrt{(m_2 + E_1)^2 - p_1^2}} = \frac{m_2 + E_1}{\sqrt{m_2^2 + 2m_2E_1 + E_1^2 - p_1^2}} = \frac{m_2 + E_1}{W}$$

(c)

In the non-relativistic limit, $E_1 \approx m_1 + \frac{p_1^2}{2m_1}$.

Therefore,

$$W^2 \approx m_1^2 + m_2^2 + 2m_2(m_1 + \frac{p_1^2}{2m_1}) = (m_1 + m_2)^2 + \frac{m_2}{m_1}p_1^2 = (m_1 + m_2)^2\left[1 + \frac{m_2}{(m_1 + m_2)^2}\frac{p_1^2}{m_1}\right]$$

$$W = (m_1 + m_2)\sqrt{1 + \frac{m_2}{(m_1 + m_2)^2}\frac{p_1^2}{m_1}} \approx (m_1 + m_2)\left[1 + \frac{m_2}{(m_1 + m_2)^2}\frac{p_1^2}{2m_1}\right] = m_1 + m_2 + \frac{m_2}{m_1 + m_2}\frac{p_1^2}{2m_1}$$

Similarly,

$$\vec{p}' = \frac{m_2}{W}\vec{p}_1 \approx \frac{m_2}{m_1 + m_2}\vec{p}_1$$

$$\vec{\beta}_{cm} = \frac{\vec{p}_1}{m_2 + E_1} \approx \frac{\vec{p}_1}{m_1 + m_2}$$

These are the familiar Galilean relativity results.

11.30

An isotropic linear material medium, characterized by the constitutive relations (in its rest frame $K'$), $\vec{D}' = \varepsilon\vec{E}'$ and $\mu\vec{H}' = \vec{B}'$, is in uniform translation with velocity $\vec{v}$ in the inertial frame $K$. By exploiting the fact that $F_{\mu\upsilon} = (\vec{E}, \vec{B})$ and $G_{\mu\upsilon} = (\vec{D}, \vec{H})$ transform as second rank 4-tensors under Lorentz transformations, show that the macroscopic fields $\vec{D}$ and $\vec{H}$ are given in terms of $\vec{E}$ and $\vec{B}$ by

$$\vec{D} = \varepsilon\vec{E} + \gamma^2\left(\varepsilon - \frac{1}{\mu}\right)\left[\beta^2\vec{E}_\perp + \beta\times\vec{B}\right]$$

$$\vec{H} = \frac{1}{\mu}\vec{B} + \gamma^2\left(\varepsilon - \frac{1}{\mu}\right)\left[-\beta^2\vec{B}_\perp + \beta\times\vec{E}\right]$$

where $\vec{E}_\perp$ and $\vec{B}_\perp$ are components perpendicular to $\vec{v}$.

Sol:

Since $F_{\mu\upsilon}$ transforms as a rank-2 tensor, we have seen that the components $\vec{E}$ and $\vec{B}$ transform according to

$$\vec{E}' = \gamma\left(\vec{E} + \vec{\beta}\times\vec{B}\right) - \frac{\gamma^2}{\gamma+1}\vec{\beta}\left(\vec{\beta}\cdot\vec{E}\right)$$

$$\vec{B}' = \gamma\left(\vec{B} - \vec{\beta}\times\vec{E}\right) - \frac{\gamma^2}{\gamma+1}\vec{\beta}\left(\vec{\beta}\cdot\vec{B}\right)$$

For this problem, it is actually convenient to rewrite these expressions in terms of the perpendicular and parallel field components

$$\vec{E}' = \gamma\left(\vec{E}_\perp + \vec{\beta}\times\vec{B}\right) - \hat{\beta}\left(\hat{\beta}\cdot\vec{E}\right)$$
$$\vec{B}' = \gamma\left(\vec{B}_\perp - \vec{\beta}\times\vec{E}\right) - \hat{\beta}\left(\hat{\beta}\cdot\vec{B}\right)$$

where $\vec{E}_\perp = \vec{E} - \hat{\beta}\left(\hat{\beta}\cdot\vec{E}\right) = -\hat{\beta}\times\left(\hat{\beta}\times\vec{E}\right)$ and similarly for $\vec{B}_\perp$. Since these are the relativistic transformations of the components of a rand-2 tensor, any other rank-2 tensor must transform similarly. In particular, since $\vec{D}$ and $\vec{H}$ are components of the $G_{\mu\upsilon}$ tensor, they also transform as

$$\vec{D}' = \gamma\left(\vec{D}_\perp + \vec{\beta}\times\vec{H}\right) - \hat{\beta}\left(\hat{\beta}\cdot\vec{D}\right)$$
$$\vec{H}' = \gamma\left(\vec{H}_\perp - \vec{\beta}\times\vec{D}\right) - \hat{\beta}\left(\hat{\beta}\cdot\vec{H}\right)$$

The inverse transformation may be obtained by taking $\hat{\beta} \to -\hat{\beta}$

$$\vec{D}=\gamma\left(\vec{D}_{\perp}'+\vec{\beta}\times\vec{H}'\right)-\hat{\beta}\left(\hat{\beta}\cdot\vec{D}'\right)$$
$$\vec{H}=\gamma\left(\vec{H}_{\perp}'-\vec{\beta}\times\vec{D}'\right)-\hat{\beta}\left(\hat{\beta}\cdot\vec{H}'\right)$$

Using the constitutive relations $\vec{D}'=\varepsilon\vec{E}'$ and $\mu\vec{H}'=\vec{B}'$ gives

$$\vec{D}=\gamma\left(\varepsilon\vec{E}_{\perp}'+\vec{\beta}\times\frac{1}{\mu}\vec{B}'\right)-\hat{\beta}\left(\hat{\beta}\cdot\varepsilon\vec{E}'\right)$$
$$\vec{H}=\gamma\left(\frac{1}{\mu}\vec{B}_{\perp}'-\vec{\beta}\times\varepsilon\vec{E}'\right)-\hat{\beta}\left(\hat{\beta}\cdot\frac{1}{\mu}\vec{B}'\right)$$

From the result we have

$$\vec{E}'=\gamma\left(\vec{E}_{\perp}+\vec{\beta}\times\vec{B}\right)-\hat{\beta}\left(\hat{\beta}\cdot\vec{E}\right)$$
$$\vec{B}'=\gamma\left(\vec{B}_{\perp}-\vec{\beta}\times\vec{E}\right)-\hat{\beta}\left(\hat{\beta}\cdot\vec{B}\right)$$

We split it into parallel and perpendicular field components

$$\vec{\beta}\cdot\vec{E}'=\vec{\beta}\cdot\vec{E}$$
$$\vec{\beta}\cdot\vec{B}'=\vec{\beta}\cdot\vec{B}$$

and

$$\vec{E}_{\perp}'=\gamma\left(\vec{E}_{\perp}+\vec{\beta}\times\vec{B}\right)$$
$$\vec{B}_{\perp}'=\gamma\left(\vec{B}_{\perp}-\vec{\beta}\times\vec{E}\right)$$

As a result, we easily see that

$$\vec{\beta}\times\vec{E}'=\gamma\left(\vec{\beta}\times\vec{E}-\beta^{2}\times\vec{B}_{\perp}\right)$$
$$\vec{\beta}\times\vec{B}'=\gamma\left(\vec{\beta}\times\vec{B}+\beta^{2}\times\vec{E}_{\perp}\right)$$

Therefore, we can obtain

$$\vec{D}=\gamma^{2}\left(\varepsilon\left(\vec{E}_{\perp}+\vec{\beta}\times\vec{B}\right)-\frac{1}{\mu}\left(\vec{\beta}\times\vec{B}+\beta^{2}\times\vec{E}_{\perp}\right)\right)+\varepsilon\hat{\beta}\left(\hat{\beta}\cdot\vec{E}\right)$$
$$\vec{H}=\gamma^{2}\left(\frac{1}{\mu}\left(\vec{B}_{\perp}-\vec{\beta}\times\vec{E}\right)+\varepsilon\left(\vec{\beta}\times\vec{E}-\beta^{2}\times\vec{B}_{\perp}\right)\right)+\frac{1}{\mu}\hat{\beta}\left(\hat{\beta}\cdot\vec{B}\right)$$

$$\Rightarrow$$

$$\vec{D}=\varepsilon\left[\vec{E}_{\perp}+\hat{\beta}\left(\hat{\beta}\cdot\vec{E}\right)\right]+\gamma^{2}\left(\varepsilon-\frac{1}{\mu}\right)\left(\beta^{2}\times\vec{E}_{\perp}+\vec{\beta}\times\vec{B}\right)$$
$$\vec{H}=\frac{1}{\mu}\left[\vec{B}_{\perp}+\hat{\beta}\left(\hat{\beta}\cdot\vec{B}\right)\right]-\gamma^{2}\left(\varepsilon-\frac{1}{\mu}\right)\left(\beta^{2}\times\vec{B}_{\perp}-\vec{\beta}\times\vec{E}\right)$$

$\Rightarrow$

$$\vec{D} = \varepsilon\vec{E} + \gamma^2\left(\varepsilon - \frac{1}{\mu}\right)\left(\beta^2 \times \vec{E}_\perp + \vec{\beta} \times \vec{B}\right)$$

$$\vec{H} = \frac{1}{\mu}\vec{B} - \gamma^2\left(\varepsilon - \frac{1}{\mu}\right)\left(-\beta^2 \times \vec{B}_\perp + \vec{\beta} \times \vec{E}\right)$$