14.1

Verify by explicit calculation that the Lienard-Wiechert expressions for all components of $\vec{E}$ and $\vec{B}$ for a particle moving with constant velocity agree with the ones obtained in the text by means of a Lorentz transformation. Follow the general method at the end of Section 14.1.

Sol:

From Jackson (14.13) and (14.14),

$$\vec{E}(\vec{x},t) = e\left[\frac{\hat{n}-\vec{\beta}}{\gamma^2\left(1-\vec{\beta}\cdot\hat{n}\right)^3 R^2}\right]_{ret} + \frac{e}{c}\left[\frac{\hat{n}\times\left\{\hat{n}-\vec{\beta}\right\}\times\dot{\vec{\beta}}}{\left(1-\vec{\beta}\cdot\hat{n}\right)^3 R}\right]_{ret}$$

Only consider $\dot{\vec{\beta}} = 0$

$$\vec{E}(\vec{x},t) = e\left[\frac{\hat{n}-\vec{\beta}}{\gamma^2\left(1-\vec{\beta}\cdot\hat{n}\right)^3 R^2}\right]_{ret}$$

$$= \frac{e\left(\hat{n}-\vec{\beta}\right)R}{\gamma^2\left(1-\vec{\beta}\cdot\hat{n}\right)^3 R^3}$$

From picture 14.2, $\hat{n}R - \vec{\beta}R = r = b\hat{y} + vt\hat{x}$ and

$$\left[\left(1-\vec{\beta}\cdot\hat{n}\right)R\right]^2 = \frac{1}{\gamma^2}\left(b^2+\gamma^2 v^2 t^2\right)$$

$\Rightarrow$

$$E_y(x,t) = \frac{\gamma e b}{\left(b^2+\gamma^2 v^2 t^2\right)^{3/2}}$$

$$E_x(x,t) = \frac{\gamma e v t}{\left(b^2+\gamma^2 v^2 t^2\right)^{3/2}}$$

$\Rightarrow$

$$B = \beta\frac{\gamma e b}{\left(b^2+\gamma^2 v^2 t^2\right)^{3/2}} = \beta E_y$$

14.4 Using the Lienard-Wiechert fields, discuss the time-averaged power radiated per unit solid angle in nonrelativistic motion of a particle with charge e, moving

(a) along the z axis with instantaneous position $z(t) = a\cos\omega_0 t$,

(b) in a circle of radius R in the x-y plane with constant angular frequency ω_0.

Sketch the angular distribution of the radiation and determine the total power radiated in each case.

Sol:

(a)

In the non-relativisitic limit, the radiated power is given by

$$\frac{dP(t)}{d\Omega} = \frac{e^2}{4\pi c}\left|\hat{n}\times\dot{\vec{\beta}}\right|^2 \text{------(1)}$$

In the case of harmonic motion along the z axis, we take

$$\vec{r} = \hat{z}a\cos\omega_0 t, \vec{\beta} = -\hat{z}\frac{a\omega_0}{c}\sin\omega_0 t, \dot{\vec{\beta}} = -\hat{z}\frac{a\omega_0^{\ 2}}{c}\cos\omega_0 t$$

By symmetry, we assume the observer is in the x-z plane tilted with angle θ from the vertical. In other words, we take

$$\hat{n} = \hat{x}\sin\theta + \hat{z}\cos\theta$$

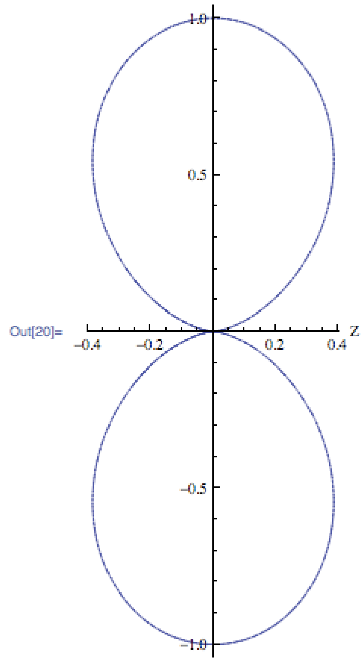
This provides enough information to simply substitute into the power expression (1)

$$\hat{n}\times\dot{\vec{\beta}} = \hat{y}\frac{a\omega_0^{\ 2}}{c}\sin\theta\cos\omega_0 t \Rightarrow \frac{dP(t)}{d\Omega} = \frac{e^2 a^2\omega_0^{\ 4}}{4\pi c^3}\sin^2\theta\cos^2\omega_0 t$$

Taking a time average $(\cos^2\omega_0 t \to 1/2)$ gives

$$\frac{dP(t)}{d\Omega} = \frac{e^2 a^2\omega_0^4}{8\pi c^3}\sin^2\theta$$

This is a familiar dipole power distribution, which looks like



(Note: the positive horizontal axis is $\theta = 0$, and the figure is plotted in the unit of $\frac{e^2 a^2 \omega_0^4}{8\pi c^3}$.)

Integrating over angles gives the total power

$$P = \int \frac{dP}{d\Omega} d\Omega = \frac{e^2 a^2 \omega_0^4}{8\pi c^3} 2\pi \int_0^\pi \sin^2\theta d(\cos\theta) = \frac{e^2 a^2 \omega_0^4}{3c^3}.$$

(b)

Here we take instead

$\vec{r} = R(\hat{x}\cos\omega_0 t + \hat{y}\sin\omega_0 t)$

$\rightarrow \vec{\beta} = \frac{R\omega_0}{c}(-\hat{x}\sin\omega_0 t + \hat{y}\cos\omega_0 t), \dot{\vec{\beta}} = -\frac{R\omega_0{}^2}{c}(\hat{x}\cos\omega_0 t + \hat{y}\sin\omega_0 t)$

Then

$$\hat{n} \times \dot{\vec{\beta}} = -\frac{R\omega_0{}^2}{c}[\hat{y}\cos\theta\cos\omega_0 t + (\hat{z}\sin\theta - \hat{x}\cos\theta)\sin\omega_0 t]$$

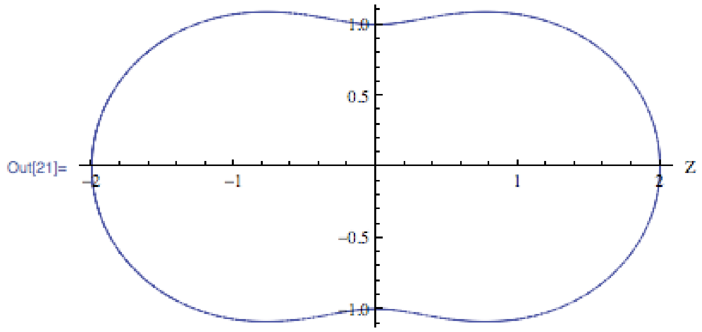
which gives

$$\frac{dP(t)}{d\Omega} = \frac{e^2 R^2 \omega_0^4}{4\pi c^3}\left(\cos^2\theta\cos^2\omega_0 t + \sin^2\omega_0 t\right)$$

Taking a time average gives

$$\frac{dP}{d\Omega} = \frac{e^2 R^2 \omega_0^4}{8\pi c^3}\left(\cos^2\theta + 1\right)$$

This distribution looks like



The total power is given by integration over angles. The result is $P = \frac{2e^2 R^2 \omega_0^4}{3c^3}$.

14.5

A nonrelativistic particle of charge ze, mass m, and kinetic energy E makes a head-on collision with a fixed central force field of finite range. The interaction is repulsive and described by a potential $V(r)$, which becomes greater than E at close distances.

(a) Show that the total energy radiated is given by

$$\Delta W = \frac{4}{3}\sqrt{\frac{m}{2}}\int_{r_{\min}}^{\infty}\left|\frac{dV}{dr}\right|^2 \frac{dr}{\sqrt{V(r_{\min}) - V(r)}}$$

where $r_{\min}$ is the closest distance of approach in the collision.

(b) If the interaction is a Coulomb potential $V(r) = zZe^2/r$, show that the total energy radiated is

$$\Delta W = \frac{8}{45}\frac{zmv_0^5}{zc^3}$$

where v_0 is the velocity of the charge at infinity.

Sol:

(a) In the non-relativistic limit, we may use Lamour's formula written in terms of $\dot{\vec{p}}$

$$P(t) = \frac{2(ze)^2}{3m^2c^3}\left|\frac{d\vec{p}}{dt}\right|^2 = \frac{2(ze)^2}{3m^2c^3}\left(\frac{dV(r)}{dr}\right)^2$$

where we have used Newton's second law to write

$$\frac{d\vec{p}}{dt} = \vec{F} = -\hat{r}\frac{dV(r)}{dr}$$

The radiated energy is given by integrating power over time

$$\Delta W = \int_{-\infty}^{\infty} P(t)dt$$

However, this can be converted to an integral over the trajectory of the particle. By symmetry, we double the value of the integral from closest approach to infinity

$$\Delta W = 2\int_{r_{\min}}^{\infty} \frac{P}{dr/dt}dr$$

The velocity dr/dt can be obtained from energy conservation. For a head-on collision, we gave simply

$$E = \frac{1}{2}m\dot{r}^2 + V(r) \Rightarrow \frac{dr}{dt} = \sqrt{\frac{2(E - V(r))}{m}}$$

Then

$$\Delta W = \frac{4(ze)^2}{3m^2c^3}\sqrt{\frac{m}{2}}\int_{r_{\min}}^{\infty}\left(\frac{dV}{dr}\right)\frac{dr}{\sqrt{E - V(r)}}$$

Since the velocity (and hence kinetic energy) vanishes at closest approach, the total energy E is the same as the potential energy at closest approach, $E = V(r_{\min})$. Using this finally gives

$$\Delta W = \frac{4(ze)^2}{3m^2c^3}\sqrt{\frac{m}{2}}\int_{r_{\min}}^{\infty}\left(\frac{dV}{dr}\right)^2\frac{dr}{\sqrt{V(r_{\min}) - V(r)}}$$

(b) Substituting

$V(r) = \frac{zZe^2}{r}$, $\frac{dV}{dr} = -\frac{zZe^2}{r^2}$ into the result of part (a) gives

$$\Delta W = \frac{4z^3Ze^5}{3m^2c^3}\sqrt{\frac{zZmr_{\min}}{2}}\int_{r_{\min}}^{\infty}\frac{1}{r^{\frac{7}{2}}}\frac{dr}{\sqrt{r - r_{\min}}}$$

$$= \frac{4z^3Ze^5}{3m^2c^3r_{\min}^3}\sqrt{\frac{zZmr_{\min}}{2}}\int_{1}^{\infty}\frac{1}{r^{\frac{7}{2}}}\frac{dr}{\sqrt{r - 1}}$$

$$= \frac{4z^3Ze^5}{3m^2c^3r_{\min}^3}\sqrt{\frac{zZmr_{\min}}{2}}\frac{16}{15}$$

$$= \frac{32z^3Ze^5}{45m^2c^3r_{\min}^3}\sqrt{\frac{zZmr_{\min}}{2}}$$

We may relate $r_{\min}$ to the velocity v_0 at infinity using energy conservation

$$\frac{zZe^2}{r_{\min}} = \frac{1}{2}mv_0^2$$

$$\Rightarrow r_{\min} = \frac{2zZe^2}{mv_0^2}$$

Then

$$\Delta W = \frac{8zmv_0^5}{45Zc^3}$$

14.9 A particle of mass m, charge q, moves in a plane perpendicular to a uniform, static, magnetic induction B.

(a) Calculate the total energy radiated per unit time, expressing it in terms of the constants already defined and the ratio γ of the particle's total energy to its rest energy.

(b) If at time t = 0 the particle has a total energy $E_0 = \gamma_0 mc^2$, show that it will have energy $E = \gamma mc^2 < E_0$ at a time t, where

$$t \simeq \frac{3m^3c^5}{2q^4B^2}\left(\frac{1}{\gamma} - \frac{1}{\gamma_0}\right)$$

provided $\gamma \gg 1$.

(c) If the particle is initially nonrelativistic and has a kinetic energy T_0 at t = 0, what is its kinetic energy at time t ?

(d) If the particle is actually trapped in the magnetic dipole field of the earth and is spiraling back and forth along a line of force, does it radiate more energy while near the equator, or while near its turning points? Why? Make quantitative statements if you can.

Sol:

(a)

From Eq. (14.24) and (14.25), radiated power is

$$P = -\frac{2}{3}\frac{q^2}{m^2c^3}\frac{dp_\mu}{d\tau}\frac{dp^\mu}{d\tau} = \frac{2q^2}{4m^2c^3}\left[\left(\frac{dp}{d\tau}\right)^2 - \frac{1}{c^2}\left(\frac{dE}{d\tau}\right)^2\right] = \frac{2q^2\gamma^2}{3m^2c^3}\left[\left(\frac{d\vec{p}}{dt}\right)^2 - \frac{1}{c^2}\left(\frac{dE}{dt}\right)^2\right]$$

From the Lorentz force law $\left|\frac{d\vec{p}}{dt}\right| = \left|\vec{F}\right| = \left|\frac{q}{c}\vec{v}\times\vec{B}\right| = \frac{q}{c}vB$ since $\vec{v}\perp\vec{B}$.

Also $\frac{dE}{dt} = 0$ because $\vec{B}$ field does no work (neglecting radiation reaction).

$$P = \frac{2}{3}\frac{q^2}{m^2c^3}\gamma^2(q^2\beta^2B^2) = \frac{2}{3}\frac{q^4B^2}{m^2c^3}(\gamma^2 - 1) \text{ since } \gamma^2 = \frac{1}{1-\beta^2} \Rightarrow \gamma^2\beta^2 = \gamma^2 - 1.$$

(b)

At time t=0, particle has total energy $E_0 = \gamma_0 mc^2$; at time t, it has total energy $E = \gamma mc^2 < E_0$.

For $\gamma \gg 1$, $P = -\frac{dE}{dt} = \frac{2}{3}\frac{q^4B^2}{m^2c^3}\gamma^2$ from part (a).

$$E = \gamma mc^2 \Rightarrow -\frac{dE}{dt} = -mc^2\frac{d\gamma}{dt} = \frac{2}{3}\frac{q^4B^2}{m^2c^3}\gamma^2$$

$$-\frac{d\gamma}{\gamma^2} = \frac{2}{3}\frac{q^4B^2}{m^3c^5}dt$$

Integrate both sides from t=0 to t. Recall $\gamma(t=0) = \gamma_0$

$$\frac{1}{\gamma} - \frac{1}{\gamma_0} = \frac{2}{3}\frac{q^4B^2}{m^3c^5}t$$

$$\Rightarrow t \simeq \frac{3}{2}\frac{m^3c^5}{q^4B^2}\left(\frac{1}{\gamma} - \frac{1}{\gamma_0}\right).$$

(c)

In the non-relativistic situation,

$$P = \frac{2}{3}\frac{q^2}{c^3}\left|\dot{V}\right|^2 = \frac{4}{3}\frac{q^4B^2}{m^3c^5}T$$

$$-\frac{dT}{dt} = P = \frac{4}{3}\frac{q^4B^2}{m^3c^5}T$$

Integrating over t, we get $T = \varepsilon_0 \exp\left(-\frac{4}{3}\frac{q^4B^2}{m^3c^5}t\right)$.

(d)

It radiates more energy while near its turning points. You can see problem 12.9 for further information.