Chapter 7: Electrodynamics

7.1 Electromotive Force   7.1.1 Ohm's Law

Pushing on the charges make a current flow. How fast the charges move depends on the nature of the materials and the forces.

current density ⟶ $\mathbf{J} = \rho \mathbf{v}$ ⟵ velocity of the charge

volume charge density

Ohm's law (an empirical equation): $\mathbf{J} = \sigma \mathbf{E}$ or $\mathbf{E} = \dfrac{\mathbf{J}}{\sigma} = \rho \mathbf{J}$

conductivity resistivity

ρ: volume charge density or resistivity?

The Lorentz force drives the charges to produce current:

$$\mathbf{J} \propto \mathbf{F} \quad \Rightarrow \quad \mathbf{J} = \sigma(\mathbf{E} + \cancel{\mathbf{v} \times \mathbf{B}}^{\,\approx 0})$$

Resistivities (ohm-meters)

Material	Resistivity	Material	Resistivity
Conductors:		*Semiconductors:*	
Silver	1.59×10^{-8}	Salt water (saturated)	4.4×10^{-2}
Copper	1.68×10^{-8}	Germanium	4.6×10^{-1}
Gold	2.21×10^{-8}	Diamond	2.7
Aluminum	2.65×10^{-8}	Silicon	2.5×10^{3}
Iron	9.61×10^{-8}	*Insulators:*	
Mercury	9.58×10^{-7}	Water (pure)	2.5×10^{5}
Nichrome	1.00×10^{-6}	Wood	$10^{8} - 10^{11}$
Manganese	1.44×10^{-6}	Glass	$10^{10} - 10^{14}$
Graphite	1.4×10^{-5}	Quartz (fused)	$\sim 10^{16}$

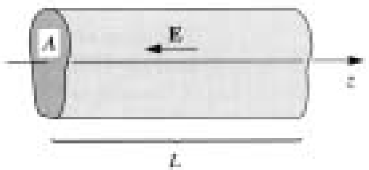
Confusion 1: $\mathbf{E}$=0 inside a conductor ➔ $\mathbf{J}$=0 ?

2: For a perfect conductor $\sigma=\infty$ ➔ $\mathbf{E}$=0 ?

Question: Can we treat the connecting wires in electric circuits as equal potentials?

Example 7.1

A cylindrical resistor of cross-sectional area A and length L is made from material with conductivity σ. If the potential is constant over each end, and the potential difference between the ends is V, what current flows?



Sol:

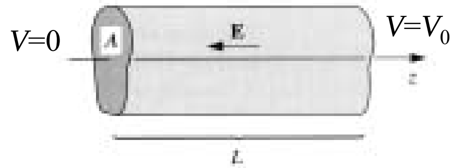
$$I = \mathbf{J} \cdot \mathbf{A} = \sigma E A = \frac{\sigma A}{L} V$$

Question: Is the electric field uniform within the wire?

To be proved in a moment, see Ex. 7.3.

Example 7.3

Prove that the electric field within the wire is uniform.



Sol:

The potential V with the cylinder obeys Laplace's equation.

On the cylinder surface $\mathbf{J} \cdot \mathbf{n} = 0 \quad \therefore \mathbf{E} \cdot \mathbf{n} = 0$, and hence $\partial V / \partial n = 0$

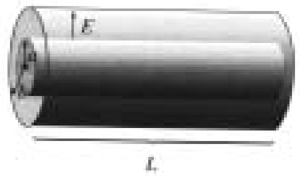
With V or its normal derivate specified on all the surfaces, the potential is uniquely determine (Prob. 3.4).

Guess: A potential obeys Laplace's equation and fits the boundary conditions.

$V(z) = \dfrac{V_0 z}{L}$ and $\mathbf{E} = -\nabla V = -\dfrac{V_0}{L}\hat{\mathbf{z}}$ ⟵ the unique solution.

Example 7.2

Two long cylinders (radii a and b) are separated by material of conductivity σ. If they are maintained at a potential different V, what current flows from one to the other, in a length L?



Sol: $\mathbf{J} = \dfrac{I}{\mathbf{A}} = \sigma \mathbf{E}$ $\quad \mathbf{E} = \dfrac{I}{2\pi\sigma L}\dfrac{1}{s}\hat{\mathbf{r}}$

$$V = \int_a^b \mathbf{E} \cdot d\mathbf{s} = \int_a^b \frac{I}{2\pi\sigma L}\frac{1}{s}ds = \frac{I}{2\pi\sigma L}\ln(b/a)$$

$$I = \frac{2\pi\sigma L}{\ln(b/a)}V$$

Ohm's Law

Ex. 7.1 $\quad V = \dfrac{L}{\sigma A} I$

Ex. 7.2 $\quad V = \dfrac{\ln(b/a)}{2\pi\sigma L} I$

$\Rightarrow V = IR$ (A more familiar version of Ohm's law.)

resistance

The total current flowing from one electrode to the other is proportional to the potential difference between them.

Resistance is measured in ohms (Ω): an ohm is a volt per ampere.

For a steady current and uniform conductivity,

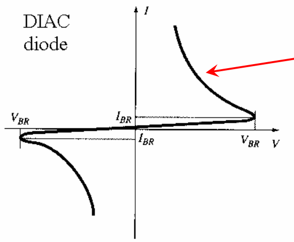
$$\rho = \varepsilon_0 \nabla \cdot \mathbf{E} = \varepsilon_0 \nabla \cdot (\frac{\mathbf{J}}{\sigma}) = \frac{\varepsilon_0}{\sigma}\nabla \cdot \mathbf{J} = 0$$

Any unbalanced charge resides on the surface.

Ohm's Law (rule of thumb)

Gauss's law or Ampere's law is really a true law, but Ohm's law is an empirical equation.

* Finding an exception won't win a Nobel prize.



negative resistance

Q1: Why the electric field does not accelerate the charge particle to a very high speed?

Q2: Ohm's law implies that a constant field produces a constant current, which suggests a constant velocity. Isn't that a contradiction of Newton's law.

Ohm's Law (a naive picture)

A naive picture: Electrons are frequently collided with ions which slow down the acceleration.

$$\lambda = \frac{1}{2}at^2 \Rightarrow t = \sqrt{\frac{2\lambda}{a}}, \text{ where } a = \frac{qE}{m}$$

$$\text{average velocity: } v_{\text{ave}} = \frac{1}{2}at = \sqrt{\frac{\lambda qE}{2m}} \propto \sqrt{E}$$

The velocity is proportional to the square root of the field. That is no good!

Q1: How to explain it correctly?

The charges in practice are already moving quite fast because of their thermal energy.

Ohm's Law (Drude model)

The net *drift velocity* is a tiny extra bit. The time between collisions is actually much shorter than we supposed.

$$\text{collision frequency: } t = \frac{\lambda}{v_{\text{thermal}}}$$

$$\text{average velocity: } \mathbf{v}_{\text{ave}} = \frac{1}{2}\mathbf{a}t = \frac{\mathbf{a}\lambda}{2v_{\text{thermal}}}$$

$$\text{acceleration: } \mathbf{a} = \frac{\mathbf{F}}{m} = \frac{q}{m}\mathbf{E}$$

(f: free electrons per molecule)

$$\mathbf{J} = n(fq)\mathbf{v}_{\text{ave}} = nfq\frac{\lambda}{2v_{\text{thermal}}}\frac{q}{m}\mathbf{E} = \left(\frac{nf\lambda q^2}{2mv_{\text{thermal}}}\right)\mathbf{E}$$

(n: molecules per unit volume)

The Joule Heating Law

$$\mathbf{J} = \left(\frac{nf\lambda q^2}{2mv_{\text{thermal}}}\right)\mathbf{E} = \sigma\mathbf{E}, \quad \text{where } \sigma = \frac{nf\lambda q^2}{2mv_{\text{thermal}}}$$

This equation correctly predicts that conductivity is proportional to the density of the moving charges and *ordinarily* decreases with increasing temperature.

The Joule heating law:

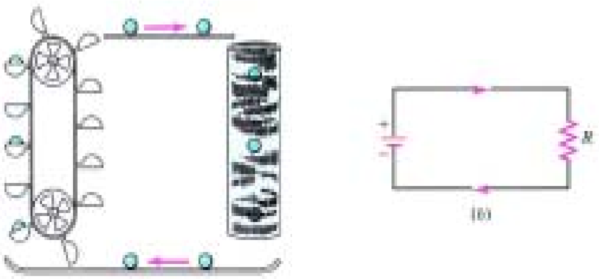
$$P = IV = I^2R = \frac{V^2}{R} \qquad \text{where} \begin{cases} I: \text{amperes} \\ R: \text{ohms} \\ V: \text{volts} \\ P: \text{watts} \end{cases}$$

7.1.2 Electromotive Force (emf)

An emf is the work per unit charge done by the source of emf in moving the charge around *a closed loop*.

$$\mathcal{E} = \frac{W_{\text{ne}}}{q}$$

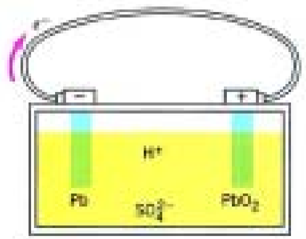
The subscript "ne" emphasizes that the work is done by some nonelectrostatic agent, such as a battery or an electrical generator.



What is the difference between emf and potential difference?

Electromotive Force: Production of a Current

What is the function of the acid solution in the voltaic pile?



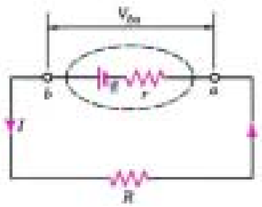
$$\text{Pb} + \text{SO}_4^- \rightarrow \text{PbSO}_4 + 2\text{e}^-$$

$$\text{PbO}_2 + 4\text{H}^+ + \text{SO}_4^- + 2\text{e}^- \rightarrow \text{PbSO}_4 + 2\text{H}_2\text{O}$$

Note that for every electron that leaves the Pb plate, another enters the PbO_2 plate.

Electromotive Force: Terminal Potential Difference

A real source of emf, such as a battery, has *internal resistance r*.



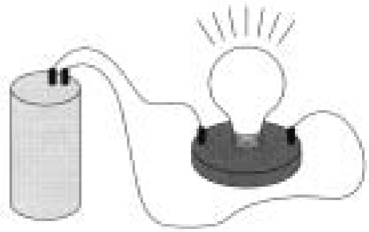
$$V_{ba} = V_b - V_a = \mathcal{E} - Ir$$

The change in potential is called the **terminal potential difference**.

Unlike the emf, which is a fixed property of the source, the terminal potential difference depends on the current flowing through it.

As a battery ages its internal resistance increases, and so, for a given output current, the terminal potential difference falls.

Electromotive Force Drives the Electrons



Example: A battery is hooked up to a light bulb.

The battery generates the force which drives the electrons move along the loop.

Snail's pace: the charges in a wire move slowly (~0.1 mm/s @ϕ=1mm, 1A, see Prob. 5.19(b)).

Q1: Why does the bulb response so fast when turning it on or off?

Q2: How do all the charges know to start moving at the same instant?

Example 5.19: The Snail's Pace

Calculate the average electron drift velocity in a copper wire 1mm in diameter, carrying a current of 1 A.

Sol: $$J = \frac{I}{\pi s^2} = \rho v_\mathrm{d} \Rightarrow v_\mathrm{d} = \frac{I}{\pi s^2 \rho} \quad (\rho\text{: volume charge density})$$

$$\rho = \frac{\text{mobile charges}}{\text{volume}} = \frac{\text{charge}}{\text{atom}}\frac{\text{atom}}{\text{mole}}\frac{\text{mole}}{\text{gram}}\frac{\text{gram}}{\text{volume}}$$
$$= (1.6\times10^{-19})(6\times10^{23})(1/64)(9) = 1.4\times10^{4}\,\mathrm{C/cm^3}$$

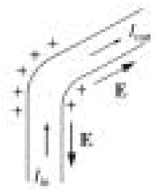
$$v_\mathrm{d} = \frac{I}{\pi s^2 \rho} = \frac{1}{\pi\times0.05^2\times1.4\times10^4} = 9.1\times10^{-3}\ \text{(cm/s)}$$

$$@\ 1\text{A},\ \phi\text{=1 mm} \Rightarrow v_\mathrm{d} = 0.091\ \text{(mm/s)}$$
$$@10\text{A},\ \phi\text{=1 mm} \Rightarrow v_\mathrm{d} = 0.91\ \text{(mm/s)}\ \text{Snail's pace}$$

Will the Charge Piling Up Somewhere?

If a current is not the same all the way around, then the charge is piling up somewhere, and the electric field of this accumulating charge is in such a direction as to even out the flow.

Charge piling up at the "knee" produces a field aiming away from the kink.



It self-corrects the current flow, and it does it all so quickly. (Why? Thermal electrons)

Forces Involved in Driving Currents Around a Circuit

Two forces involved in driving currents around a circuit.
$\mathbf{f}_s$: ordinarily confined to one portion of the loop (a battery, say).
$\mathbf{E}$: the electrostatic force: smooth out the flow and communicate the influence of the source to distant parts of the circuit.

$$\mathbf{f} = \mathbf{f}_s + \mathbf{E}$$

What is the physical agency responsible for $\mathbf{f}_s$?

- Battery → a chemical force
- Piezoelectric crystal → mechanical pressure
- Thermal couple → temperature gradient
- Photoelectric cell → light

The Electromotive Force

The net effect of the electromotive force is determined by the line integral of $\mathbf{f}$ around the circuit:

$$\mathcal{E} \equiv \oint \mathbf{f}\cdot d\mathbf{l} = \oint \mathbf{f}_s\cdot d\mathbf{l} + \underbrace{\cancel{\oint \mathbf{E}\cdot d\mathbf{l}}}_{=0} = \oint \mathbf{f}_s\cdot d\mathbf{l}$$

(the electromotive force, or emf)

Emf is a lousy term, since it is not a force at all --- it is the integral of a force per unit charge.

$$\mathcal{E} = \frac{W_{ne}}{q}$$

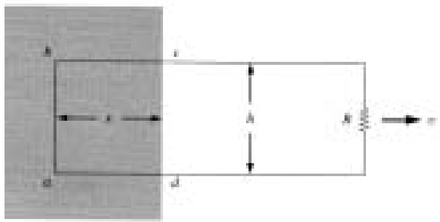
An emf is the work per unit charge done by the source of emf in moving the charge around *a closed loop*.

7.1.3 Motional emf

The most common source of the emfs: the generator

Generators exploit motional emf's, which arise when you move a wire through a magnetic field.

A primitive model for a generator



Shaded region: uniform B-field pointing into the page.

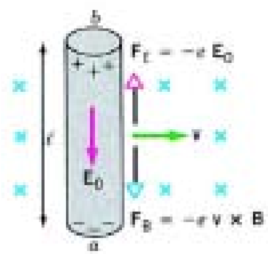
R: whatever it is, we are trying to drive current through.

$$\mathcal{E} = \oint \mathbf{f}_{mag}\cdot d\mathbf{l} = \oint (\mathbf{v}\times\mathbf{B})\cdot d\mathbf{l} = vBh$$

Motional emf (another example)

When the magnetic field is constant in time, there is no induced electric field.

When a mental rod moving perpendicular to magnetic field lines. There is a separation of charge and an associated electrostatic potential difference set up.



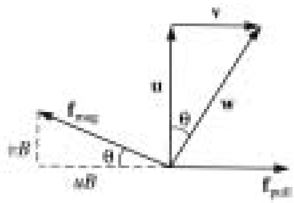
The potential difference associated with this electrostatic field is given by $V_b - V_a = E_0 L = BLv$.

$$\mathcal{E} = \oint (\mathbf{v}\times\mathbf{B})\cdot d\ell$$

Since there is no current flowing, the "terminal potential difference" is equal to the **motional emf**.

Magnetic Force Does No Work

A person exerts a force per unit charge on the wire by pulling it. The force counteracts the force generated by the magnetic force quB.



$$f_{\text{pull}} = uB$$

This force is transmitted to the charge by the structure of the wire.

The work done per unit charge is:

$$\int \mathbf{f}_{\text{pull}} \cdot d\mathbf{l} = (uB)\left(\frac{h}{\cos\theta}\right)\cos(90° - \theta) = vBh = \mathcal{E}$$

The work done per unit charge is exactly equal to the emf.

Instantaneous emf

$$\mathcal{E} = vBh = Bhv(t) = \mathcal{E}(t)$$

E: carried out at one instant of time – take a “snapshot” of the loop, if you like, and work from that.

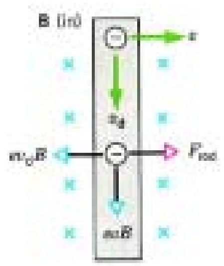
The magnetic force is responsible for establishing the emf and the emf seems to heat the resistor (i.e. do work), but magnetic fields never do work.

Who is supplying the energy that heats the resistor.

The person who’s pulling on the loop!

Magnetic Force Does No Work (II)

In the previous viewgraph, we find a source of emf converts some form of energy into electrostatic energy and does work on charges. Can magnetic forces do work? No.



The magnetic field acts, in a sense, as an intermediary in the transfer of the energy from the external agent to the rod.

The Flux Rule

There is a particular nice way of expressing the emf generated in a moving loop ➔ the flux rule.

Let $\Phi \equiv \int \mathbf{B} \cdot d\mathbf{a}$ the flux of B through the loop

The flux of a rectangular loop $\Phi = Bhx$

The flux change rate $\frac{d\Phi}{dt} = Bh\frac{dx}{dt} = -Bhv$

The minus sign accounts for the fact that dx/dt is negative.

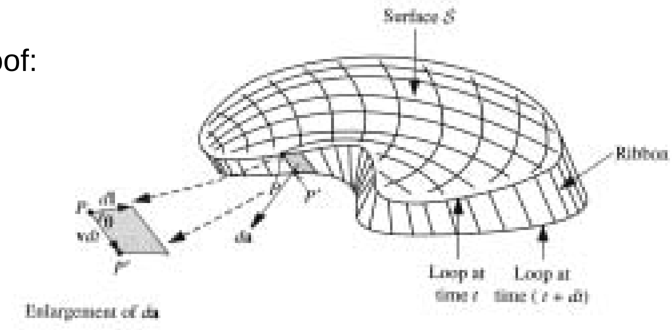
The flux rule for the motional emf: $-\frac{d\Phi}{dt} = Bhv = \mathcal{E}$

Next step proves: $\mathcal{E} = -\frac{d\Phi}{dt}$

The Flux Rule (Generalized)

The flux rule can be applied to non-rectangular loop through non-uniform magnetic field.

Proof:



Compute the flux at time t using surface S, and the flux at time $t+dt$, using the surface consisting of S plus the “ribbon” that connects the new position of the loop to the old.

The Flux Rule (Generalized II)

The change in flux is

$$d\Phi = \Phi(t+dt) - \Phi(t) = \Phi_{\text{ribbon}} = \int_{\text{ribbon}} \mathbf{B}\cdot d\mathbf{a}$$

The infinitesimal element of area on the ribbon

$$d\mathbf{a} = \mathbf{v}dt \times d\mathbf{l} = (\mathbf{v}\times d\mathbf{l})dt$$

$$d\Phi = \int_{\text{ribbon}} \mathbf{B}\cdot d\mathbf{a} = \int_{\text{ribbon}} \mathbf{B}\cdot(\mathbf{v}\times d\mathbf{l})dt$$

$$\boxed{\begin{aligned}\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) &= \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})\\ &= -(\mathbf{B}\times\mathbf{A})\cdot\mathbf{C}\end{aligned}}$$

$$\frac{d\Phi}{dt} = \int_{\text{ribbon}} \mathbf{B}\cdot(\mathbf{v}\times d\mathbf{l}) = -\int_{\text{ribbon}} (\mathbf{v}\times\mathbf{B})\cdot d\mathbf{l}$$

$$= -\int_{\text{ribbon}} [(\mathbf{v}+\mathbf{u})\times\mathbf{B}]\cdot d\mathbf{l} = -\oint_{\text{ribbon}} (\mathbf{W}\times\mathbf{B})\cdot d\mathbf{l}$$

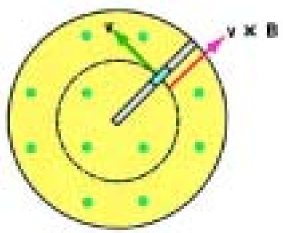
magnetic force per unit charge

$$\frac{d\Phi}{dt} = -\oint_{\text{ribbon}} \mathbf{f}_{\text{mag}}\cdot d\mathbf{l} = -\mathcal{E} \qquad \text{qed}$$

Example 7.4

In a homopolar generator a conducting disk of radius R rotates at angular velocity ω rad/s. Its plane is perpendicular to a uniform and constant magnetic field **B**. What is the emf generated between the center and the rim?

Solution:



$$\mathcal{E} = \oint (v\times B)\cdot d\ell = \int_0^R vBdr$$

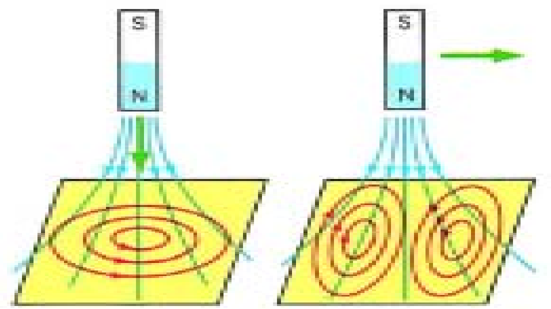
$$= \int_0^R \omega rBdr = \frac{1}{2}\omega BR^2$$

Hint1: How to choose a proper closed loop?

Hint2: The total magnetic flux passing through the disk is constant in time. Where is the induced emf coming from? (Ref. Benson & Feyman)

Eddy Currents (I)

What happens when a bar magnet approaches or moves parallel to a conducting plate? It induces eddy current.

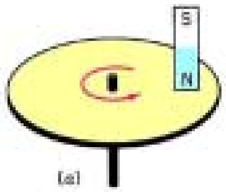


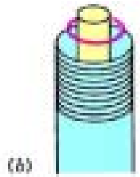
The eddy current is distributed throughout the plate.

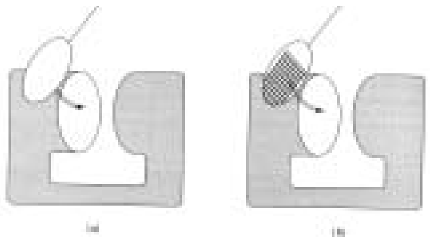
Eddy Currents (II)

Applications of the eddy current:

1. The braking system of a train.
2. Eddy current generated in copper pots can also be used for "inductive cooking".
3. Project a metal ring. The ring gets very hot when projected.



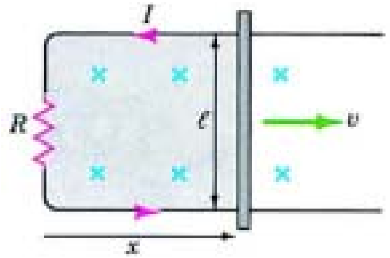




Example

A metal rod of length L sides at constant velocity v on conducting rails that terminate in a resistor R. There is a uniform and constant magnetic field perpendicular to the plane of the rails. Find: (a) the current in the resistor; (b) the power dissipated in the resistor; (c) the mechanical power needed to pull the rod.

Solution:



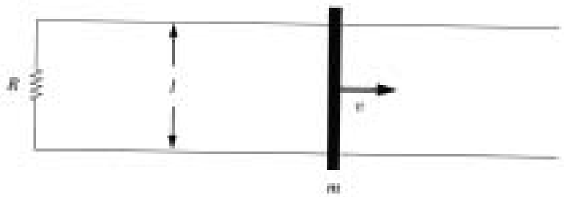
$$\text{(a)}\ |V_{\text{emf}}| = \frac{d\Phi}{dt} = B\frac{dA}{dt} = Blv$$

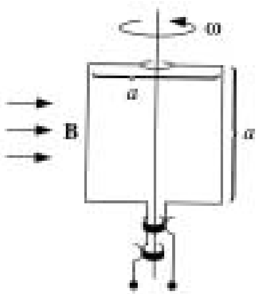
$$\text{I} = \frac{|V_{\text{emf}}|}{R} = \frac{Blv}{R}$$

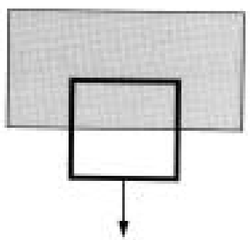
$$\text{(b)}\ P_{elec} = I^2 R = \frac{(Blv)^2}{R}$$

$$\text{(c)}\ P_{mech} = \mathbf{F}_{ext}\cdot\mathbf{v} = \frac{(Blv)^2}{R}$$

More Examples







Homework of Chap.7 (part I)

Prob. 2, 6, 8