### 7.2.3 Inductance

Two loops of wire at rest.
A steady current $I_{1}$ around loop $1 \rightarrow \mathbf{B}_{1}$ Some $\mathbf{B}_{1}$ passes through loop $2 \boldsymbol{\rightarrow} \Phi_{2}$


$$
\begin{gathered}
\Phi_{2}=\int \mathbf{B}_{1} \cdot d \mathbf{a} \text { and } \mathbf{B}_{1}=\frac{\mu_{0} I_{1}}{4 \pi} \oint \frac{d \mathbf{l}_{1} \times \boldsymbol{r}}{\mathbf{r}^{2}} \\
\Phi_{2}=\left[\frac{\mu_{0}}{4 \pi} \int \oint \frac{d \mathbf{l}_{1} \times \boldsymbol{r}}{\mathbf{r}^{2}} \cdot d \mathbf{a}\right] I_{1}=M_{21} I_{1}
\end{gathered}
$$



The constant of proportionality: mutual inductance of the two loops.

## Neumann Formula for the Mutual Inductance

$\Phi_{2}=\int \mathbf{B}_{1} \cdot d \mathbf{a}=\int\left(\nabla \times \mathbf{A}_{1}\right) \cdot d \mathbf{a}=\oint \mathbf{A}_{1} \cdot d \mathbf{I}_{2}$
$\mathbf{A}_{1}=\frac{\mu_{0} I_{1}}{4 \pi} \oint \frac{d \mathbf{I}_{1}}{\mathrm{r}}$
$\Phi_{2}=\frac{\mu_{0} I_{1}}{4 \pi} \oint \oint \frac{d \mathbf{l}_{1} \cdot d \mathbf{l}_{2}}{\mathrm{r}}$
$M_{21}=\frac{\mu_{0}}{4 \pi} \oint \oint \frac{d \mathbf{l}_{1} \cdot d \mathbf{l}_{2}}{r} \Leftarrow$ Neumann formula $\uparrow$
It involves a double line integral --one integration around loop 1, the other around loop 2.

## Important Things about Mutual Inductance

$$
M_{21}=\frac{\mu_{0}}{4 \pi} \oint \oint \frac{d \mathbf{I}_{1} \cdot d \mathbf{I}_{2}}{\mathrm{r}} \longleftarrow \begin{aligned}
& \text { It is not very useful for practical } \\
& \text { calculation, but it reveals two } \\
& \text { important features. }
\end{aligned}
$$

1. $M_{21}$ is purely geometrical quantity, having to do with the size, shape, and relative position.
2. $M_{21}=M_{12}$, so we can drop the subscripts and call them $M$.

Whatever the shapes and positions of the loops, the flux through 2 when we run current $I$ around 1 is identical to the flux through 1 when we send the same current $I$ around 2 .

Advantage of $M_{21}=M_{12}$, see the following examples.

## Example

A circular coil with a cross-sectional area of $4 \mathrm{~cm}^{2}$ has 10 turns. It is placed at the center of a long solenoid that has 15 turns/cm and a cross-sectional area of $10 \mathrm{~cm}^{2}$, as shown below. The axis of the coil coincides with the axis of the solenoid. What is their mutual inductance?
Solution:

$$
\begin{aligned}
\Phi_{12} & =B_{2} A_{1}=\mu_{0} n_{2} I_{2} A_{1} \\
M & =\frac{N_{1} \Phi_{12}}{I_{2}}=\mu_{0} n_{2} N_{1} A_{1} \\
& =\left(4 \pi \times 10^{-7}\right)(1500)(10)(0.000 \\
& =7.54 u \mathrm{~F}
\end{aligned}
$$

Notice that although $M_{12}=M_{21}$, it would have been much difficult to find $\Phi_{21}$ because the field due to the coil is quite nonuniform.

## Self-Inductance

It is convenient to express the induced emf in terms of a current rather than the magnetic flux through it.
The magnetic flux is directly proportional to the current flowing through it.

$$
N_{1} \Phi_{11}=L_{1} I_{1}
$$

where $L_{1}$ is a constant of proportionality called the selfinductance of coil 1. The SI unit of self-inductance is the henry $(H)$. The self-inductance of a circuit depends on its size and its shape.
The self-induced emf in coil 1 due to changes in $I_{1}$ takes the form

$$
\mathrm{E}=-L_{1} \frac{d I_{1}}{d t}
$$

## Example 7.11

Find the self-inductance of a toroidal coil with rectangular cross section (inner radius $a$, outer radius $b$, height $h$ ), which carries a total $n$ turns.

Sol: $\begin{aligned} & \text { magnetic field } \\ & \text { Inside a toroidal }\end{aligned} \quad B=\frac{\mu_{0} N I}{2 \pi s}$

$$
L_{1}=\frac{N \Phi_{11}}{I_{1}} \text { and } \Phi_{11}=h \int_{a}^{b} \frac{\mu_{0} N I_{1}}{2 \pi s} d s=\frac{\mu_{0} h N I_{1}}{2 \pi} \ln (b / a)
$$

$$
\therefore L_{1}=\frac{\mu_{0} h N^{2}}{2 \pi} \ln \left(\frac{b}{a}\right)
$$

## Example

A coaxial cable consists of an inner wire of radius a that carries a current I upward, and an outer cylindrical conductor of radius $b$ that carries the same current downward. Find the self-inductance of a coaxial cable of length $L$. Ignore the magnetic flux within the inner wire.
Solution:
$B=\frac{\mu_{0} I}{2 \pi x}, \quad d \Phi=B d A=\frac{\mu_{0} I}{2 \pi x} \ell d x$
$\Phi=\int_{a}^{b} \frac{\mu_{0} I}{2 \pi x} \ell d x=\frac{\mu_{0} I \ell}{2 \pi} \ln \frac{b}{a}=L I$
$L=\frac{\mu_{0} \ell}{2 \pi} \ln \frac{b}{a}$


Hint1: The direction of the magnetic field.
Hint2: What happens when considers the inner flux?

## LR Circuits

How does the current rise and fall as a function of time in a circuit containing an inductor and a resistor in series?
Rise

$$
\begin{aligned}
& V_{\mathrm{emf}}-I R-L \frac{d I}{d t}=0 \\
& \text { Let } I=I_{0} e^{-\alpha t}+\beta \Rightarrow \frac{d I}{d t}=-\alpha I_{0} e^{-\alpha t} \\
& e^{-\alpha t}: \alpha=\frac{R}{L} \\
& 0 \quad \mathrm{E}-R \beta=0 \Rightarrow \beta=\frac{\mathrm{E}}{R} \\
& t=0: I_{0}=-\beta=-\frac{\mathrm{E}}{R} \\
& \therefore I=\frac{\mathrm{E}}{R}\left(1-e^{-\frac{R}{L} t}\right)
\end{aligned}
$$

## LR Circuits

## Decay

$-I R-L \frac{d I}{d t}=0$
Let $I=I_{0} e^{-\alpha t} \Rightarrow \frac{d I}{d t}=-\alpha I_{0} e^{-\alpha t}$

$$
\begin{cases}e^{-\alpha t} & : \alpha=\frac{R}{L} \\ t=0: & I_{0}=\frac{\mathrm{E}}{R}\end{cases}
$$

$$
\therefore I=\frac{\mathrm{E}}{R} e^{-\frac{R}{L} t}=\frac{\mathrm{E}}{R} e^{-\frac{t}{\tau}}
$$


(a)

(b)

The quantity $\tau=L / R$ is called the time constant.

### 7.2.4 Energy in Magnetic Field

Inductance (like capacitance) is an intrinsically positive quantity. Lenz's law dictates that the emf is in such a direction as to oppose any change in current $\boldsymbol{\rightarrow}$ back emf.
It takes a certain amount of energy to start a current flowing in a circuit.
What we are concerned with are the work you must do against the back emf to get the current going.

Is this a fixed amount? Is it recoverable?
Yes, you get it back when the current is turned off.
It represents energy latent in the circuit or it can be regard as energy stored in the magnetic field.

## Energy Stored in an Inductor

The battery that establishes the current in an inductor has to do work against the opposing induced emf. The energy supplied by the battery is stored in the inductor.
In Kirchhoff's loop rule, we obtain


## The Power

The work done on a unit charge, against the back emf, in one trip around the circuit is -E.

> the work done by you against the emf

The total work done on per unit time is

$$
\frac{d W}{d t}=\frac{d(-\mathrm{E} Q)}{d t}=-\mathrm{E} I=L I \frac{d I}{d t}
$$

The total work is

$$
W=\int_{0}^{I_{0}} L I \frac{d I}{d t}=\frac{1}{2} L I_{0}^{2}
$$

Depends only on the geometry of the loop (in the form of $L$ ) and the final current $I_{0}$.

Energy Density of the Magnetic Field
We have expressed the total energy stored in the inductor in terms of the current and we know the magnetic field is proportional to the current. Can we express the total magnetic energy in terms of the B-field? Yes.
Let's consider the case of solenoid.
$L=\mu_{0} n^{2} A \ell$
$U_{L}=\frac{1}{2} L I^{2}=\frac{1}{2 \mu_{0}}\left(\mu_{0} n I\right)^{2} A \ell=\frac{B^{2}}{2 \mu_{0}} A \ell$
$u_{B}=\frac{B^{2}}{2 \mu_{0}}$ (The energy density of a magnetic field in free space)
Although this relation has been obtained from a special case, the expression is valid for any magnetic field.

## Generalized Total Energy

There is a nicer way to write the total magnetic energy $W$.

$$
\Phi=\int_{J_{S}} \mathbf{B} \cdot d \mathbf{a}=\int_{S}(\nabla \times \mathbf{A}) \cdot d \mathbf{a}=\oint_{P_{R}} \mathbf{A} \cdot d \mathbf{l}=L I
$$

$S$ : surface bounded by $P \quad P$ : perimeter of the loop

$$
W=\frac{1}{2} L I^{2}=\frac{1}{2} \Phi I=\frac{1}{2} I \oint_{P} \mathbf{A} \cdot d \mathbf{l}=\frac{1}{2} \oint_{P}(\mathbf{A} \cdot \mathbf{I}) d l
$$

generalize to the volume current

$$
\begin{gathered}
W=\frac{1}{2} \oint_{P}(\mathbf{A} \cdot \mathbf{I}) d l=\frac{1}{2} \int_{V}(\mathbf{A} \cdot \mathbf{J}) d \tau, \quad \text { where } \mathbf{J}=\frac{1}{\mu_{0}} \nabla \times \mathbf{B} \\
W=\frac{1}{2} \int_{V}(\mathbf{A} \cdot \mathbf{J}) d \tau=\frac{1}{2 \mu_{0}} \int_{V}[\mathbf{A} \cdot(\nabla \times \mathbf{B})] d \tau
\end{gathered}
$$

## Generalized Total Energy II

Product rule $6, \nabla \cdot(\mathbf{A} \times \mathbf{B})=\mathbf{B} \cdot(\nabla \times \mathbf{A})-\mathbf{A} \cdot(\nabla \times \mathbf{B})$

$$
\mathbf{A} \cdot(\nabla \times \mathbf{B})=\mathbf{B} \cdot \underbrace{(\nabla \times \mathbf{A})}_{\mathbf{B}}-\nabla \cdot(\mathbf{A} \times \mathbf{B})
$$

$W=\frac{1}{2 \mu_{0}} \int_{V}[\mathbf{A} \cdot(\nabla \times \mathbf{B})] d \tau=\frac{1}{2 \mu_{0}} \int_{V}[\mathbf{B} \cdot \mathbf{B}-\nabla \cdot(\mathbf{A} \times \mathbf{B})] d \tau$ $=\frac{1}{2 \mu_{0}} \int_{V} B^{2} d \tau-\frac{1}{2 \mu_{0}} \oint_{S}(\mathbf{A} \times \mathbf{B}) \cdot d \mathbf{a} \quad$ divergence theorem
$V \rightarrow$ all space $\frac{1}{2 \mu_{0}} \oint_{S}(\mathbf{A} \times \mathbf{B}) \cdot d \mathbf{a} \rightarrow 0$
$W=\frac{1}{2 \mu_{0}} \int_{\text {all space }} B^{2} d \tau$

## Electric and Magnetic Field Energy

## Electric field energy

$$
W_{\mathrm{elec}}=\frac{1}{2} \int(V \rho) d \tau=\frac{\varepsilon_{0}}{2} \int E^{2} d \tau
$$

Magnetic field energy

$$
W_{\mathrm{mag}}=\frac{1}{2} \int(\mathbf{A} \cdot \mathbf{J}) d \tau=\frac{1}{2 \mu_{0}} \int B^{2} d \tau, \quad u_{B}=\frac{1}{2 \mu_{0}} B^{2}
$$

Magnetic fields themselves do no work. Where does the energy come from?
A changing magnetic field induces an electric field which can do work.

## Example

The breakdown electric field strength of air is $3 \times 10^{6} \mathrm{~V} / \mathrm{m}$. A very large magnetic field strength is 20 T . compare the energy densities of the field.

$$
\text { Solution: } \begin{aligned}
U_{E} & =\frac{1}{2} \varepsilon_{0} E^{2}=(0.5)\left(8.85 \times 10^{-12}\right)\left(3 \times 10^{6}\right)^{2} \\
& =40 \mathrm{~J} / \mathrm{m}^{3} \\
U_{B} & =\frac{1}{2 \mu_{0}} B^{2}=\frac{20^{2}}{2 \times 4 \pi \times 10^{-7}} \\
& =3.2 \times 10^{8} \mathrm{~J} / \mathrm{m}^{3}
\end{aligned}
$$

Magnetic fields are an effective means of storing energy without breakdown of the air. However, it is difficult to produce such large fields over large regions.

## Example

Use the expression for the energy density of the magnetic field to calculate the self-inductance of a toroid with a rectangular cross section.

## Solution:

$$
\begin{aligned}
B & =\frac{\mu_{0} N I}{2 \pi r} \\
d U_{B} & =\frac{B^{2}}{2 \mu_{0}} d V=\frac{B^{2}}{2 \mu_{0}} h(2 \pi r d r)=\frac{\mu_{0} h(N I)^{2}}{4 \pi r} d r \\
U_{B} & =\int_{a}^{b} \frac{\mu_{0} h(N I)^{2}}{4 \pi r} d r=\frac{\mu_{0} h N^{2} I^{2}}{4 \pi} \ln \left(\frac{b}{a}\right)=\frac{1}{2} L I^{2} \\
L & =\frac{\mu_{0} N^{2} h}{2 \pi} \ln \left(\frac{b}{a}\right)
\end{aligned}
$$

Can we use the concept of magnetic flux to derive the selfinductance?

## Example 7.13

A long coaxial cable carries current $I$ (the current flows down the surface of the inner cylinder, radius $a$, and back along the outer cylinder, radius $b$ ) as shown in the Figure. Find the magnetic energy stored in a section of length $l$.
Sol:
magnetic field $\quad \mathbf{B}=\frac{\mu_{0} I}{2 \pi s} \hat{\boldsymbol{\varphi}}$

energy density $u_{B}=\frac{1}{2 \mu_{0}} B^{2}=\frac{\mu_{0} I^{2}}{8 \pi^{2} s^{2}}$
magnetic energy $\quad W_{B}=\int_{V} u_{B} d \tau=\int_{V} \frac{\mu_{0} I^{2}}{8 \pi^{2} s^{2}} l 2 \pi s d s=\frac{\mu_{0} I^{2}}{4 \pi} l \ln \left(\frac{b}{a}\right)$
self-inductance

$$
W_{B}=\frac{1}{2} L I^{2} \Rightarrow L=\frac{\mu_{0} l}{2 \pi} \ln \left(\frac{b}{a}\right)
$$

### 7.3 Maxwell's Equations

7.3.1 Electrodynamics before Maxwell
$\left\{\begin{array}{ccl}\nabla \cdot \mathbf{E}=\frac{1}{\varepsilon_{0}} \rho & \text { (Gauss's law) } & \\ \nabla \cdot \mathbf{B}=0 & \text { (no name) } & \text { electromagnetic theory } \\ \nabla \times \mathbf{E}=-\frac{\partial \mathbf{B}}{\partial t} & \text { (Faraday's law) } & \\ \nabla \times \mathbf{B}=\mu_{0} \mathbf{J} & \text { (Ampere's law) century ago } & \end{array}\right.$

A fatal inconsistency in Ampere's law

$$
\begin{gathered}
\nabla \cdot(\nabla \times \mathbf{B})=\mu_{0} \nabla \cdot \mathbf{J} \\
\downarrow \\
=0
\end{gathered} \quad \neq 0
$$

Ampere's law is incorrect for the nonsteady current.

## The Electric and Magnetic Fields

Two distinct kinds of electric fields:
( $E$ (in static case): attributed to electric charges, using Coulomb's law.
$E$ (in nonsteady case): associated with changing magnetic field, using Faraday's law.

Two distinct kinds of magnetic fields:
$B$ (in static case): attributed to electric currents, using Ampere's law.
$B$ (in nonsteady case): associated with changing electric field, using?

## Another Inconsistency of Ampere's Law

How do we determine the enclosed current $I_{\text {enc }}$ ?

$\int$ * The simplest surface---the wire puncture this surface so $I_{\text {enc }}=I \leftarrow$ Ampere's law is ok.

* A bollon-shaped surface---no current passes through this surface. so $I_{\text {enc }}=0 \longleftarrow$ Ampere's law is not valid!

For nonsteady current, "the current enclosed by a loop" is an ill-defined.

## How Maxwell Fixed Ampere's Law

Applying the continuity equation and Gauss's law, the offending term can be rewritten:
$\nabla \cdot \mathbf{J}=-\frac{\partial \rho}{\partial t}=-\frac{\partial\left(\varepsilon_{0} \nabla \cdot \mathbf{E}\right)}{\partial t}=\nabla \cdot\left(-\varepsilon_{0} \frac{\partial \mathbf{E}}{\partial t}\right)$
A new current $\mathbf{J}^{\prime}=\mathbf{J}+\varepsilon_{0} \frac{\partial \mathbf{E}}{\partial t} \leftarrow$ kills off the extra divergence
$\nabla \cdot(\nabla \times \mathbf{B})=\mu_{0}\left(\nabla \cdot \mathbf{J}^{\prime}\right)=\mu_{0} \nabla \cdot\left(\mathbf{J}+\varepsilon_{0} \frac{\partial \mathbf{E}}{\partial t}\right)=0$

When $E$ is constant (electrostatic+magnetostatic), we will have $\nabla \times \mathbf{B}=\mu_{0} \mathbf{J}$.
$\varepsilon_{0} \frac{\partial \mathbf{E}}{\partial t}$ plays a crucial role in the EM wave propagation.

## Electric Analogy of Faraday's Law

Maxwell's term cures the defect in Ampere's law, and moreover, it has a certain aesthetic appeal.

Faraday's law $\rangle$
A changing magnetic field induces a electric field.

## A changing electric field induces a magnetic field.

Maxwell called this extra term "the displacement current".

$$
\mathbf{J}_{\mathrm{d}} \equiv \varepsilon_{0} \frac{\partial \mathbf{E}}{\partial t}
$$ a misleading name, nothing to do with current

## The Displacement Current

How the displacement current resolves the paradox of the charging capacitor.

The electric field between the two capacitor plates is


$$
\begin{aligned}
& E=\frac{\sigma}{\varepsilon_{0}}=\frac{1}{\varepsilon_{0}} \frac{Q}{A} \longleftarrow \text { the charge on the plate area of the plate } \\
& \varepsilon_{0} \frac{\partial E}{\partial t}=\frac{1}{A} \frac{\partial Q}{\partial t}=\frac{I}{A}=J \\
& \mathbf{J}_{\text {tot }}=\mathbf{J}+\mathbf{J}_{d}\left\{\begin{array}{l}
\mathbf{J}=\mathrm{J}, \mathbf{J}_{\mathrm{d}}=0 \text { at the flat surface } \\
\mathbf{J}=0, \mathbf{J}_{\mathrm{d}}=\mathrm{J} \text { at the balloon-shaped surface }
\end{array}\right.
\end{aligned}
$$

## Maxwell's Equations (II)

Another expression of the Maxwell's equations.

$$
\begin{array}{lc}
\nabla \cdot \mathbf{E}=\frac{\rho}{\varepsilon_{0}} & \nabla \times \mathbf{E}+\frac{\partial \mathbf{B}}{\partial t}=0 \\
\nabla \cdot \mathbf{B}=0 & \nabla \times \mathbf{B}-\mu_{0} \varepsilon_{0} \frac{\partial \mathbf{E}}{\partial t}=\mu_{0} \mathbf{J}
\end{array}
$$

The fields (E and B) on the left and the sources ( $\rho$ and $\mathbf{J}$ ) on the right.

Maxwell's equations tell you how sources produce fields; reciprocally, the Lorentz force law tells you how fields affect sources. $\leftarrow$ A nonlinear feedback

### 7.3.3 Maxwell's Equations

Maxwell's equations in the traditional way.

$$
\begin{aligned}
& \left\{\begin{array}{cc}
\nabla \cdot \mathbf{E}=\frac{1}{\varepsilon_{0}} \rho & \text { (Gauss's law) } \\
\nabla \cdot \mathbf{B}=0 & \text { (no name) } \\
\nabla \times \mathbf{E}=-\frac{\partial \mathbf{B}}{\partial t} & \text { (Faraday's law) } \\
\nabla \times \mathbf{B}=\mu_{0} \mathbf{J}+\mu_{0} \varepsilon_{0} \frac{\partial \mathbf{E}}{\partial t} & \text { (Ampere's law with } \\
\text { Maxwell's correction) }
\end{array}\right. \\
& \text { Lorentz force law } \\
& \mathbf{F}=q(\mathbf{E}+\mathbf{v} \times \mathbf{B})
\end{aligned} \begin{aligned}
& \text { Continuity equation } \\
& \nabla \cdot \mathbf{J}=-\frac{\partial \rho}{\partial t}
\end{aligned}
$$

### 7.3.4 Magnetic Charge

If there is a magnetic "charge" $\rho_{\mathrm{m}}$ and the corresponding current of the magnetic "current" $\mathbf{J}_{\mathrm{m}}$, the Maxwell's equations read

$$
\begin{array}{rcl}
\nabla \cdot \mathbf{E}=\frac{\rho_{\mathrm{e}}}{\varepsilon_{0}} & \nabla \times \mathbf{E}+\frac{\partial \mathbf{B}}{\partial t}=-\varepsilon_{0} \mathbf{J}_{\mathrm{m}} & \begin{array}{l}
\text { A symmetric } \\
\text { between } \mathbf{E} \text { an } \\
\nabla \cdot \mathbf{B}=\mu_{0} \rho_{\mathrm{m}} \\
\end{array} \quad \nabla \times \mathbf{B}-\mu_{0} \varepsilon_{0} \frac{\partial \mathbf{E}}{\partial t}=\mu_{0} \mathbf{J}_{\mathrm{e}} \\
\begin{array}{l}
\mathbf{E} \rightarrow \mathbf{B} \\
\mathbf{B} \rightarrow-\mu_{0} \varepsilon_{0} \mathbf{E}
\end{array}
\end{array}
$$

Both charges would be conserved:

$$
\nabla \cdot \mathbf{J}_{\mathrm{e}}=-\frac{\partial \rho_{\mathrm{e}}}{\partial t}, \text { and } \nabla \cdot \mathbf{J}_{\mathrm{m}}=-\frac{\partial \rho_{\mathrm{m}}}{\partial t}
$$

Q: Has any one ever found the magnetic charge?
No.

### 7.3.5 Maxwell's Equations in Matter

When working with materials that are subject to electric and magnetic polarization, there is a more convenient way to write the Maxwell's equations.

## Static case:

An electric polarization produces a bound charge: $\rho_{\mathrm{b}}=-\nabla \cdot \mathbf{P}$
A magnetic polarization results in a bound current: $\mathbf{J}_{\mathrm{b}}=\nabla \times \mathbf{M}$

## Nonstatic case:

Any change in the electric polarization involves a flow of bound charge.

$$
\begin{aligned}
& \mathbf{J}_{\mathrm{p}}=\frac{d I}{d a_{\perp}}=\frac{d \sigma_{\mathrm{b}}}{d t} \frac{d a_{\perp}}{d a_{\perp}}=\frac{\partial \mathbf{P}}{\partial t} \text { where } \sigma_{\mathrm{b}}=\mathbf{P} \cdot \hat{\mathbf{n}} \\
& \text { polarization current } \\
& \text { (nothing to do with the bound current). }
\end{aligned}
$$

## Maxwell's Equations in Matter

In terms of free charges and currents, Maxwell's equations read

$$
\begin{array}{ll}
\nabla \cdot \mathbf{D}=\rho_{\mathrm{f}} & \nabla \times \mathbf{E}+\frac{\partial \mathbf{B}}{\partial t}=0 \\
\nabla \cdot \mathbf{B}=0 & \nabla \times \mathbf{H}-\frac{\partial \mathbf{D}}{\partial t}=\mathbf{J}_{\mathrm{f}}
\end{array}
$$

The constitutive relations: $\quad \mathbf{P}=\varepsilon_{0} \chi_{e} \mathbf{E}$

$$
\mathbf{M}=\mu_{0} \chi_{\mathrm{m}} \mathbf{H}
$$

So

$$
\begin{aligned}
& \mathbf{D}=\varepsilon_{0} \mathbf{E}+\mathbf{P}=\varepsilon_{0}\left(1+\chi_{e}\right) \mathbf{E}=\varepsilon \mathbf{E} \\
& \mathbf{H}=\frac{1}{\mu_{0}} \mathbf{B}-\mathbf{M} \Rightarrow \mathbf{B}=\mu_{0}\left(1+\chi_{\mathrm{m}}\right) \mathbf{H}=\mu \mathbf{H}
\end{aligned}
$$

## Polarization and Bound Currents

Bound current $\mathbf{J}_{\mathrm{b}}$ : magnetization of the material involving the spin and orbital motion of electrons.
Polarization current $\mathbf{J}_{\mathrm{p}}$ : the linear motion of charge when the electric polarization changes.

Now $\quad \rho=\rho_{\mathrm{f}}+\rho_{\mathrm{b}}=\rho_{\mathrm{f}}-\nabla \cdot \mathbf{P}$

$$
\mathbf{J}=\mathbf{J}_{\mathrm{f}}+\mathbf{J}_{\mathrm{b}}+\mathbf{J}_{\mathrm{p}}=\mathbf{J}_{\mathrm{f}}+\nabla \times \mathbf{M}+\frac{\partial \mathbf{P}}{\partial t}
$$

Gauss's law: $\quad \nabla \cdot \mathbf{E}=\frac{1}{\varepsilon_{0}}\left(\rho_{\mathrm{f}}-\nabla \cdot \mathbf{P}\right) \Rightarrow \nabla \cdot\left(\varepsilon_{0} \mathbf{E}+\mathbf{P}\right)=\rho_{\mathrm{f}}$
Ampere's law: $\quad \nabla \times \mathbf{B}=\mu_{0}\left(\mathbf{J}_{\mathrm{f}}+\nabla \times \mathbf{M}+\frac{\partial \mathbf{P}}{\partial t}\right)+\mu_{0} \varepsilon_{0} \frac{\partial \mathbf{E}}{\partial t}$

$$
\Rightarrow \nabla \times\left(\frac{1}{\mu_{0}} \mathbf{B}-\mathbf{M}\right)=\mathbf{J}_{\mathrm{f}}+\frac{\partial}{\partial t}\left(\varepsilon_{0} \mathbf{E}+\mathbf{P}\right)
$$

### 7.3.6 Boundary Conditions (I)

Differential form

$$
\begin{aligned}
& \nabla \cdot \mathbf{D}=\rho_{\mathrm{f}} \\
& \nabla \cdot \mathbf{B}=0
\end{aligned}
$$

$$
\left.\begin{array}{c}
\text { Integral form } \\
\oint_{S} \mathbf{D} \cdot d \mathbf{a}=\rho_{\mathrm{f}} \\
\oint_{S} \mathbf{B} \cdot d \mathbf{a}=0
\end{array}\right\}
$$

over any enclosed surface $S$.
wafer thin
Gaussian pillbox

$$
\begin{aligned}
& \oint_{S} \mathbf{D} \cdot d \mathbf{a}=\rho_{\mathrm{f}} \\
& \mathbf{D}_{1} \cdot \mathbf{a}-\mathbf{D}_{2} \cdot \mathbf{a}=\sigma_{\mathrm{f}} \mathbf{a} \Rightarrow D_{1}^{\perp}-D_{2}^{\perp}=\sigma_{\mathrm{f}}
\end{aligned}
$$

$$
\begin{aligned}
& \oint_{S} \mathbf{B} \cdot d \mathbf{a}=0 \\
& \mathbf{B}_{1} \cdot \mathbf{a}-\mathbf{B}_{2} \cdot \mathbf{a}=\sigma_{\mathrm{f}} \mathbf{a} \Rightarrow B_{1}^{\perp}-B_{2}^{\perp}=0
\end{aligned}
$$

## Boundary Conditions (II)

## Differential form

$\nabla \times \mathbf{E}+\frac{\partial \mathbf{B}}{\partial t}=0$
$\nabla \times \mathbf{H}-\frac{\partial \mathbf{D}}{\partial t}=\mathbf{J}_{\mathrm{f}}$
Integral form
$\oint_{P} \mathbf{E} \cdot d \mathbf{l}=-\frac{\partial}{\partial t} \int_{S} \mathbf{B} \cdot d \mathbf{a}$
$\left.\oint_{P} \mathbf{H} \cdot d \mathbf{l}=\mathbf{J}_{\mathrm{f}}+\frac{\partial}{\partial t} \int_{S} \mathbf{D} \cdot d \mathbf{a}\right\}$
for any surface $S$ bounded by the closed loop $P$.

## Boundary Conditions in Linear Media

$$
\begin{array}{cc}
D_{1}^{\perp}-D_{2}^{\perp}=\sigma_{\mathrm{f}} & \mathbf{E}_{1}^{\prime \prime}-\mathbf{E}_{2}^{\prime \prime}=0 \\
B_{1}^{\perp}-B_{2}^{\perp}=0 & \mathbf{H}_{1}^{\perp}-\mathbf{H}_{2}^{\perp}=\left(\mathbf{K}_{f} \times \hat{\mathbf{n}}\right)
\end{array}
$$

In case of linear media, $\mathbf{D}$ and $\mathbf{H}$ can be express in terms of $\mathbf{E}$ and $\mathbf{B}$.

$$
\begin{array}{cc}
\varepsilon_{1} E_{1}^{\perp}-\varepsilon_{2} E_{2}^{\perp}=\sigma_{\mathrm{f}} & \mathbf{E}_{1}^{\prime \prime}-\mathbf{E}_{2}^{\prime \prime}=0 \\
B_{1}^{\perp}-B_{2}^{\perp}=0 & \frac{1}{\mu_{1}} \mathbf{B}_{1}^{\perp}-\frac{1}{\mu_{2}} \mathbf{B}_{2}^{\perp}=\mathbf{K}_{f} \times \hat{\mathbf{n}}
\end{array}
$$

If there is no free charge or free current at the interface,
then

$$
\begin{array}{cc}
\varepsilon_{1} E_{1}^{\perp}-\varepsilon_{2} E_{2}^{\perp}=0 & \mathbf{E}_{1}^{\prime \prime}-\mathbf{E}_{2}^{\prime \prime}=0 \\
B_{1}^{\perp}-B_{2}^{\perp}=0 & \frac{1}{\mu_{1}} \mathbf{B}_{1}^{\perp}-\frac{1}{\mu_{2}} \mathbf{B}_{2}^{\perp}=0
\end{array}
$$

