Chapter 11: Radiation

11.1 Dipole Radiation 11.1.1 What is Radiation?

A charge at rest does not generate electromagnetic wave; nor does a steady current. It takes *accelerating charges*, and/or *changing currents*.
The purpose of this chapter is to show you how such configurations produce electromagnetic wave.

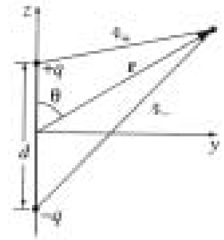
How charges radiate? Consider Jefimenko's equations.

$$\begin{cases} \mathbf{E} = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int [\frac{\rho \hat{\mathscr{r}}}{\mathscr{r}^2} + \frac{\dot{\rho}\hat{\mathscr{r}}}{c\mathscr{r}} - \frac{\dot{\mathbf{J}}}{c^2\mathscr{r}}] d\tau' \\ \mathbf{B} = \dfrac{\mu_0}{4\pi}\displaystyle\int [\frac{\mathbf{J}}{\mathscr{r}^2} + \frac{1}{c\mathscr{r}}\dot{\mathbf{J}}] \times \hat{\mathscr{r}} d\tau' \end{cases}$$

$\dot{\rho}$ and $\dot{\mathbf{J}}$ are responsible for electromagnetic radiation (i.e. EM field at large distance).

11.1.2 Electric Dipole Radiation

Consider two point charges of $+q$ and $-q$ separating by a distance $d(t)$. Assume $d(t)$ can be expressed in sinusoidal form.



The result is an oscillating electric dipole:

$$\mathbf{P}(t) = qd(t)\hat{\mathbf{z}} = qd\cos(\omega t)\hat{\mathbf{z}} = P_0\cos(\omega t)\hat{\mathbf{z}}, \quad \text{where } P_0 \equiv qd.$$

The retarded potential is:

$$V(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r}',t_r)}{\mathscr{r}} d\tau'$$

$$= \frac{1}{4\pi\varepsilon_0}\left\{ \frac{q_0\cos[\omega(t-\mathscr{r}_+/c)]}{\mathscr{r}_+} - \frac{q_0\cos[\omega(t-\mathscr{r}_-/c)]}{\mathscr{r}_-} \right\}$$

Electric Dipole Radiation: Approximations

Approximation #1: Make this physical dipole into a perfect dipole. $d << r$

Estimate the spearation distances by the law of cosines.

$$\mathscr{r}_{\pm} = \sqrt{r^2 \mp rd\cos\theta + (d/2)^2} \cong r(1 \mp \frac{d}{2r}\cos\theta)$$

$$\frac{1}{\mathscr{r}_{\pm}} \cong \frac{1}{r}(1 \pm \frac{d}{2r}\cos\theta)$$

$$\cos[\omega(t-\mathscr{r}_{\pm}/c)] \cong \cos[\omega(t-\frac{r}{c}) \pm \frac{\omega d}{2c}\cos\theta]$$

$$= \cos[\omega(t-\frac{r}{c})]\cos(\frac{\omega d}{2c}\cos\theta) \mp \sin[\omega(t-\frac{r}{c})]\sin(\frac{\omega d}{2c}\cos\theta)$$

Approximation #2: The wavelength is much longer than the dipole size. $d << \dfrac{c}{\omega} = \dfrac{\lambda}{2\pi}$

The Retarded Scalar Potential

$$\cos[\omega(t-\mathscr{r}_{\pm}/c)] \cong \cos[\omega(t-\frac{r}{c})]\underbrace{\cos(\frac{\omega d}{2c}\cos\theta)}_{=1} \mp \sin[\omega(t-\frac{r}{c})]\underbrace{\sin(\frac{\omega d}{2c}\cos\theta)}_{\frac{\omega d}{2c}\cos\theta}$$

$$= \cos[\omega(t-\frac{r}{c})] \mp \sin[\omega(t-\frac{r}{c})]\frac{\omega d}{2c}\cos\theta$$

The retarded scalar potential is:

$$V(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\left\{ \begin{array}{l} \left[\cos[\omega(t-\frac{r}{c})] - \sin[\omega(t-\frac{r}{c})]\frac{\omega d}{2c}\cos\theta\right]\frac{1}{r}(1+\frac{d}{2r}\cos\theta) \\ -\left[\cos[\omega(t-\frac{r}{c})] + \sin[\omega(t-\frac{r}{c})]\frac{\omega d}{2c}\cos\theta\right]\frac{1}{r}(1-\frac{d}{2r}\cos\theta) \end{array} \right\}$$

$$\cong \frac{p_0\cos\theta}{4\pi\varepsilon_0 r}\left[-\frac{\omega}{c}\sin[\omega(t-\frac{r}{c}) + \frac{1}{r}\cos[\omega(t-\frac{r}{c})]\right]$$

The Retarded Scalar Potential

Approximation #3: at the radiation zone. $\frac{c}{\omega} << r$

The retarded scalar potential is:

$$V(\mathbf{r},t) \cong \frac{p_0 \cos\theta}{4\pi\varepsilon_0 r}\left[-\frac{\omega}{c}\sin[\omega(t-\frac{r}{c})\right]$$

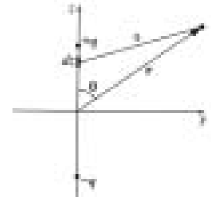
Three approximations

$$d << r \qquad d << \frac{c}{\omega}(=\frac{\lambda}{2\pi}) \qquad \frac{c}{\omega} << r$$

$$\Rightarrow d << \lambda << r$$

The Retarded Vector Potential

The retarded vector potential is determined by the current density.



$$I(t) = \frac{dq}{dt}\hat{\mathbf{z}} = -q_0\omega \sin\omega t\hat{\mathbf{z}}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}',t_r)}{\text{r}}d\tau' = \frac{\mu_0}{4\pi}\int_{-d/2}^{d/2}\frac{-q\omega\sin[\omega(t-r/c)]\hat{\mathbf{z}}}{\text{r}}dz$$

$$\cong -\frac{\mu_0 p_0\omega}{4\pi r}\sin[\omega(t-\frac{r}{c})]\hat{\mathbf{z}} \qquad @\, d << \lambda << r$$

Retarded potentials:

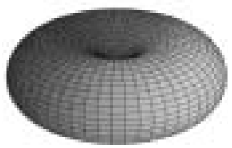
$$\begin{cases} V(\mathbf{r},t) = -\frac{p_0\omega}{4\pi\varepsilon_0 c}\left[\frac{\cos\theta}{r}\sin[\omega(t-\frac{r}{c})\right] \\ \mathbf{A}(\mathbf{r},t) = -\frac{\mu_0 p_0\omega}{4\pi r}\sin[\omega(t-\frac{r}{c})]\hat{\mathbf{z}} \end{cases} \qquad \begin{cases} \mathbf{E} = -\nabla V - \frac{\partial\mathbf{A}}{\partial t} \\ \mathbf{B} = \nabla\times\mathbf{A} \end{cases}$$

The Electromagnetic Fields and Poynting Vector

$$\begin{cases} \mathbf{E} = -\nabla V - \frac{\partial\mathbf{A}}{\partial t} = -\frac{\mu_0 p_0\omega^2}{4\pi\varepsilon_0 c}(\frac{\sin\theta}{r})\cos[\omega(t-\frac{r}{c})]\hat{\boldsymbol{\theta}} \\ \mathbf{B} = \nabla\times\mathbf{A} = -\frac{\mu_0 p_0\omega^2}{4\pi c}(\frac{\sin\theta}{r})\cos[\omega(t-\frac{r}{c})]\hat{\boldsymbol{\varphi}} \end{cases}$$

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = \frac{\mu_0}{c}\left\{\frac{p_0\omega^2}{4\pi}(\frac{\sin\theta}{r})\cos[\omega(t-\frac{r}{c})]\right\}^2\hat{\mathbf{r}}$$

The total power radiated is

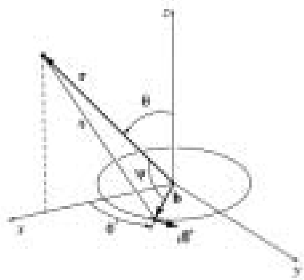


$$< P > = \int\langle\mathbf{S}\rangle\cdot d\mathbf{a} = \frac{\mu_0 p_0^2\omega^4}{32\pi^2 c}\int(\frac{\sin\theta}{r})^2 r^2\sin\theta d\theta d\phi$$

$$= \frac{\mu_0 p_0^2\omega^4}{12\pi c}$$

11.1.3 Magnetic Dipole Radiation

Suppose we have a loop of radius *b*, around which we drive an alternating current.



$$I(t) = I_0\cos\omega t$$

This is a model for an oscillating magnetic dipole,

$$\mathbf{m}(t) = \pi b^2 I(t)\hat{\mathbf{z}} = m_0\cos\omega t\hat{\mathbf{z}}$$

The loop is uncharged, so the retarded scalar potential is zero. *V*=0

The retarded vector potential

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}',t_r)}{\text{r}}d\tau' = \frac{\mu_0}{4\pi}\int\frac{I_0\cos[\omega(t-r/c)]}{\text{r}}d\mathbf{l}'$$

Retarded Vector Potential with Three Approximations

$$\mathbf{A}(\mathbf{r},t)=\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}',t_r)}{\text{r}}d\tau'=\frac{\mu_0}{4\pi}\int\frac{I_0\cos[\omega(t-\text{r}/c)]}{\text{r}}d\mathbf{l}'$$

Approximation #1: Make this physical dipole into a perfect dipole. $b<<r$

Estimate the spearation distances by the law of cosines.

$$\text{r}=\sqrt{r^2+b^2-2rb\cos\psi},$$

where ψ is the angle between the vectors **r** and **b**:

$$rb\cos\psi=r\cdot b=rb\sin\theta\cos\phi'$$

$$\text{r}=\sqrt{r^2+b^2-2rb\sin\theta\cos\phi'}\cong r(1-\frac{b}{r}\sin\theta\cos\phi')$$

$$\frac{1}{\text{r}}\cong\frac{1}{r}(1+\frac{b}{r}\sin\theta\cos\phi')$$

Retarded Vector Potential with Three Approximations

$$\begin{aligned}\cos[\omega(t-\text{r}/c)]&=\cos[\omega(t-\frac{r}{c})+\frac{\omega b}{c}\sin\theta\cos\phi']\\&=\cos[\omega(t-\frac{r}{c})]\cos[\frac{\omega b}{c}\sin\theta\cos\phi']\\&\quad-\sin[\omega(t-\frac{r}{c})]\sin[\frac{\omega b}{c}\sin\theta\cos\phi']\end{aligned}$$

Approximation #2: The size of the dipole is small compared to the wavelength radiated.

$$b<<\frac{c}{\omega}(=\frac{\lambda}{2\pi})$$

$$\cos[\omega(t-\text{r}/c)]\cong\cos[\omega(t-\frac{r}{c})]-\left(\frac{\omega b}{c}\sin\theta\cos\phi'\right)\sin[\omega(t-\frac{r}{c})]$$

The Retarded Vector Potential

$$\mathbf{A}(\mathbf{r},t)=\frac{\mu_0 I_0 b}{4\pi r}\int_0^{2\pi}\left\{\begin{array}{l}\cos[\omega(t-\frac{r}{c})]\\+b\sin\theta\cos\phi'\\\left(\frac{1}{r}\cos[\omega(t-\frac{r}{c})]-\frac{\omega}{c}\sin[\omega(t-\frac{r}{c})]\right)\end{array}\right\}\cos\phi' d\phi'$$

The second-order term is dropped.

The first term integrates to zero: $\int_0^{2\pi}\cos\phi' d\phi'=0$

The second term involves the integral of cosine squared. $\int_0^{2\pi}\cos^2\phi' d\phi'=\pi$

Putting this in, and noting that **A** points in the ϕ–direction.

The Retarded Vector Potential

The vector potential of an oscillating perfect magnetic dipole is:

$$\mathbf{A}(\mathbf{r},t)=\frac{\mu_0 m_0}{4\pi}\frac{\sin\theta}{r}\left\{\frac{1}{r}\cos[\omega(t-\frac{r}{c})]-\frac{\omega}{c}\sin[\omega(t-\frac{r}{c})]\right\}\hat{\boldsymbol{\varphi}}$$

Approximation #3: at the radiation zone. $\frac{c}{\omega}<<r$

$$\mathbf{A}(\mathbf{r},t)=-\frac{\mu_0 m_0\omega}{4\pi c}\frac{\sin\theta}{r}\sin[\omega(t-\frac{r}{c})]\hat{\boldsymbol{\varphi}}$$

$$\begin{cases}\mathbf{E}=-\dfrac{\partial\mathbf{A}}{\partial t}=-\dfrac{\mu_0 m_0\omega^2}{4\pi c}\dfrac{\sin\theta}{r}\cos[\omega(t-\frac{r}{c})]\hat{\boldsymbol{\varphi}}\\\mathbf{B}=\nabla\times\mathbf{A}=-\dfrac{\mu_0 m_0\omega^2}{4\pi c^2}\dfrac{\sin\theta}{r}\cos[\omega(t-\frac{r}{c})]\hat{\boldsymbol{\theta}}\end{cases}$$

The Electromagnetic Fields and Poynting Vector

$$\begin{cases} \mathbf{E} = -\dfrac{\partial \mathbf{A}}{\partial t} = -\dfrac{\mu_0 m_0 \omega^2}{4\pi c}\dfrac{\sin\theta}{r}\cos[\omega(t-\dfrac{r}{c})]\hat{\boldsymbol{\varphi}} \\ \mathbf{B} = \nabla\times\mathbf{A} = -\dfrac{\mu_0 m_0 \omega^2}{4\pi c^2}\dfrac{\sin\theta}{r}\cos[\omega(t-\dfrac{r}{c})]\hat{\boldsymbol{\theta}} \end{cases}$$

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = \frac{\mu_0}{c}\left\{\frac{m_0\omega^2}{4\pi c}(\frac{\sin\theta}{r})\cos[\omega(t-\frac{r}{c})]\right\}^2\hat{\mathbf{r}}$$

The total power radiated is: $< P >= \int \langle \mathbf{S} \rangle \cdot d\mathbf{a} = \dfrac{\mu_0 m_0^2 \omega^4}{12\pi c^3}$

$$\frac{P_{\text{magnetic}}}{P_{\text{electric}}} = \frac{m_0^2}{p_0^2 c^2} \ll 1 \text{ (Electric dipole radiation dominates)}$$

Homework of Chap.11

Prob. 1, 2, 5, 6