Chapter 12 Electrodynamics and Relativity

12.1 The Special Theory of Relativity

Ether: Since mechanical waves require a medium to propagate, it was generally accepted that light also require a medium. This medium, called the ether, was assumed to pervade all mater and space in the universe.

“Absolute” frame: The Maxwell’s equation was inferred that the speed of light should equal *c* only with respect to ether. This meant that the ether was a “preferred” or “absolute” reference frame.

$$\frac{\partial^2 E}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0$$

Ether

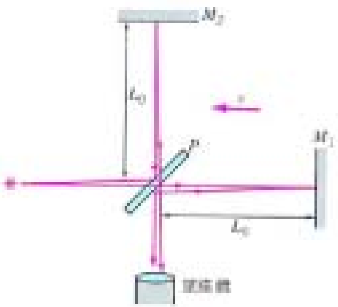
Properties of the ether: Since the light speed c is enormous, the ether had to be extremely rigid. So it did not impede the motion of light. For a substance so crucial to electro-magnetism , it was embarrassingly elusive. Despite the peculiar property just mentioned, no one could detect its ghostly presence.

Efforts to detect the ether: Michelson inspired by the Maxwell took the problem of detecting the ether as a challenge. He developed his interferometer and used it to try to detect the earth’s motion relative to the ether. The result were not conclusive.

The Michelson-Morley Experiment

Michelson and Morley wanted to detect the speed of the earth relative to the ether. If the earth were moving relative to the ether, there should be an “ether wind” blowing at the same speed relative to the earth but in the opposite direction.

Michelson-Morley interferometer: Use light speed variation to verify the existence of ether.



The Michelson-Morley Experiment (II)

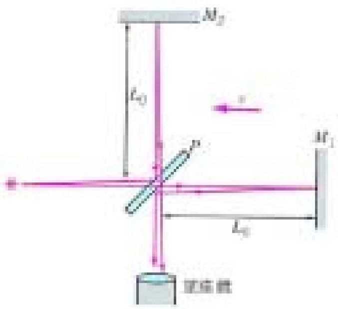
Parallel:

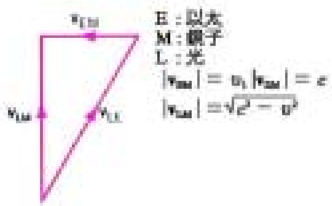
$$T_1 = \frac{L_0}{(c-v)} + \frac{L_0}{(c+v)} = \frac{(2L_0/c)}{(1-v^2/c^2)}$$

Perpendicular:

$$T_2 = \frac{2L_0}{(c^2-v^2)^{1/2}} = \frac{(2L_0/c)}{(1-v^2/c^2)^{1/2}}$$

$$\Delta T = T_1 - T_2 \cong \frac{L_0}{c}\left(\frac{v^2}{c^2}\right)$$





The Michelson-Morley Experiment (III)

Using v=30 km/s, the expected shift was about 0.4 fringe. Even though they were able to detect shifts smaller than 1/20 of a fringe, ***they found nothing***.

Possibilities:

- The ether was dragged with the Earth. ✗
- No ether. ✓
- Constant light speed. ✓

The Two Postulates

The two postulates in the theory of special relativity are:

1. **The principle of relativity**: All physical laws have the same form in all inertia frames.
2. **The universal speed of light**: The speed of light in free space is the same in all inertial frames. It does not depend on the motion of the source or the observer.

Both postulates are restricted to inertial frames. This is why the theory is special.

•The principle of relativity extends the concept of covariance from mechanics to all physical laws.

•The constancy of the speed of light is difficult to accept at first.

All the experimental consequences have confirmed its correctness.

Some Preliminaries

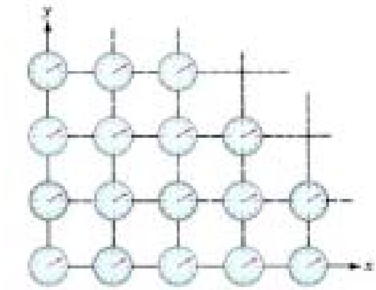
Event: Event is something that occurs at a single point in space at a single instant in time.

Observer: An observer is ether a person, or an automatic device, with a clock and a meter stick. Each observers can record events only in the immediate vicinity.

Reference frame: A reference frame is a whole set of observers uniformly distributed in space. The frame in which an object is at rest is called its **rest frame**.

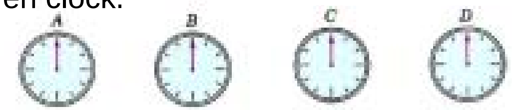
Synchronization of clocks: It is extremely important to define precisely what is meant by the time in a given reference frame. This requires a careful procedure for the synchronization of clocks.

Some Preliminaries (II)



A reference frame is assumed to consist of many observers uniformly spread through the space. Each observer has a meter stick and a clock to make measurements only in the immediate vicinity.

To synchronize four equally spaced clocks, a signal is sent out by clock A to trigger the other clocks---each of which has been set ahead by the amount of time it takes to travels from A to the given clock.

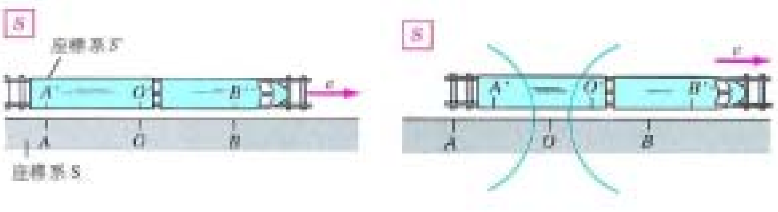


Relativity of Simultaneity

How can we determine whether two events at different locations are simultaneous?

Two events at different locations are simultaneous if an observer midway between them receives the flashes at the same instant.

Relativity of Simultaneity: Spatially separated events that are simultaneous in one frame are not simultaneous in another, moving relative to the first.



Relativity of Simultaneity
(another example)

Two events that are simultaneous in one inertial system are not, in general, simultaneous in another.

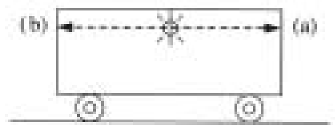


Figure 12.4

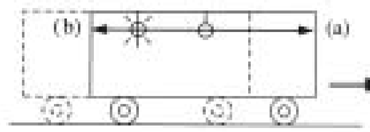
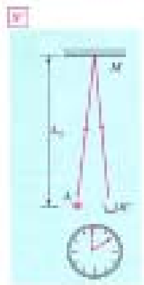


Figure 12.5

Geometry of Relativity: Time Dilation

How does the relative motion of two frames affects the measured time interval between two events?

$$\tau = \frac{2L_0}{c}$$



A **proper time,** τ, is the time interval between two events as measured in the rest frame of a clock. In this frame both events occur at the same position. (Note: proper→own),

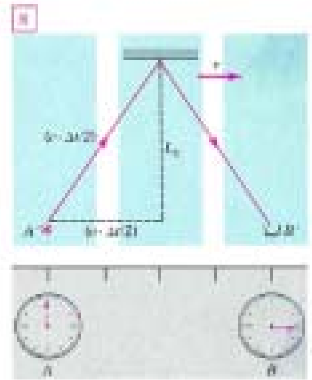
Time Dilation (II)

Now let us find the time interval recorded in the frame *S*, in which the clock has velocity v. The time interval Δt in frame *S* measured by two observers A and B at different positions.

$$(c\cdot\frac{\Delta t}{2})^2 = L_0^2 + (v\cdot\frac{\Delta t}{2})^2$$

$$T = \Delta t = \frac{2L_0}{c}\cdot\left(\frac{1}{\sqrt{1-v^2/c^2}}\right)$$

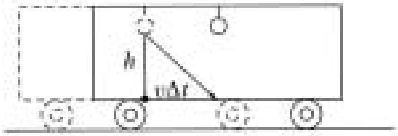
$$T = \gamma T_0 \quad \text{where } \gamma = \frac{1}{\sqrt{1-v^2/c^2}}$$



Note that we have used *c* as the speed of light in both frames---in accord with the second postulate.

Time Dilation (III)

Another example:



$$\tau = \frac{h}{c} \quad \text{(proper time)}$$

$$\Delta t = \frac{\sqrt{h^2 + (v\Delta t)^2}}{c} \quad \Rightarrow \Delta t = \frac{1}{\sqrt{1 - v^2/c^2}} \frac{h}{c}$$

$$\Delta t = \gamma \tau, \quad \text{where } \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

Moving clocks run slow.

Time Dilation (IV)

Since $\gamma>1$, the time interval T measured in frame S (by two clocks) is greater than the proper time, T_0, registed by the clock in its rest frame S'. The effect is called time dilation.

Two spatially separated clocks, A and B, record a greater time interval between two events than the proper time recorded by a single clock that moves from A to B and is present at both events.

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

v/c	γ
0.6	5/4
0.8	5/3
0.98	5
0.995	10
0.9965	12
0.9992	25

*有關 γ 的數值已經加以化約。

Example of Time Dilation

Experimental evidence (muon decay):

The reality of time dilation was verified in an experiment performed in 1941.

Rest frame at ground: An elementary particle, the muon (μ), decays into other particle. The particle decay rate is

$$N = N_0 e^{-t/\tau}$$

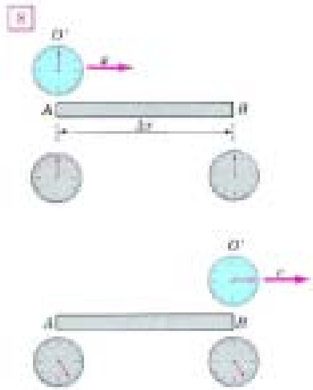
where $\tau = 2.2\ \mu s$ is called the mean lifetime.

Moving frame at the upper atmosphere: Another source of generating muon is bombarded with cosmic ray protons. The muon generated with this method has the speed of v=0.995c. The mean lifetime is 10 times longer than their cousins that decay at rest in the laboratory.

Geometry of Relativity: Length Contraction

Consider a rod *AB* at rest in frame *S*, as shown below. The distance between its ends is its proper length L_0:

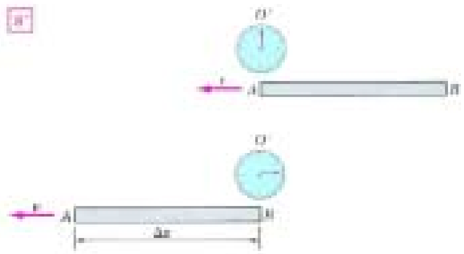
The **proper length**, L_0, of an object is the space interval between its ends measured in the rest frame of the object.



Length Contraction (II)

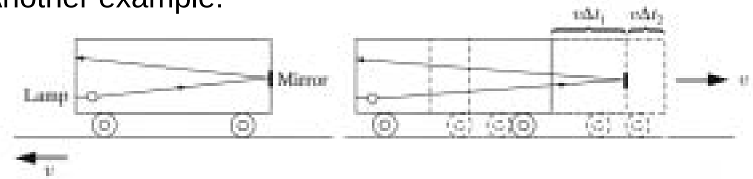
An observer O' in Frame S', which moves at velocity v relative to frame S, can measure the rod's length. By recording the interval b times at which O' passes A and B. The measurements in the two frames are

$$\left.\begin{array}{l}\text{Frame } S: \ L_0 = \Delta x = v\Delta t \\ \text{Frame } S': \ L = \Delta x' = v\Delta t'\end{array}\right\} \Rightarrow L = \frac{1}{\gamma} L_0$$



Length Contraction (III)

Another example:



$$L_0 = \frac{c\tau}{2} \quad (L_0\text{: proper length})$$

$$\Delta t_1 = \frac{L + v\Delta t_1}{c}, \quad \Delta t_2 = \frac{L - v\Delta t_2}{c}$$

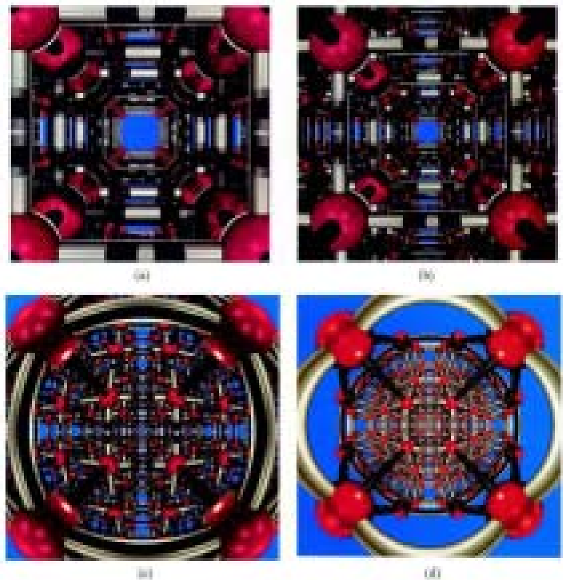
$$\Delta t_1 = \frac{L}{c - v}, \quad \Delta t_2 = \frac{L}{c + v} \quad \Rightarrow \Delta t = \Delta t_1 + \Delta t_2 = 2\frac{L}{c}\frac{1}{1 - v^2/c^2}$$

$$L = \frac{c}{2}\frac{1}{\gamma^2}\Delta t = \frac{c}{2}\frac{1}{\gamma^2}\gamma\tau = \frac{1}{\gamma}L_0$$

Moving objects are shortened.

Effects of Length Contraction (I)

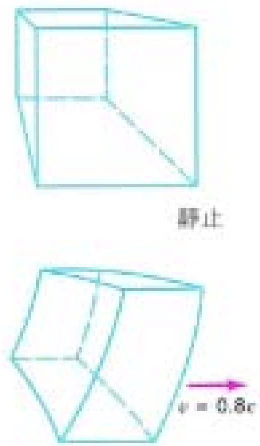
$v = 0.0\,c$ $v = 0.5\,c$



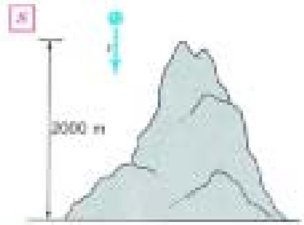
$v = 0.95\,c$ $v = 0.99\,c$

Length Contraction Effects (II)

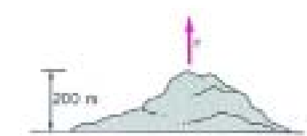
distortion



rest frame



muon frame



The Twin Paradox

Nothing in the theory of relativity catches the imagination more than the so-called twin paradox.

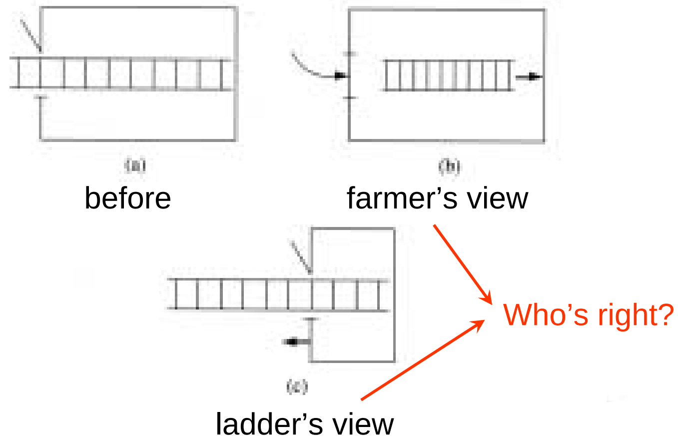
Twin A stays on earth while twin B travels at high speed to a nearby start. When B returns, they both find that A has aged more than B.

The paradox arises because of the apparent symmetry of the situation: In B's frame, it is A that leaves and returns, so one should also find that B has aged more than A.

$$\left.\begin{matrix} A > B \\ B > A \end{matrix}\right\} ?$$

What's going on?

The Barn and Ladder Paradox



The Lorentz Transformation

The laws of electromagnetism are not covariant with respect to the Galilean transformation. However with Lorentz transformation they are covariant. The space and time are related shown as follows:

Rest frame

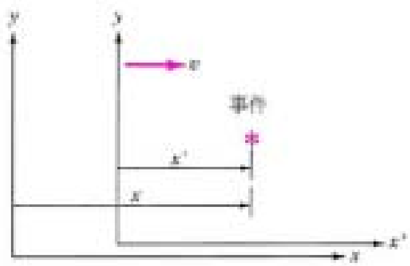
$$x' = \gamma(x - vt)$$

$$t' = \gamma(t - \frac{vx}{c^2})$$

Moving frame

$$x = \gamma(x' + vt')$$

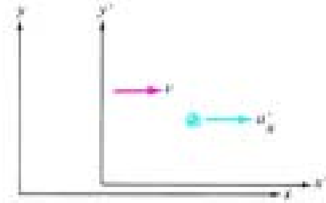
$$t = \gamma(t' + \frac{vx'}{c^2})$$



The Addition of Velocity

$$dx = \gamma(dx' + vt') = \gamma dt'(u'_x + v)$$

$$dt = \gamma(dt' + \frac{vdx'}{c^2}) = \gamma dt'(1 + \frac{u'_x v}{c^2})$$



Taking the ratio of these equations we find

$$u_x = \frac{u'_x + v}{1 + u'_x v / c^2}$$

A extreme case

when $u'_x = c$, we have $u_x = \dfrac{c + v}{1 + cv/c^2} = c$

The Structure of Spacetime: (i) Four-vectors

$$x^0 \equiv ct,\ x^1 = x,\ x^2 = y,\ x^3 = z,\ \text{and } \beta = \frac{v}{c}$$

$$\left.\begin{aligned} \bar{x}^0 &= \gamma(x^0 - \beta x^1), \\ \bar{x}^1 &= \gamma(x^1 - \beta x^2) \\ \bar{x}^2 &= x^2 \\ \bar{x}^3 &= x^3 \end{aligned}\right\} \text{the Lorentz transformations}$$

$$\begin{pmatrix} \bar{x}^0 \\ \bar{x}^1 \\ \bar{x}^2 \\ \bar{x}^3 \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{the Lorentz transformation matrix}} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \qquad \bar{x}^\mu = \sum_{\nu=0}^{3} \Lambda^\mu_\nu x^\nu$$

Covariant Vector, Contravariant Vector, and Invariant Quantity

the covariant vector (row): a_μ $\quad a_\mu = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \end{pmatrix} \equiv \begin{pmatrix} -a^0 & a^1 & a^2 & a^3 \end{pmatrix}$

the contravariant vector (column): a^μ $\quad a^\mu = \begin{pmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix}$

invariant quantity under Lorentz transformation

the Einstein summation convention

$$a_\mu b^\mu = \sum_{\nu=0}^{3} a_\mu b^\mu = \begin{pmatrix} -a^0 & a^1 & a^2 & a^3 \end{pmatrix} \begin{pmatrix} b^0 \\ b^1 \\ b^2 \\ b^3 \end{pmatrix} \qquad \boxed{a_\mu b^\mu = a^\mu b_\mu}$$

$$= -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3 = -\bar{a}^0 \bar{b}^0 + \bar{a}^1 \bar{b}^1 + \bar{a}^2 \bar{b}^2 + \bar{a}^3 \bar{b}^3$$

The Invariant Interval

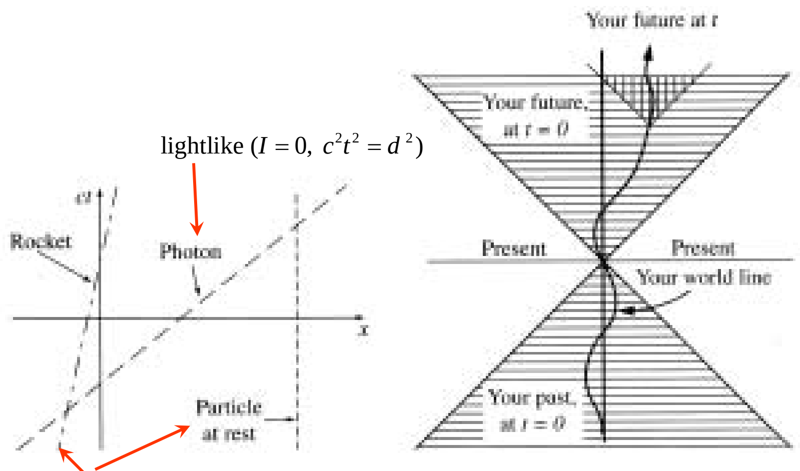
Two event A and B occurs at $\left(x_A^0, x_A^1, x_A^2, x_A^3\right)$ and $\left(x_B^0, x_B^1, x_B^2, x_B^3\right)$

the displacement 4-vector: $\Delta x^\mu \equiv x_A^\mu - x_B^\mu$

the interval between two events: $I \equiv \Delta x_\mu \Delta x^\mu = -c^2 t^2 + d^2$

$$\begin{aligned} &\text{timelike} && I < 0 \ \ (c^2 t^2 > d^2) \\ &\text{spacelike} && I > 0 \ \ (c^2 t^2 < d^2) \\ &\text{lightlike} && I = 0 \ \ (c^2 t^2 = d^2) \end{aligned}$$

Space-Time Diagrams (Minkowski Diagrams)



timelike ($I < 0,\ c^2t^2 > d^2$)

spacelike ($I > 0,\ c^2t^2 < d^2$)? Occurs at the same time.

12.2 Relativistic Mechanics

12.2.1 Proper Time and Proper Velocity

How to define the velocity?

Imagine you are on a flight to Moon, and the pilot announces that the plane's velocity relative to ground is 4/5c.

$$\mathbf{u} = \frac{d\mathbf{l}}{dt} \qquad \text{the ordinary velocity}$$

However, your watch runs slow due to time dilation. You might be more interested in the distance covered per unit proper time.

$$\boldsymbol{\eta} = \frac{d\mathbf{l}}{d\tau} = \frac{1}{\sqrt{1-u^2/c^2}}\mathbf{u} \qquad \text{the proper velocity}$$

Which definition is more preferable/useful?

Proper Velocity 4-Vector (4-velocity)

Proper time τ is invariant, whereas "ordinary" time t depends on the particular reference frame.

Proper velocity has an enormous advantage over ordinary velocity: it transforms simply.

$$\eta^\mu \equiv \frac{dx^\mu}{d\tau} \qquad \begin{array}{l}\text{The numerator, } dx^\mu\text{, is a displacement 4-vector;}\\ \text{The denominator, } d\tau\text{, is invariant.}\end{array}$$

$$\left.\begin{array}{l}\bar{\eta}^0 = \gamma(\eta^0 - \beta\eta^1),\\ \bar{\eta}^1 = \gamma(\eta^1 - \beta\eta^2)\\ \bar{\eta}^2 = \eta^2\\ \bar{\eta}^3 = \eta^3\end{array}\right\} \text{More generally, } \bar{\eta}^\mu = \Lambda^\mu_\nu \eta^\nu$$

$$\eta^0 = \frac{dct}{d\tau} = c\frac{dt}{d\tau} = \frac{c}{\sqrt{1-u^2/c^2}}$$

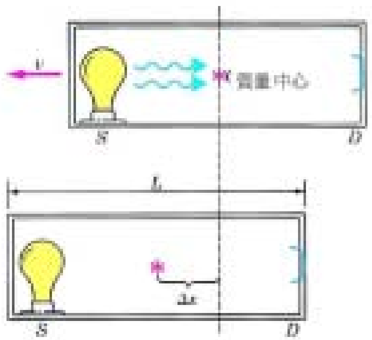
Mass-Energy Equivalence

Mass-energy equivalence: Since no internal process can even move the center of mass of a system, we can derive mass-energy equivalence.

$$\frac{E}{c} = mv, \quad \Delta x = v\Delta t = \frac{EL}{Mc^2}$$

$$-M\Delta x + mL = 0$$

$$E = mc^2$$



12.2.2 Relativistic Energy and Momentum

How to define the momentum?

In classical mechanics Momentum is mass times velocity, but immediately a question arise: Should we use *ordinary* velocity or *proper* velocity? There is no priori reason to favor one over the other.

$$\mathbf{p} \equiv m\boldsymbol{\eta} = \frac{m\mathbf{u}}{\sqrt{1-u^2/c^2}} \qquad \text{the relativistic momentum}$$

$$= m_{\text{rel}}\mathbf{u} \qquad m_{\text{rel}}\text{: the relativistic mass}$$

$$p^0 = m\eta^0 = mc\frac{dt}{d\tau} = \frac{mc}{\sqrt{1-u^2/c^2}} = \frac{E}{c}$$

$$\text{where } E \equiv \frac{mc^2}{\sqrt{1-u^2/c^2}} \qquad \text{relativistic energy}$$

Kinetic Energy

How to define the kinetic energy?

The **relativistic kinetic energy** is the **total energy** minus the **rest energy**:

$$E_{\text{kin}} = E - E_{res} = \frac{mc^2}{\sqrt{1-u^2/c^2}} - mc^2$$
$$= mc^2\left(1+\frac{1}{2}\frac{u^2}{c^2}+\frac{3}{8}\frac{u^4}{c^4}+\ldots-1\right)$$
$$= \frac{1}{2}mu^2+\frac{3}{8}\frac{mu^4}{c^2}+\ldots$$

$$K = \frac{1}{2}mu^2 \quad \text{the classical defination of the kinetic energy}$$

Conservation and Invariant

Conserved quantity: same value before and after some process.
Invariant quantity: same value in all inertia frame.

Mass is invariant, but not conserved.
Charge is both conserved and invariant.
Energy is conserved, but not invariant.
Momentum is conserved, but not invariant.

Velocity is neither conserved nor invariant.

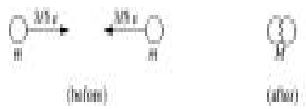
Invariant: $p^\mu p_\mu = -(p^0)^2 + (p\cdot p) = -m^2c^2$

$$\Rightarrow \frac{E^2}{c^2} - p^2 = m^2c^2$$

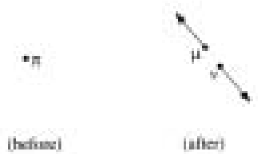
12.2.3 Relativistic Kinematics

Explore some applications of the conservation law to particle decays and collisions.

Example 12.7 Two lumps of clay, each of (rest) mass m, collide head-on at $3/5c$. They stick together. Question: what is the mass (M) of the composite lump?



Example 12.8 A pion at rest decays into a muon and a neutrino. Find the energy of the outgoing muon, in terms of the two masses, $m\pi$ and $m\mu$ (assume $m\mu$=0)



Does it make any sense?

Massless Particle: Photon

In classical mechanics there is no such thing as a massless particle.

In special relativity, $\mathbf{p}$ and E are still proportional to m. If u=c, then the zero numerator is balanced by a zero in the denominator, leaving $\mathbf{p}$ and E indeterminate (zero over zero).

$$\text{When } u = c \text{ and } m = 0, \quad \Rightarrow \begin{cases} \mathbf{p} = \dfrac{m\mathbf{u}}{\sqrt{1-u^2/c^2}} = \dfrac{0}{0} \\ E = \dfrac{mc^2}{\sqrt{1-u^2/c^2}} = \dfrac{0}{0} \end{cases}$$

A massless particle could carry energy and momentum, provided it always travels at the speed of light.

$E = pc = h\nu$ photon

The Compton Effect (Example 12.9)

The derivation of Compotn's scattering: Classically an electromagnetic wave carries moment given by $p=E/c$.

Conservation of linear momentum:

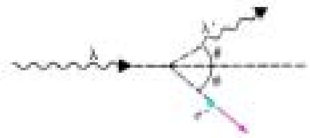
$$\begin{cases} p_x: \ \ p_\lambda = p_{\lambda'}\cos\theta + p\cos\phi \\ p_y: \ \ 0 = p_{\lambda'}\sin\theta - p\sin\phi \end{cases} \qquad p = \frac{hf}{c} = \frac{h}{\lambda}$$

$$(p_\lambda - p_{\lambda'}\cos\theta)^2 + (p_{\lambda'}\sin\theta)^2 = p^2 \qquad (1)$$

Conservation of energy:

$$hf = hf' + K; \ \ K = (\gamma - 1)m_0c^2$$

$$\begin{cases} (cp_\lambda - cp_{\lambda'}) = K \\ K^2 + 2Km_0c^2 = c^2p^2 \end{cases}$$



$$(p_\lambda - p_{\lambda'})^2 + 2(p_\lambda - p_{\lambda'})m_0c = p^2 \qquad (2)$$

The Compton Effect (ii)

For known X ray frequency and final particle momentums We can further solve these two equations.

$$(p_\lambda - p_{\lambda'}\cos\theta)^2 + (p_{\lambda'}\sin\theta)^2 = p^2$$

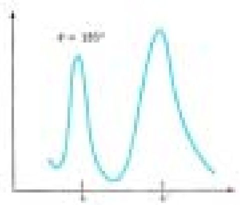
$$(p_\lambda - p_{\lambda'})^2 + 2(p_\lambda - p_{\lambda'})m_0c = p^2$$

Further solving these two equations, we obtain

$$(p_\lambda - p_{\lambda'})m_0c = p_\lambda p_{\lambda'}(1-\cos\theta)$$

$$\frac{1}{p_{\lambda'}} - \frac{1}{p_\lambda} = \frac{1}{m_0c}(1-\cos\theta)$$

$$\Rightarrow \Delta\lambda = \frac{h}{m_0c}(1-\cos\theta)$$



$\frac{h}{m_0c} = 0.00243$ nm is called the Compton wavelength.

12.2.4 Relativistic Dynamics
Newton's laws

Newton's *first* law is built into the principle of relativity.

Newton's *second* law retains its validity in relativistic mechanics, provided we use the relativistic momentum.

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \qquad \text{where } \mathbf{p} = \frac{m\mathbf{u}}{\sqrt{1-u^2/c^2}} = m_{\text{rel}}\mathbf{u}$$

Newton's *third* law does not, in general, extend to the relativistic domain due to the relativity of simultaneously.

Only in the case of contact interactions, where the two forces are applied at the same physical point, can the third law be retained.

Work-Energy Theorem

The work-energy theorem ("the net work done on a particle equals the increase in its kinetic energy") holds relativistically.

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}, \text{ where } \mathbf{p} = \frac{m\mathbf{u}}{\sqrt{1-u^2/c^2}} = m_{\text{rel}}\mathbf{u}$$

$$W \equiv \int \mathbf{F}\cdot d\mathbf{l} = \int \frac{d\mathbf{p}}{dt}\cdot d\mathbf{l} = \int \frac{d\mathbf{p}}{dt}\cdot\frac{d\mathbf{l}}{dt}dt = \int \frac{d}{dx}\left(\frac{m\mathbf{u}}{\sqrt{1-u^2/c^2}}\right)\cdot\mathbf{u}dt$$

$$\frac{d}{dt}\left(\frac{m\mathbf{u}}{\sqrt{1-u^2/c^2}}\right)\cdot\mathbf{u} = \frac{1}{1-u^2/c^2}\left(\sqrt{1-u^2/c^2}\frac{dm\mathbf{u}}{dt} - m\mathbf{u}\frac{1}{\sqrt{1-u^2/c^2}}\frac{-udu}{c^2dt}\right)\cdot\mathbf{u}$$

$$= \frac{1}{(1-u^2/c^2)^{\frac{3}{2}}}\left((1-\frac{u^2}{c^2})mu\frac{du}{dt} + \frac{u^2}{c^2}mu\frac{du}{dt}\right) = \frac{1}{(1-u^2/c^2)^{\frac{3}{2}}}\left(mu\frac{du}{dt}\right)$$

$$= \frac{d}{dt}\left(\frac{mc^2}{\sqrt{1-u^2/c^2}}\right) = \frac{dE}{dt} \qquad \Rightarrow \quad W = \int\frac{dE}{dt}dt = E_{\text{final}} - E_{\text{initial}}$$

The Ordinary Force and The Minkowski Force

The *ordinary* force: **F** is the derivative of momentum with respect to ordinary time, transformation is ugly (both the numerator and denominator must be transformed).

$$\bar{F}_y = \frac{d\bar{p}_y}{d\bar{t}} = \frac{dp_y}{\gamma(dt - \frac{\beta}{c}dx)} = \frac{dp_y/dt}{\gamma(1-\frac{\beta}{c}u_x)} = \frac{F_y}{\gamma(1-\frac{\beta}{c}u_x)}$$

$$\bar{F}_z = \frac{d\bar{p}_z}{d\bar{t}} = \frac{dp_z}{\gamma(dt - \frac{\beta}{c}dx)} = \frac{dp_z/dt}{\gamma(1-\frac{\beta}{c}u_x)} = \frac{F_z}{\gamma(1-\frac{\beta}{c}u_x)}$$

$$\bar{F}_x = \frac{d\bar{p}_x}{d\bar{t}} = \frac{\gamma(dp_x - \beta dp^0)}{\gamma(dt - \frac{\beta}{c}dx)} = \frac{\frac{dp_x}{dt} - \beta\frac{dp^0}{dt}}{(1-\frac{\beta}{c}u_x)} = \frac{\frac{dp_x}{dt} - \frac{\beta}{c}\frac{dE}{dt}}{(1-\frac{\beta}{c}u_x)}$$

The *Minkowski* force: **K** is the derivative of momentum with respect to proper time.

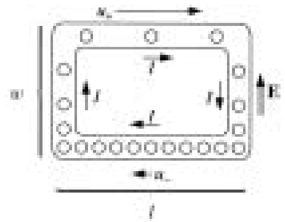
$$\mathbf{K} \equiv \frac{d\mathbf{p}}{d\tau} = \frac{dt}{d\tau}\frac{d\mathbf{p}}{dt} = \frac{\mathbf{F}}{\sqrt{1-u^2/c^2}}, \qquad K^\mu \equiv \frac{dp^\mu}{d\tau}$$

Example 12.12: Hidden momentum

As a model for a magnetic dipole **m**, consider a rectangular loop of wire carrying a steady current. Picture the current as a stream of noninteracting positive charges that move freely within the wire. When a uniform electric field **E** is applied, the charges accelerate in the left segment and deccelerate in the right one. Find the total momentum of all charges in the loop.

Solution:

The current is the same in all four segments $I=\lambda u$.



$$I = \frac{eN_+}{l}u_+ = \frac{eN_-}{l}u_- \quad \text{so} \quad N_\pm u_\pm = \frac{Il}{e}$$

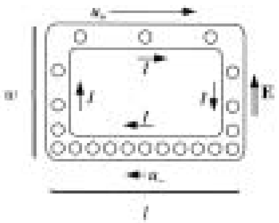
Relativistic momentum is

$$p = \gamma_+ m N_+ u_+ - \gamma_- m N_- u_- = (\gamma_+ - \gamma_-) m \frac{Il}{e} \neq 0$$

Hidden momentum (relativistic effect)

The gain in energy (γmc^2) is equal to the work done by the electric force **E**.

$$(\gamma_+ - \gamma_-)mc^2 = eEw \quad \Rightarrow \quad p = \frac{IlEw}{c^2}$$



Ilw is the magnetic dipole moment of the loop as vectors **m** points into the page, and **p** is to the right, so

$$\mathbf{p} = \frac{1}{c^2}(\mathbf{m} \times \mathbf{E})$$

A magnetic dipole in an electric field carries linear momentum, even though it is not moving.

This so-called *hidden momentum* is strictly relativistic, and purely mechanical.

A more realistic model for a current-carrying wire can be found in the supplement.
See V. Hnizdo, *Am. J. Phys*. **65**, 92 (1997).

12.3 Relativistic Electrodynamics
12.3.2 How the Fields Transform

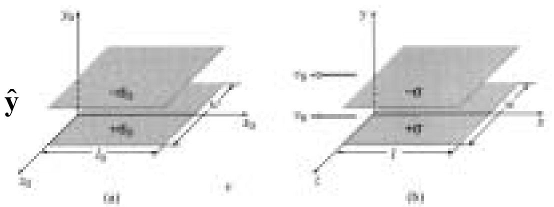
We have learned that one observer's electric field is another's magnetic field.

What are the general transformation rules for electromagnetic fields?

Let's start with "***Charge invariant***".

Consider the simplest possible electric field.

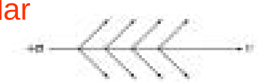
$$\mathbf{E}_0 = \frac{\sigma_0}{\varepsilon_0}\hat{\mathbf{y}}$$



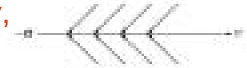
$$\mathbf{E} = \frac{\sigma}{\varepsilon_0}\hat{\mathbf{y}}$$

The Transformation of The Electric Field

Are we sure that the field is still perpendicular to the plates?



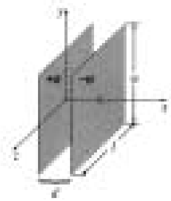
What if the field of a moving plane tilted, say, in the direction of motion? It doesn't.



The total charge on each plate is invariant.

$$Q=\sigma_0 l_0 w_0=\sigma l w \quad \text{where } l=\sqrt{1-v_0^2/c^2}\,l_0 \text{ and } w=w_0$$

$$\sigma=\frac{1}{\sqrt{1-v_0^2/c^2}}\sigma_0=\gamma_0\sigma_0 \quad\Rightarrow\quad \mathbf{E}^{\perp}=\gamma_0\mathbf{E}_0^{\perp} \longleftarrow$$ perpendicular components



$$Q=\sigma_0 l_0 w_0=\sigma l w,$$
$$\text{where } l=l_0 \text{ and } w=w_0$$
$$\sigma=\sigma_0 \quad\Rightarrow\quad \mathbf{E}^{//}=\mathbf{E}_0^{//} \longleftarrow$$ parallel components

Example 12.13: The E-field of a moving point chagre.

A point charge q is at rest at the origin in system S_0. Question: What is the electric field of this same charge in system S, which moves to the right at speed v_0 relative to S_0?

Solution:

$$\mathbf{E}_0=\frac{1}{4\pi\epsilon_0}\frac{q}{r_0^2}\hat{\mathbf{r}}_0,$$

$$\begin{cases} E_{x0} = \dfrac{1}{4\pi\epsilon_0}\dfrac{qx_0}{(x_0^2+y_0^2+z_0^2)^{3/2}}, \\ E_{y0} = \dfrac{1}{4\pi\epsilon_0}\dfrac{qy_0}{(x_0^2+y_0^2+z_0^2)^{3/2}}, \\ E_{z0} = \dfrac{1}{4\pi\epsilon_0}\dfrac{qz_0}{(x_0^2+y_0^2+z_0^2)^{3/2}}. \end{cases} \qquad \begin{cases} E_x = E_{x0} = \dfrac{1}{4\pi\epsilon_0}\dfrac{qx_0}{(x_0^2+y_0^2+z_0^2)^{3/2}}, \\ E_y = \gamma_0 E_{y0} = \dfrac{1}{4\pi\epsilon_0}\dfrac{\gamma_0 q y_0}{(x_0^2+y_0^2+z_0^2)^{3/2}}, \\ E_z = \gamma_0 E_{z0} = \dfrac{1}{4\pi\epsilon_0}\dfrac{\gamma_0 q z_0}{(x_0^2+y_0^2+z_0^2)^{3/2}}. \end{cases}$$

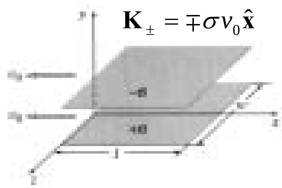
Very efficient as compared with Chap.10 Eq. 10.68.

The Transformation of The Magnetic Field

To derive the general rule we must start out in a system with both electric and magnetic fields.

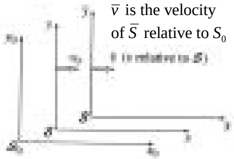
$$E_y=\frac{\sigma}{\varepsilon_0} \quad\text{and}\quad B_z=-\mu_0\sigma v_0$$

$$\mathbf{K}_{\pm}=\mp\sigma v_0\hat{\mathbf{x}}$$



In a third system, $\bar{S}$, traveling to the right with speed v relative to S, the field would be

$$\bar{E}_y=\frac{\bar{\sigma}}{\varepsilon_0} \quad\text{and}\quad \bar{B}_z=-\mu_0\bar{\sigma}\bar{v}$$

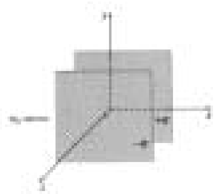


$$\bar{v}=\frac{v+v_0}{1+vv_0/c^2}, \quad \bar{\gamma}=\frac{1}{\sqrt{1-\bar{v}^2/c^2}}, \quad \bar{\sigma}=\bar{\gamma}\sigma_0$$

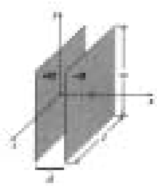
$$\Rightarrow \quad \bar{E}_y=\frac{\bar{\gamma}\sigma_0}{\varepsilon_0}=\left(\frac{\bar{\gamma}}{\gamma_0}\right)\frac{\sigma}{\varepsilon_0}, \quad \text{where } \gamma=\frac{1}{\sqrt{1-v^2/c^2}}$$

The Transformation of The Magnetic Field

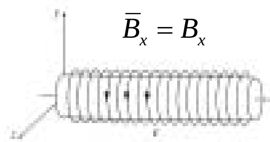
$$\frac{\bar{\gamma}}{\gamma_0}=\gamma\left(1+\frac{vv_0}{c^2}\right) \text{ and } c^2=\frac{1}{\varepsilon_0\mu_0} \quad \begin{rcases} \Rightarrow \bar{E}_y=\gamma(E_y-vB_z) \\ \Rightarrow \bar{B}_z=\gamma\left(B_z-\dfrac{v}{c^2}E_y\right) \end{rcases}$$



$$\begin{rcases} \Rightarrow \bar{E}_z=\gamma(E_z+vB_y) \\ \Rightarrow \bar{B}_y=\gamma\left(B_y+\dfrac{v}{c^2}E_z\right) \end{rcases}$$



$$\bar{E}_x=E_x$$



$$\bar{B}_x=B_x$$

$$\boxed{\begin{array}{lll} \bar{E}_x=E_x & \bar{E}_y=\gamma(E_y-vB_z) & \bar{E}_z=\gamma(E_z+vB_y) \\ \bar{B}_x=B_x & \bar{B}_y=\gamma\left(B_y+\dfrac{v}{c^2}E_z\right) & \bar{B}_z=\gamma\left(B_z-\dfrac{v}{c^2}E_y\right) \end{array}}$$

Two Special Cases

$$\bar{E}_x = E_x \quad \bar{E}_y = \gamma(E_y - vB_z) \quad \bar{E}_z = \gamma(E_z + vB_y)$$

$$\bar{B}_x = B_x \quad \bar{B}_y = \gamma(B_y + \frac{v}{c^2}E_z) \quad \bar{B}_z = \gamma(B_z - \frac{v}{c^2}E_y)$$

1. If **B**=0 in S, then

$$\bar{\mathbf{B}} = \gamma(\frac{v}{c^2}E_z)\hat{\mathbf{y}} - \gamma(\frac{v}{c^2}E_z)\hat{\mathbf{z}}$$
$$= -\frac{1}{c^2}(\mathbf{v}\times\bar{\mathbf{E}}) \qquad \text{where } \mathbf{v} = v\hat{\mathbf{x}}$$

2. If **E**=0 in S, then

$$\bar{\mathbf{E}} = -\gamma v(B_z\hat{\mathbf{y}} - B_y\hat{\mathbf{z}}) = -v(\bar{B}_z\hat{\mathbf{y}} - \bar{B}_y\hat{\mathbf{z}})$$
$$= \mathbf{v}\times\bar{\mathbf{B}}$$
$$\text{where } \mathbf{v} = v\hat{\mathbf{x}}$$

12.3.3 The Field Tensor

E and **B** certainly do not transform like the spatial parts of the two 4-vectors (4-velocity and 4-momentum).

What sort of an object is this, which has six components and transforms according previous results?

Answer: Antisymmetric, second-rank tensor.

$$t^{\mu\nu} = \begin{Bmatrix} t^{00} & t^{01} & t^{02} & t^{03} \\ t^{10} & t^{11} & t^{12} & t^{13} \\ t^{20} & t^{21} & t^{22} & t^{23} \\ t^{30} & t^{31} & t^{32} & t^{33} \end{Bmatrix}$$

$t^{\mu\nu} = t^{\nu\mu}$ (symmetric tensor, 10 distinct components)

$t^{\mu\nu} = -t^{\nu\mu}$ (antisymmetric tensor, 6 distinct components)

$$t^{\mu\nu} = \begin{Bmatrix} 0 & t^{01} & t^{02} & t^{03} \\ -t^{01} & 0 & t^{12} & t^{13} \\ -t^{02} & -t^{12} & 0 & t^{23} \\ -t^{03} & -t^{13} & -t^{23} & 0 \end{Bmatrix}$$

The Tensor Transformation

$\bar{a}^{\nu} = \Lambda^{\nu}_{\lambda}a^{\lambda}$ 4-vector transformation

$\bar{t}^{\mu\nu} = \Lambda^{\mu}_{\lambda}\Lambda^{\nu}_{\sigma}t^{\lambda\sigma}$ tensor transformation

$$\Lambda = \begin{Bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Bmatrix}$$

Work out the following transformation:

$$\bar{t}^{01} = t^{01} \quad \bar{t}^{02} = \gamma(t^{02} - \beta t^{12}) \quad \bar{t}^{03} = \gamma(t^{03} + \beta t^{31})$$
$$\bar{t}^{23} = t^{23} \quad \bar{t}^{31} = \gamma(t^{31} + \beta t^{03}) \quad \bar{t}^{12} = \gamma(t^{12} - \beta t^{02})$$

By direct comparison, we can construct the **field tensor** $F^{\mu\nu}$

$$\bar{E}_x = E_x \quad \bar{E}_y = \gamma(E_y - vB_z) \quad \bar{E}_z = \gamma(E_z + vB_y)$$

$$\bar{B}_x = B_x \quad \bar{B}_y = \gamma(B_y + \frac{v}{c^2}E_z) \quad \bar{B}_z = \gamma(B_z - \frac{v}{c^2}E_y)$$

The Field Tensor and The Dual Tensor

$$F^{01} \equiv \frac{E_x}{c},\ F^{02} \equiv \frac{E_y}{c},\ F^{03} \equiv \frac{E_z}{c},\ F^{12} \equiv B_z,\ F^{31} \equiv B_y,\ F^{23} \equiv B_x.$$

$$F^{\mu\nu} = \begin{Bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{Bmatrix}$$

the **field tensor**

There was a different way of imbedding **E** and **B** in an antisymmetric tensor.

$$G^{\mu\nu} = \begin{Bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{Bmatrix}$$

the **dual tensor**

12.3.4 Electrodynamics in Tensor Notation

Reformulate the laws of electrodynamics (Maxwell's equations and the Lorentz force law) in relativistic notation.

How the sources of the fields, ρ and $\mathbf{J}$, transform?

$$\rho = \frac{Q}{V} \text{ and } \mathbf{J} = \rho\mathbf{u}, \text{ where } \rho_0 = \frac{Q}{V_0} \text{ (the proper charge density)}$$

$$\rho = \rho_0 \frac{V_0}{V} = \gamma\rho_0, \text{ where } V = \sqrt{1-u^2/c^2}\,V_0 \text{ (length contraction)}$$

$$\mathbf{J} = \rho\mathbf{u} = \gamma\rho_0\mathbf{u} = \rho_0(\gamma\mathbf{u}) = \rho_0\boldsymbol{\eta}, \text{ where } \boldsymbol{\eta} = \gamma\mathbf{u} \text{ (proper velocity)}$$

The current density 4-vector: $J^\mu = (c\rho, J_x, J_y, J_z)$

Conservation of charge:

$$\nabla\cdot\mathbf{J} = -\frac{\partial\rho}{\partial t}$$

$$\nabla\cdot\mathbf{J} = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} = \sum_{i=0}^{3}\frac{\partial J^i}{\partial x^i}$$

$$-\frac{\partial\rho}{\partial t} = -\frac{\partial(c\rho)}{\partial(ct)} = -\frac{\partial J^0}{\partial x^0}$$

$$\boxed{\frac{\partial J^\mu}{\partial x^\mu} = 0}$$

Maxwell's Equations in Tensor Notation (i)

Maxwell's equations can be written in the following forms.

$$\boxed{\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu}$$

Gauss's law
Ampere's law with Maxwell's correction

$$\mu = 0 \quad \frac{\partial F^{\mu\nu}}{\partial x^\nu} = \frac{\partial F^{00}}{\partial x^0} + \frac{\partial F^{01}}{\partial x^1} + \frac{\partial F^{02}}{\partial x^2} + \frac{\partial F^{03}}{\partial x^3} = \mu_0 J^0$$

$$\frac{1}{c}\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) = \mu_0 c\rho \quad \Rightarrow \quad \underline{\underline{\nabla\cdot\mathrm{E} = \frac{\rho}{\varepsilon_0}}}$$

$$\mu = 1 \quad \frac{\partial F^{1\nu}}{\partial x^\nu} = \frac{\partial F^{10}}{\partial x^0} + \frac{\partial F^{11}}{\partial x^1} + \frac{\partial F^{12}}{\partial x^2} + \frac{\partial F^{13}}{\partial x^3} = \mu_0 J^1$$

$$-\frac{1}{c^2}\frac{\partial E_x}{\partial t} + \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \mu_0 J_x \quad \rightarrow \quad \left(-\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \nabla\times\mathbf{B}\right)_x = \mu_0(\mathbf{J})_x$$

$$+\mu = 2 \text{ and } 3 \quad \left(-\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \nabla\times\mathbf{B}\right)_{y,z} = \mu_0\mathbf{J}_{y,z} \quad \Rightarrow \underline{\underline{\nabla\times\mathbf{B} - \mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t} = \mu_0\mathbf{J}}}$$

Maxwell's Equations in Tensor Notation (ii)

Maxwell's equations can be written in the following forms.

$$\boxed{\frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0}$$

Faraday's law
Gauss's law for magnetic field

$$\mu = 0 \quad \frac{\partial G^{0\nu}}{\partial x^\nu} = \frac{\partial G^{00}}{\partial x^0} + \frac{\partial G^{01}}{\partial x^1} + \frac{\partial G^{02}}{\partial x^2} + \frac{\partial G^{03}}{\partial x^3} = 0$$

$$\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right) = 0 \quad \Rightarrow \quad \underline{\underline{\nabla\cdot\mathbf{B} = 0}}$$

$$\mu = 1 \quad \frac{\partial G^{1\nu}}{\partial x^\nu} = \frac{\partial G^{10}}{\partial x^0} + \frac{\partial G^{11}}{\partial x^1} + \frac{\partial G^{12}}{\partial x^2} + \frac{\partial G^{13}}{\partial x^3} = 0$$

$$-\frac{1}{c}\frac{\partial B_x}{\partial t} - \frac{1}{c}\frac{\partial E_z}{\partial y} + \frac{1}{c}\frac{\partial E_y}{\partial z} = 0 \quad \rightarrow \quad \left(\frac{\partial\mathbf{B}}{\partial t} + \nabla\times\mathbf{E}\right)_x = 0$$

$$+\mu = 2 \text{ and } 3 \quad \left(\frac{\partial\mathbf{B}}{\partial t} + \nabla\times\mathbf{E}\right)_{y,z} = 0 \quad \Rightarrow \quad \underline{\underline{\nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0}}$$

The Minkouski Force and Relativistic Potentials

The Minkowski force on a charge q is given by

$$\mathbf{K} = \frac{1}{\sqrt{1-u^2/c^2}}q[\mathbf{E} + (\mathbf{u}\times\mathbf{B})] = \frac{1}{\sqrt{1-u^2/c^2}}\mathbf{F}$$

The electric and magnetic fields can be expressed in terms of a scalar potential and a vector potential.

$$\mathbf{E} = -\nabla V - \frac{\partial\mathbf{A}}{\partial t} \qquad \mathbf{B} = \nabla\times\mathbf{A}$$

$$A^\mu = (V/c, A_x, A_y, A_z) \quad \text{4-vector potential}$$

$$F^{\mu\nu} = \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu} \quad \text{the definition of the field tensor}$$

$$\frac{\partial A^\mu}{\partial x^\mu} = 0 \quad \text{the Lorentz gauge}$$

Homework of Chap.12

Prob. 3, 4, 6, 25, 30, 33, 38, 46

39.3 Covariance

Covariance: The laws of mechanics are covariance---*they retain their form*---with respective to Galiliean transformation. Newton's second low, *F=ma*, in one frame has the same form, *F'=ma'*, in another. However, the Maxwell's equations does not satisfy this requirement when applying the Galiliean transformation.

$$x' = x - vt; \quad t' = t$$

Three Problems:

1. The force between the charge depends on the frame of reference employed.
2. Maxwell's equations are valid in only one special frame with the Galiliean transformation.
3. The applied electromagnetism law will change with reference frame.

The Relativistic Doppler Effect

In the classical Doppler effect for sound waves, the observed frequency depends differently on the velocities of the source and the observer. The underlying reason is that for sound there is a medium (the air) that serves as an "**absolute" reference frame**.

(a) Source at rest, observer moves (velocity modulation)

$$f' = \frac{v'}{\lambda_o} = \frac{v \pm v_o}{v} f_o$$

(b) Source moves, observer at rest (wavelength modulation)

$$f' = \frac{v}{\lambda'} = \frac{v}{v \pm v_s} f_o$$

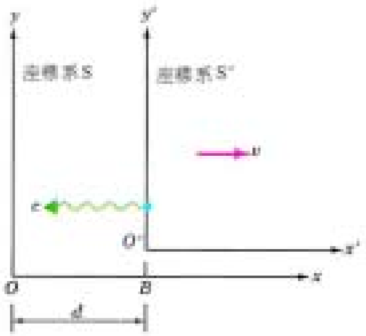
In contrast, for light there is **no absolute frame**: The relativistic Doppler effect for light depends only on the relative velocity between the source and the observer.

The Relativistic Doppler Effect (II)

In order to obtain the Doppler effect, we have to calculate the time interval measured in two frames. Note that the time dilation is assumed in the following calculation.

$$T = \Delta t + \frac{d}{c} = \gamma(1 + \frac{v}{c})T_0$$

$$T = \sqrt{\frac{c+v}{c-v}}T_0$$



Longitudinal

$$f = \sqrt{\frac{c-v}{c+v}} f_o$$

Transverse

$$f = \frac{1}{\gamma} f_o$$