

Chapter 10 Systems of Particles

Consider a system consisting of a large number of particles. It is almost impossible to clearly describe the motion of each particle, even though their collisions are elastic.

How do we apply our understanding on force, momentum, kinetic and potential energy, and conserved quantities to such a system?

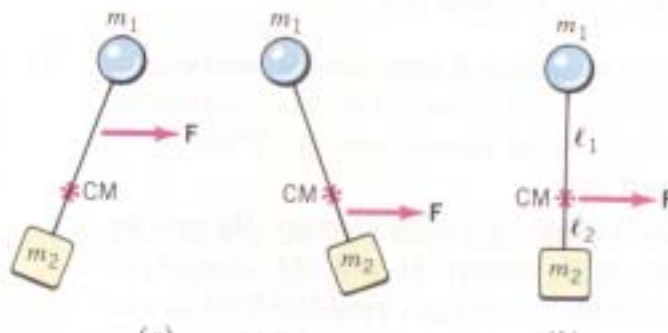
Center of Mass (CM)

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10.1 Center of Mass

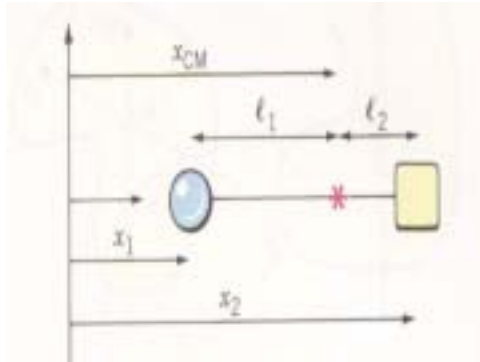
When the motion of a body involves not only translation but also rotation or vibration or both, we must treat it as a system of particles.

Despite the complex motions of which a system is capable, there is a single point, the **center of mass (CM)**, whose translational motion is characteristic of the system as a whole.



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Where is the Center of Mass?



$$m_1(X_{cm} - X_1) = m_2(X_2 - X_{cm})$$

$$X_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

The position x_{cm} is a weighted average in which each coordinated is weighted by the mass located at that point.

For N particles the position of CM is:

$$\mathbf{r}_{cm} = \frac{m_1 x_1 + m_2 x_2 + \dots + m_N x_N}{m_1 + m_2 + \dots + m_N} = \frac{\sum m_i \mathbf{r}_i}{M}$$

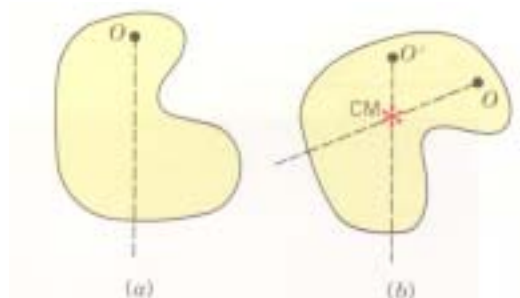
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Center of Mass and Center of Gravity

Under uniform gravity, the position of the center of mass is exactly the same as that of center of gravity.

$$\mathbf{r}_{cm} = \frac{m_1 g x_1 + m_2 g x_2 + \dots + m_N g x_N}{m_1 g + m_2 g + \dots + m_N g} = \frac{\sum m_i \mathbf{r}_i}{M}$$

The center of mass of a symmetric body always lies on an axis or a plane of symmetry.



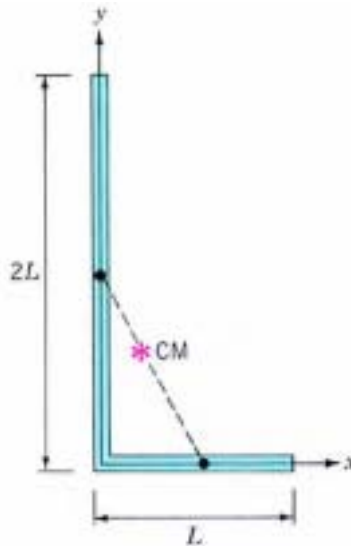
Locating the center of mass of a planar body.

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Example 10.2

A thin rod of length $3L$ is bent at right angles at a distance L from one end (see Fig. 10.7). Locate the CM with respect to the corner. Take $L=1.2$ m.

Sol:



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10.2 Center of Mass of Continuous Bodies

To find the center of mass of a continuous body one must integrate the contributions of each mass element dm .

$$\begin{aligned}\mathbf{r}_{cm} &= \frac{1}{M} \int_{body} \mathbf{r} dm = \frac{1}{M} \int_{body} \mathbf{r} \lambda(\mathbf{r}) dr \\ &= \frac{1}{M} \int_{body} \mathbf{r} \sigma(\mathbf{r}) d^2 r \\ &= \frac{1}{M} \int_{body} \mathbf{r} \rho(\mathbf{r}) d^3 r\end{aligned}$$

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Example 10.3

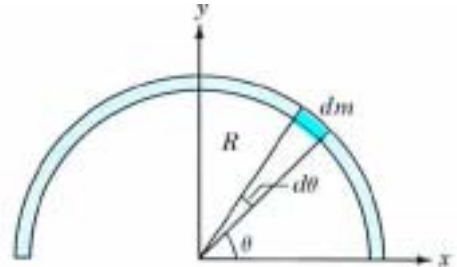
Find the CM of a semicircular rod of radius R and linear density λ kg/m as shown in Fig. 10.10.

Sol:

$$\mathbf{r}_{cm} = (0, y_{cm})$$

$$y_{cm} = \frac{2}{M} \int_0^{\pi/2} R \sin \theta \cdot R \lambda d\theta$$

$$= \frac{2R^2 \lambda}{\pi R \lambda} (-\cos \theta) \Big|_0^{\pi/2} = \frac{2}{\pi} R$$



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Example 10.4

Find the CM of a uniform solid cone of height h and semiangle α , as in Fig. 10.11.

Sol:

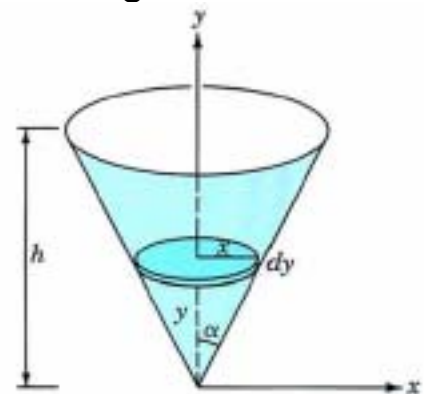
$$\mathbf{r}_{cm} = (0, y_{cm})$$

$$y_{cm} = \frac{1}{M} \int_0^h y \cdot \rho \pi x^2 dy; \quad x = y \tan \alpha$$

$$= \frac{1}{M} \int_0^h y \cdot \rho \pi (\tan \alpha)^2 y^2 dy = \frac{1}{4M} \rho \pi (\tan \alpha)^2 h^4$$

$$M = \int_0^h \rho \pi x^2 dy = \int_0^h \rho \pi (\tan \alpha)^2 y^2 dy = \frac{1}{3} \rho \pi (\tan \alpha)^2 h^3$$

$$\therefore y_{cm} = 3h/4$$



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Questions

Is it possible for the center of mass of a high jumper or pole vaulter to pass under the bar while the torso passes over it? If so, how?

Is it possible to have a system with zero kinetic energy but zero total linear momentum? If so, give an example.
(b) How about nonzero linear momentum but no kinetic energy.

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10.3 Motion of the Center of Mass (I)

Velocity of the center of mass

$$\mathbf{v}_{cm} = \frac{d\mathbf{r}_{cm}}{dt} = \frac{\sum m_i \mathbf{v}_i}{M}$$

Total linear momentum of a system of particles

$$\mathbf{P} = M\mathbf{v}_{cm} = \sum m_i \mathbf{v}_i$$

The total momentum of a system of particles is equivalent to that of a single (imaginary) particle of mass $M = \sum m_i$ moving at the velocity of the center of mass \mathbf{v}_{cm} .

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10.3 Motion of the Center of Mass (II)

Force of the center of mass

$$\mathbf{F} = \frac{d\mathbf{P}}{dt} = M \frac{d\mathbf{v}_{cm}}{dt} = M\mathbf{a}_{cm} = \sum m_i \mathbf{a}_i$$

The rate of change of the total momentum of a system is equal to the net external force.

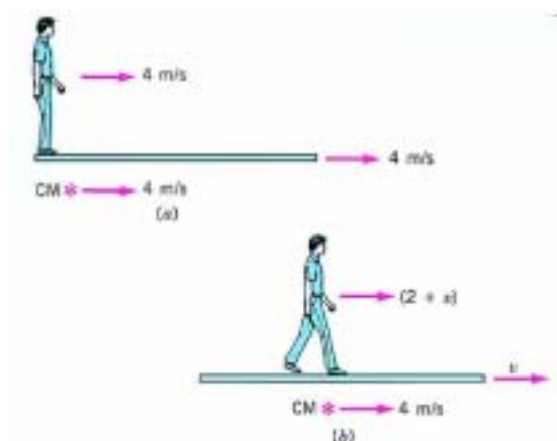
$$\text{If } \mathbf{F}_{ext} = 0, \text{ then } \mathbf{v}_{cm} = \text{constant}$$

If the net external force on a system of particles is zero, the velocity of the center of mass remains constant.

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Example 10.7

A 75-kg man sits at the rear end of a platform of mass 25 kg and length 4 m, which moves initially at 4 m/s over a frictionless surface. At $t=0$, he walks at 2 m/s relative to the platform and then sits down at the front end. During the period he is walking, find the displacements of: (a) the platform, (b) the man, and (c) the center of mass.



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10.4 Kinetic Energy of a System of Particles (I)

The position \mathbf{r}_i of the i -th particle can be re-written as the sum of the position of center of mass \mathbf{r}_{cm} plus the position relative to the center of mass.

$$\mathbf{r}_i = \mathbf{r}_{cm} + \mathbf{r}'_i$$

Taking the time derivative of above equation, the velocity of the i -th particle is

$$\mathbf{v}_i = \mathbf{v}_{cm} + \mathbf{v}'_i$$

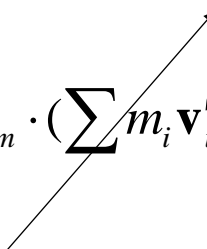
The kinetic energy of the i -th particle is

$$K_i = \frac{1}{2} m_i (\mathbf{v}_i \cdot \mathbf{v}_i) = \frac{1}{2} m_i (v_{cm}^2 + v_i'^2 + 2\mathbf{v}_{cm} \cdot \mathbf{v}'_i)$$

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10.4 Kinetic Energy of a System of Particles (II)

The total kinetic energy of the system is

$$K = \sum \frac{1}{2} m_i (\mathbf{v}_i \cdot \mathbf{v}_i) = \frac{1}{2} M v_{cm}^2 + \sum \frac{1}{2} m_i v_i'^2 + \mathbf{v}_{cm} \cdot \left(\sum m_i \mathbf{v}'_i \right)$$


The total kinetic energy is $K = K_{cm} + K_{rel}$

$$K_{cm} = \frac{1}{2} M v_{cm}^2$$

The kinetic energy of the CM relative to the fixed origin O

$$K_{rel} = \sum \frac{1}{2} m_i v_i'^2$$

The kinetic energy of the particle relative to CM.

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Historical Note: Mass-Energy Equivalence

Einstein imagined a closed and isolated box with a light bulb at one end and a detector at the other. Once the bulb emits a flash of light toward the detector, the box recoils, because wave carries momentum. When the flash is received by the detector, the box will experience an equal and opposite impulse, and so the whole system will again come to rest. The CM of this isolated system seems not be fixed. Why?

Conservation of linear momentum led Einstein to note the flash of light is not weightless. He then went on to derive the famous equation $E=MC^2$ that relates the mass of a particle to its total energy.

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10.5 Work-Energy Theorem For a System of Particles (I)

1. Review work-energy theorem in one dimension with kinetic energy only (Sec. 7.3).
2. Review work-energy theorem with both kinetic energy and potential energy (Sec. 8.6)

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7.3 Work-Energy Theorem in One Dimension

Work-energy theorem states that the net work done on a particle is equal to the resulting change in its (translational) kinetic energy.

$$W_{net} = \Delta K$$

Whereas **force** and **acceleration** are **vectors**, **work** and **energy** are **scalars**, which make them easier to deal with.

True/false: If the kinetic energy of a body is fixed, the net force on it is zero. Explain your response.

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8.6 Mechanical Energy and Non-conservative Force

The conservation of mechanical energy may be applied to a system *only when there is no work done by any non-conservative force.*

$$\Delta E = E_f - E_i = W_{nc} = \Delta K + \Delta U$$

The above equation is the modified conservation of mechanical energy *when work is done by non-conservative force.*

For example, press or pull a spring and lift a stone by hand.

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10.5 Work-Energy Theorem for a System of Particles (II)

From Sec. 10.4, we know the kinetic energy for a system of Particles can be decomposed as $K=K_{cm}+K_{rel}$.

Newton's third law states that the internal forces cancel in pair, that is, $\Sigma F_{int}=0$. However, this statement does not imply the internal forces will not do work; that is, $\Sigma W_{int}\neq 0$.

Consider, for example, a stationary, isolated system of two equal blocks held against a compressed spring. When the spring is released, the work done by internal force of the spring change kinetic energy relative to the CM, while the CM itself stays fixed.

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10.5 Work-Energy Theorem for a System of Particles (III)

The work-energy theorem for a system must include both external and internal works:

$$W_{ext} + W_{int} = \Delta K_{cm} + \Delta K_{rel}$$

Since all the basic interactions are conservative, one can always express internal work by the change of internal potential energy.

$$W_{int} = -\Delta U_{int}$$

$$\begin{aligned} W_{ext} &= \Delta K_{cm} + (\Delta K_{rel} + \Delta U_{int}) \\ &= \Delta K_{cm} + \Delta E_{int} \end{aligned}$$

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10.5 Work-Energy Theorem for a System of Particles (IV)

$$W_{\text{ext}} = \Delta K_{\text{cm}} + \Delta E_{\text{int}}$$

The above equation says that the external work on a system can change the *translational kinetic energy* of the CM and whatever forms of *internal energy* the system possesses.

This internal energy includes: *elastic potential energy, gravitational potential energy, electromagnetic energy, chemical energy, nuclear energy, thermal energy, and so on.*

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The CM Equation

The external work W_i done by external force F_i on the i -th particle can be expressed as:

$$W_i = \int \mathbf{F}_i \cdot d\mathbf{r}_i = \int \mathbf{F}_i \cdot d\mathbf{r}_{\text{cm}} + \int \mathbf{F}_i \cdot d\mathbf{r}'_i$$

The total external work W_{ext} is therefore the sum of two terms:

$$W_{\text{ext}} = W_{\text{cm}} + W_{\text{rel}}$$

Where, W_{cm} is the external work associated with the displacement of the CM and W_{rel} is the external work associated with displacements relative to the CM.

One can relate W_{cm} to ΔK_{cm} through Newton's second law.

$$W_{\text{cm}} = \Delta K_{\text{cm}}$$

(CM equation)

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10.7 Systems of Variable Mass

How do we deal with the dynamics of a system whose mass is not constant?

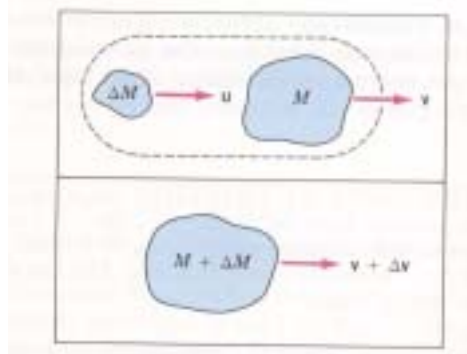
The force can be expressed as the time derivative of momentum

$$\mathbf{F}_{\text{ext}} = \frac{d\mathbf{P}}{dt} = \frac{dM\mathbf{v}}{dt} = M \frac{d\mathbf{v}}{dt} + \mathbf{v} \frac{dM}{dt}$$

However, this equation is **valid only when single particle is considered.**

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Two Masses Stick Together



The change in momentum of the system in time Δt is

$$\begin{aligned} \Delta\mathbf{P} &= (M + \Delta M)(\mathbf{v} + \Delta\mathbf{v}) - M\mathbf{v} - \Delta M\mathbf{u} \\ &\cong M\Delta\mathbf{v} - (\mathbf{u} - \mathbf{v})\Delta M \end{aligned}$$

The external force equals

$$\mathbf{F}_{\text{ext}} = \frac{d\mathbf{P}}{dt} = M \frac{d\mathbf{v}}{dt} - \mathbf{v}_{\text{rel}} \frac{dM}{dt}$$

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Rocket Thrust

In the case of a rocket, external force would be the gravitational force, the force due to air resistance and rocket thrust. The **thrust** is:

$$\text{Thrust} = \mathbf{v}_{\text{rel}} \frac{dM}{dt}$$

Since the exhaust gases are expelled backward, $V_{\text{rel}} < 0$ and the mass of the rocket is decreasing, thus the thrust is in the positive direction.

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Questions

Only a net external force can change the velocity of the CM of a system. So what purpose does the engine of a car serve?

How do we handle a system consisting of multiple particles?

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Example 10.11

The mass of the Saturn V rocket is 2.8×10^6 kg at launch time. Of this, 2×10^6 kg is fuel. The exhaust speed is 2500 m/s and the fuel is ejected at the rate of 1.4×10^4 kg/s. (a) Find the thrust of the rocket. (b) What is its initial acceleration at launch time?

Sol: (a) The magnitude of the thrust is given by

$$\text{Thrust} = \mathbf{v}_{\text{rel}} \frac{dM}{dt} = 3.5 \times 10^7 \text{ N}$$

(b) The acceleration is given by dividing both side by mass.

$$a = \frac{dv}{dt} = -g + \frac{\mathbf{v}_{\text{rel}}}{M} \frac{dM}{dt} = -9.8 + 12.5 = 2.7 \text{ m/s}^2$$

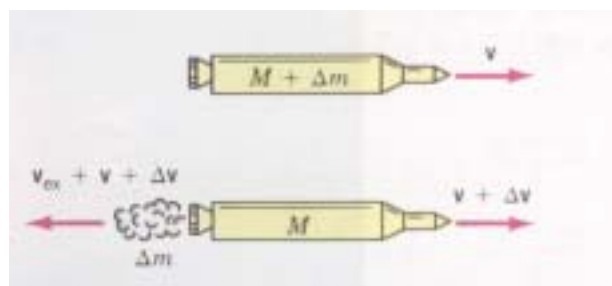
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9.7 Rocket Propulsion (I)

Consider a rocket of mass M with fuel of mass Δm . when the rocket engines are fired, the gases are expelled back with an exhaust velocity of \mathbf{v}_{ex} relative to the rocket. The acceleration can be derived from thrust:

$$a = \frac{dv}{dt} = -\frac{v_{\text{ex}}}{M} \frac{dM}{dt}$$

$$dv = -v_{\text{ex}} \frac{dM}{M}$$



On integrating both sides, we find

$$v_f = v_i - v_{\text{ex}} \ln\left(\frac{M_i}{M_f}\right)$$

9.7 Rocket Propulsion (II)

$$v_f = v_i - v_{ex} \ln\left(\frac{M_i}{M_f}\right)$$

1. The change in the velocity of the rocket is directly proportional to the exhaust velocity.
2. The final velocity v_f is depended on the mass ratio of rocket (M_f) to rocket plus fuel (M_i).
3. For fixed mass ratio, increasing the exhaust velocity v_{ex} will raise the final velocity.

At what velocity, the rocket gains more kinetic energy from a given quantity of fuel?

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Exercises and Problems

Ch.10: Ex.5, 15, 17, 27, 34

Prob. 2, 3, 8, 9, 12

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