Chapter 11 Rotation of a Rigid Body about a Fixed Axis

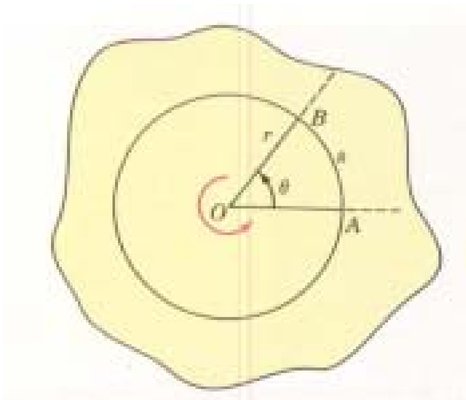
We now broaden our interest to include the rotation of a **rigid body** about a *fixed axis* of rotation.

A **rigid body** is defined as an object that has fixed size and shape. In other words, the relative positions of its constituent particles remain constant. Although a perfectly rigid body does not exist, it is a useful idealization.

By "*fixed axis*" we mean that the axis must be fixed relative to the body and fixed in direction relative to an inertia frame.

The discussion of general rotation, in which both the position and the direction of the axis change, is quite complex.

11.1 Rotational Kinematics (I)



Form the definition of a radian (arc length/radius) we know.

$$\theta = s / r$$

Disk 5: radian disk

11.1 Rotational Kinematics (II)

The average angular velocity $\omega_{av} = \frac{\Delta\theta}{\Delta t} = \frac{\theta_f - \theta_i}{t_f - t_i}$

The instantaneous angular velocity $\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$

The *period T* is the time for one revolution and the *frequency f* is the number of revolutions per second (rev/s).

$$\omega = \frac{2\pi}{T} = 2\pi f$$

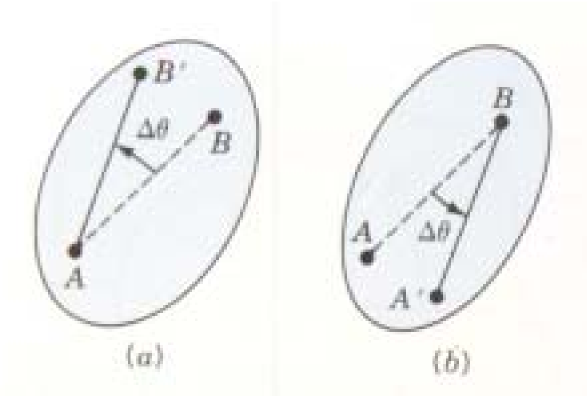
11.1 Rotational Kinematics (III)

The average angular acceleration $\alpha_{av} = \frac{\Delta\omega}{\Delta t}$

The instantaneous angular acceleration $\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$

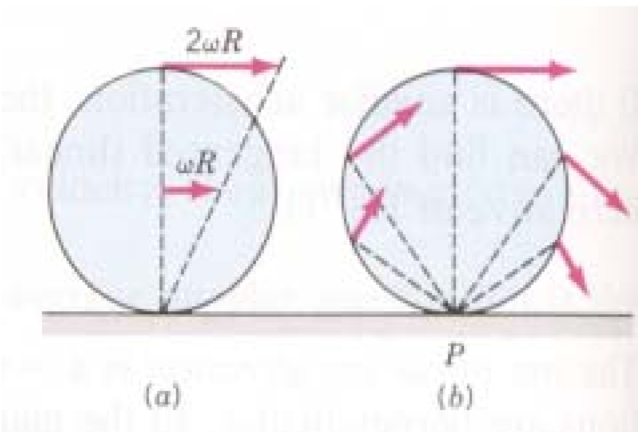
translational kinematics	rotational kinematics
$v = v_o + at$	$\omega = \omega_o + \alpha t$
$x = x_o + v_o t + \frac{1}{2} a t^2$	$\theta = \theta_o + \omega_o t + \frac{1}{2} \alpha t^2$
$v^2 = v_o^2 + 2a(x - x_o)$	$\omega^2 = \omega_o^2 + 2\alpha(\theta - \theta_o)$

11.1 Rotational Kinematics (IV)



For future reference let us establish the following: *The angular velocity of a rotating body is the same relative to any point on it.*

Rolling



The increase in speed with distance from the point of contact is easily seen in the spokes of a bicycle wheel: *The ends of the spokes need the road are fairly distinct, whereas those at the top are blurred.*

Example 11.1

A flywheel of radius 20 cm starts from rest, and has a constant angular acceleration of 60 rad/s^2. Find: (a) the magnitude of net linear acceleration of a point on the rim after 0.15 s; (b) the number of revolutions completed in 0.25 s.

Solution:

(a) Find out tangential and radial (centripetal) acceleration.

$a_t=\alpha r=12\ m/s^2$, $a_r=\omega^2 r=16.2\ m/s^2$

(b) $\theta=\alpha t^2/2=1.88$ rad=0.3 rev.

11.2 Rotational Kinetic Energy and Moment of Inertia

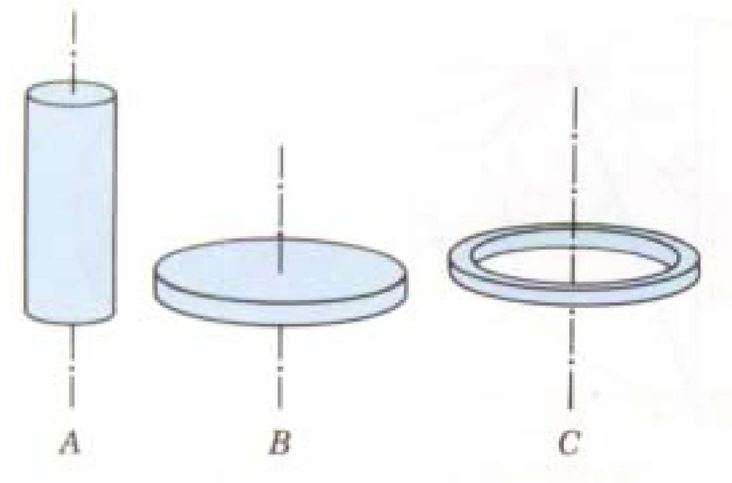
The kinetic energy of i-th particle is: $K_i = \frac{1}{2} m_i v_i^2 = \frac{1}{2} m_i r_i^2 \omega^2$

Total kinetic energy is: $K = \sum K_i = \frac{1}{2} (\sum m_i r_i^2)\, \omega^2$

Moment of Inertia: $I = \sum m_i r_i^2$

The moment of inertia of a body is a measure of its rotational inertia, that is, its resistance to change in its angular velocity.

Moment of Inertia

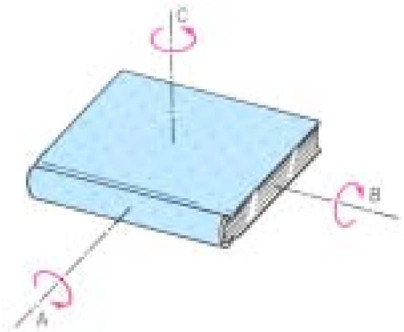


A cylinder, a disk, and a ring with same mass. The moments of inertia about the central axis depend on how the mass is distributed relative to the axis: $I_C > I_B > I_A$.

Questions (I)

1. True or false: A quick way of computing the moment of inertia of a body is to consider its mass as being concentrated at the center of mass.

4. The book in Fig. 11.39 has the same shape as this text. About which axis is the moment of inertia (a) the largest; (b) the smallest?

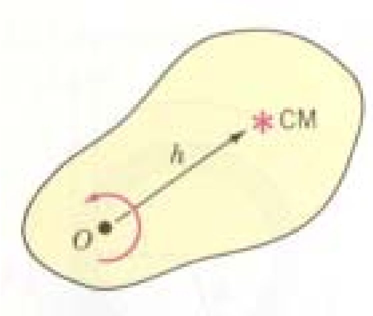


Kinetic Energy and Parallel Axis Theorem

The total kinetic energy of the system is $K=K_{cm}+K_{rel}$ or

$$K = \tfrac{1}{2}Mv_{\rm cm}^2 + \tfrac{1}{2}I_{\rm cm}\omega^2$$

If the axis of rotation is located at a perpendicular distance h from the CM, as shown in figure below.



The total kinetic energy is:

$$K = \tfrac{1}{2}Mv_{\rm cm}^2 + \tfrac{1}{2}I_{\rm cm}\omega^2 = \tfrac{1}{2}Mh^2\omega^2 + \tfrac{1}{2}I_{\rm cm}\omega^2$$

$$I = I_{\rm cm} + Mh^2$$

This relationship is called the ***parallel axis theorem***.

11.3 Moment of Inertia of Continuous Bodies

The moment of the inertia of the whole body takes the form

$$I = \int r^2 dm$$

Keep in mind that here the quantity *r* is the perpendicular distance to an axis, not the distance to an origin.

Please see the following two examples.

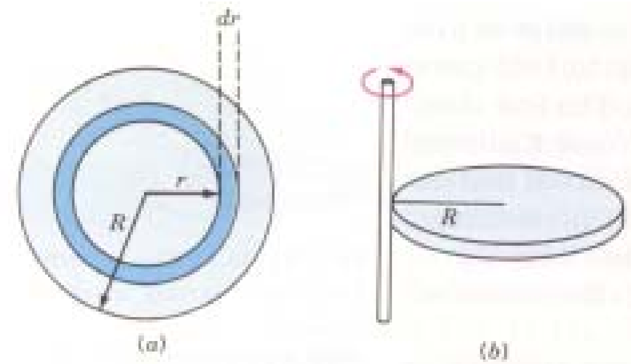
Example 11.6

Find the moment of inertia of a circular disk or solid cylinder of radius R about the following axes: (a) through the center and perpendicular to the flat surface; (b) at the rim and perpendicular to the flat surface.

Solution:

(a) $dI=r^2dm=2\pi\sigma r^3dr$, For the whole body, $I=1/2MR^2$

(b) parallel axis theorem $I_{rim}=I_{cm}+MR^2=3/2MR^2$



Example 11.7

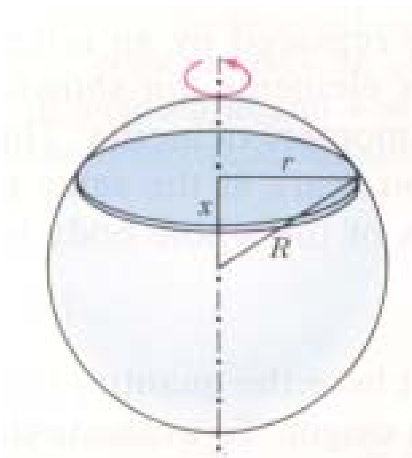
Find the moment of inertia of a uniform solid sphere of mass M and Radius R about the diameter.

Solution:

$$dI=\tfrac{1}{2}r^2dm=\tfrac{1}{2}\underbrace{(R^2-x^2)}_{r^2}\underbrace{\rho(R^2-x^2)\pi\,dx}_{dm}$$

$$I=\frac{1}{2}\int_{-R}^{R}(R^2-x^2)\rho(R^2-x^2)\pi\,dx$$

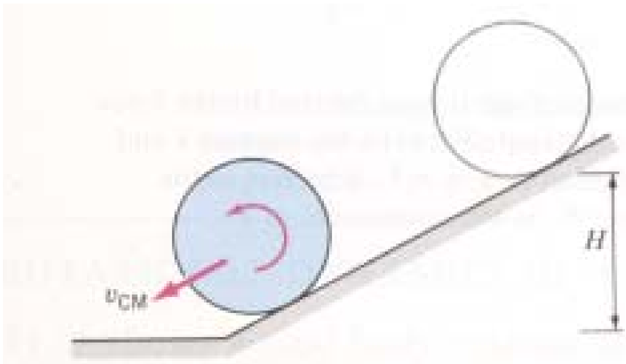
$$=\frac{8}{15}\rho\pi R^5=\frac{2}{5}MR^2$$



Example 11.8

A solid sphere and a disk are released from the same point on an incline, as shown in Fig.11.19. Given that they roll without slipping, which has greater speed at the bottom? Ignore dissipative effects.

Solution:



$$E_i = MgH$$

$$E_f = \tfrac{1}{2}Mv_{\mathrm{cm}}^2 + \tfrac{1}{2}I_{\mathrm{cm}}\omega^2$$

$$v_{\mathrm{cm}} = \omega R$$

$$\omega = \frac{2MgH}{MR^2 + I_{\mathrm{cm}}}$$

The higher I_{cm}, the lower angular velocity.

Disk 6: Rolling Bodies on Incline

Questions (II)

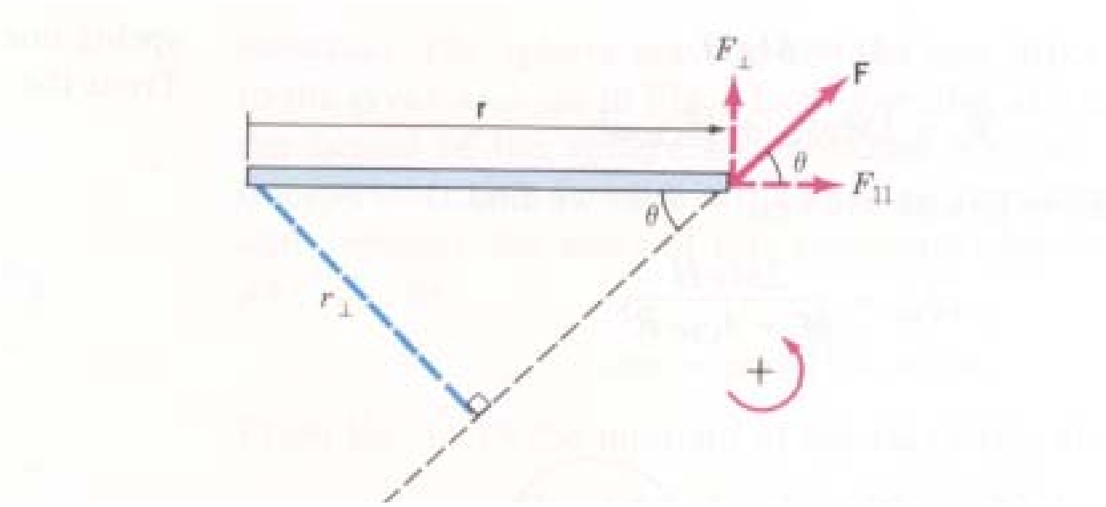
5. Two identical cans of concentrated orange juice are released at the top of an incline. One is frozen and the other has defrosted. Which reaches the bottom first?

6. The spokes of wagon wheels sometimes appear to rotate backward on film or on TV. Why is this?

11.5 Torque (I)

The torque is the rotational analog of force: force causes linear acceleration; torque causes angular acceleration.

The "turning ability" of a force about an axis or pivot is called its torque.



11.5 Torque (II)

Viewpoint 1: Only the perpendicular component of the force $F_\perp$ contribute to the turning effect.

Viewpoint 2: The lever arm $r_\perp$ is the perpendicular distance from the origin (pivot or axis) to the line of action of the force *F*.

$$\tau = rF_\perp = r_\perp F$$
$$= rF \sin\theta$$

Although torque has the same dimension as energy, these two concepts are unrelated.

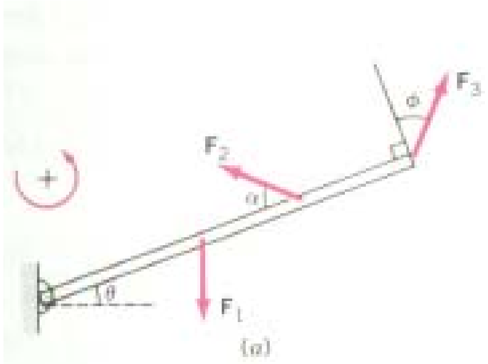
Energy is a scalar, whereas torque is a vector.

Example 11.10

Three forces F1, F2 and F3 act on a rod at distances r1, r2, and r3 from the pivoted end, as shown in fig. 11.24a. Find the torque due to each force about the pivot.

Solution:

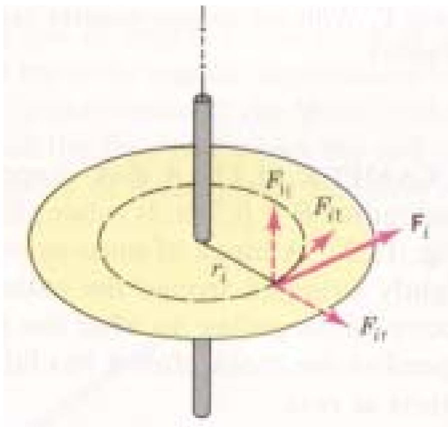
$$\tau_1 = -r_1 F_1 \sin(90^\circ + \theta) = -r_1 F_1 \cos(\theta)$$
$$\tau_2 = +r_2 F_2 \sin(180^\circ - \alpha) = +r_2 F_2 \sin(\alpha)$$
$$\tau_3 = +r_3 F_3 \sin(90^\circ - \Phi) = r_3 F_3 \cos(\Phi)$$



We adapt the convention that a torque tending to produce a counterclockwise rotation is positive (right-hand rule).

11.6 Rotational Dynamics of a Rigid Body (Fixed Axis) (I)

Any component parallel to the axis ($F_{i//}$) is counteracted by the reaction of the supports. For the same reason any radial component (F_{ir}) is also balanced. Only the component tangential to the circular path (F_{it}) will accelerate the particle.



The linear acceleration of the particle is related to the angular acceleration through $a_t = \alpha r$. Thus, the second law becomes

$$F_{it} = m_i a_{it} = m_i r_i \alpha$$

11.6 Rotational Dynamics of a Rigid Body (Fixed Axis) (II)

The torque on the particle about the axis is

$$\tau_i = r_i F_{it} = m_i r_i^2 \alpha$$

The torques on all the particles is

$$\tau = \sum r_i F_{it} = \underbrace{\sum m_i r_i^2}_{I} \alpha$$

These two equations are not vector equations.

where *I* is the moment of inertia about the given axis. The above equation is valid in two situations:

1. The axis is fixed in position and direction.
2. The axis passes through the CM and is fixed in direction only.

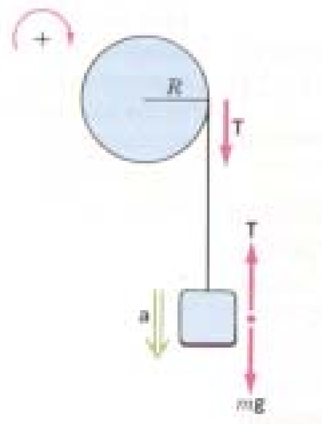
Questions (III)

9. Snow tires have a slightly greater diameter than summer tires. Is the reading of the speedometer affected?

11. About what axis is the moment of inertia of a person (a) the greatest? (b) the least? Are your answers subject to conditions?

Example 11.12

A disk-shaped pulley has mass M=4 kg and radius R=0.5 m. It rotates freely on a horizontal axis, as in Fig. 11.27. A block of mass m=2 kg hangs by a string that is tightly wrapped around the pulley. (a) What is the angular velocity of the pulley 3 s after the block is released? (b) Find the speed of the block after it has fallen 1.6 m. Assume the system starts at rest.



Solution:

$$\text{Block}\ (F = ma) \qquad mg\text{-}T = ma = mR\alpha$$

$$\text{Pulley}\ (\tau = I\alpha) \qquad TR = (\tfrac{1}{2}MR^2)\alpha$$

From these two equations, we can find two unknowns (α and T).

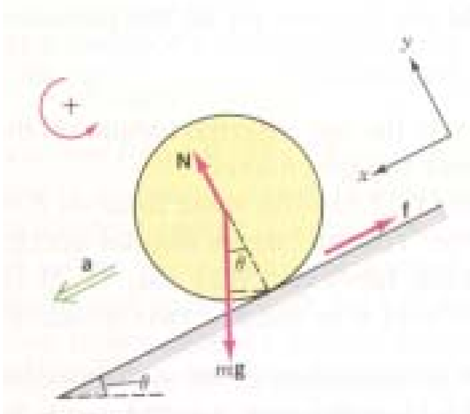
Example 11.13

Figure 11.28 shows a sphere of mass M and radius R that rolls without slipping down an incline. Its moment of inertia about a central axis is $2/5MR^2$. (a) Find the linear acceleration of the CM. (b) Which is the minimum coefficient of friction required for the sphere to roll without slipping.

Solution:

$$\text{Force} \qquad Mg\sin\theta - f = M(R\alpha)$$

$$\text{Torque} \qquad fR = I\alpha$$



From these two equations, we can find two unknowns (α and *f*).

$$\alpha = \frac{5}{7R} g\sin\theta \quad \text{and} \quad f = \frac{2}{7} Mg\sin\theta$$

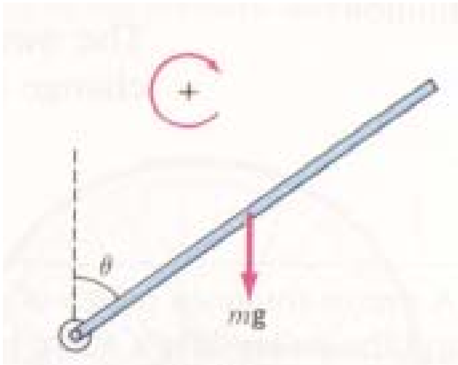
Example 11.15

A uniform rod of length L and mss M is pivoted freely at one end. (a) what is the angular acceleration of the rod when it is at angle θ to the vertical? (b) What is the tangential linear acceleration of the free end when the rod is horizontal? The moment of inertia of a rod about one end is $1/3ML^2$.

Solution:

Torque $mg\frac{L}{2}\sin\theta = \frac{ML^2}{3}\alpha$

$\alpha = \frac{3g\sin\theta}{2L}, \quad a = \frac{3g\sin\theta}{2}$



The acceleration of the free end might be greater than that of a free-fall!

11.7 Work and Power (I)

The work down by tangential component is

$$dW = (F_t)(rd\theta) = \tau d\theta$$

Thus, the power can be expressed as

$$P = \frac{dW}{dt} = \tau\omega$$

which is analogous to $P = fv$

11.7 Work and Power (II)

To derive the work-energy theorem for rotational motion, we first express torque in a convenient form. Using the chain rule we have

$$\tau = I\frac{d\omega}{dt} = I\frac{d\omega}{d\theta}\frac{d\theta}{dt} = I\omega\frac{d\omega}{d\theta}$$

We next use this result in $dW = \tau d\theta$ and integrate to find

$$W = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2$$

The work done by a torque on a rigid body rotating about a fixed axis leads to a change in its rotational kinetic energy.

Questions (IV)

14. A spool of thread is on a rough surface (see Fig. 11.42). The axle has radius r while the rim has radius R. Discuss the motion of the spool for the various directions in which the thread is pulled, (a) For what angle is there sliding but no rolling? (b) What is the condition for the spool to wind up ?

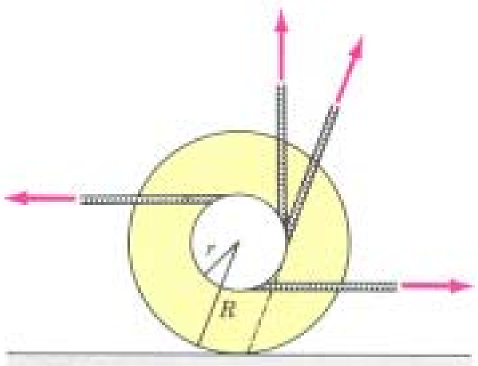


FIGURE 11.42 Question 14.

Disk 6: Spool with Wrapped Ribbon

Exercises and Problems

Ch.11:

Ex.16, 32, 34, 36, 39, 53, 56, 58, 61

Prob. 3