Chapter 12 Angular Momentum and Statics

In the last chapter we specify *just the sense*, clockwise and counter clockwise, of a torque.

In this chapter **torque** is defined in a more general way that properly expresses its **vector** nature.

Just as **force** is related to the **linear momentum**, we will see that **torque** is related to a quantity called **angular momentum**.

The importance of **angular momentum** lies in the fact that it is a **conserved quantity**.

11.5 Torque (II)

Viewpoint 1: Only the perpendicular component of the force $F_\perp$ contribute to the turning effect.

Viewpoint 2: The lever arm $r_\perp$ is the perpendicular distance from the origin (pivot or axis) to the line of action of the force *F*.

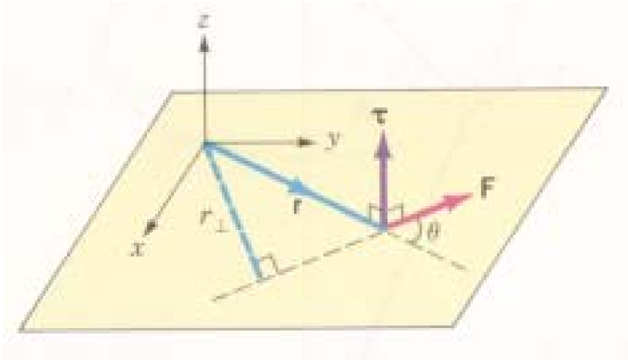
$$\tau = rF_\perp = r_\perp F$$
$$= rF\sin\theta$$

Although torque has the same dimension as energy, these two concepts are unrelated.

Energy is a scalar, whereas torque is a vector.

12.1 The Torque Vector

Torque is in fact a vector quantity, and so its direction must be specified relative to a coordinate system.



The definition of torque as a vector quantity is

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = rF \sin\theta \, \hat{\mathbf{n}}$$

Where **n** is a unit vector normal to the plane of **r** and **F**. Its direction is given by the right-hand rule.

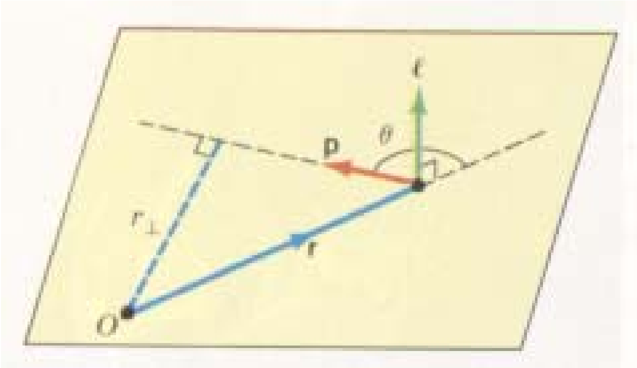
Where is the Appropriate Origin?

Choosing the origin will affect the position vector **r** and subsequently torque τ.

Since the position vector **r** is measured relative to an origin *O*, the torque is also measured relative to this point.

In many instances the origin is chosen to lie on the axis.

12.2 Angular Momentum



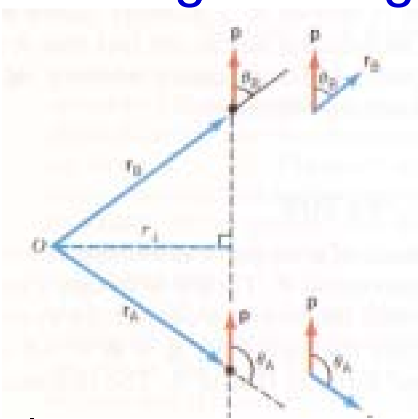
Single particle angular momentum: $\boldsymbol{\ell} = \mathbf{r} \times \mathbf{P}$

Angular momentum is measured with respect to a point, the origin of the position vector **r**.

"Moment arm" $r_\perp$ expression: $\ell = rp\sin\theta = r_\perp p$

The SI unit of the angular momentum is kg·m^2/s.

Motion along a straight line

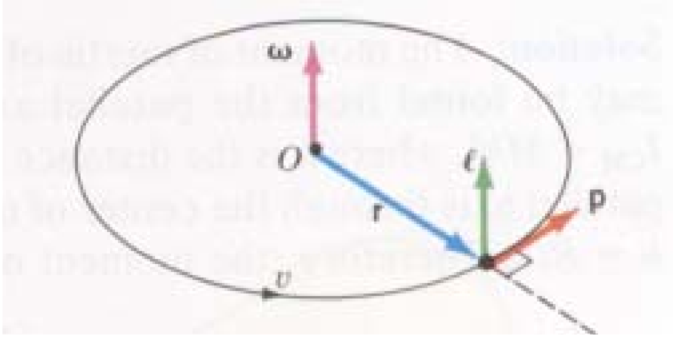


Magnitude of angular momentum at position A $\ell_A = r_A p \sin\theta_A = (r_A \sin\theta_A) p$

Magnitude of angular momentum at position B $\ell_B = r_B p \sin\theta_B = (r_B \sin\theta_B) p$

Since $r_\perp$ ($r_A \sin\theta_A = r_B \sin\theta_B$) and *p* are constant, the angular momentum is also constant.

Motion along a circle



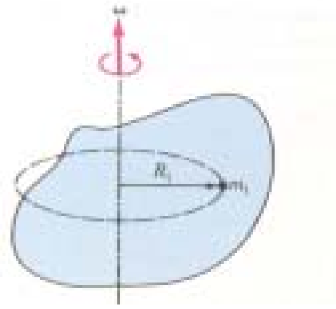
If we choose the origin to be at the center, the magnitude of angular momentum is

$$\ell = R(mv) = mR^2\omega$$

The directions of the angular momentum *L* and angular frequency ω are generally not parallel. However, the z-component of *L* does lie along ω.

$$\ell_z = mR^2\omega$$

System of particles



The total angular momentum **L** of a system of particles relative to a given origin is the sum of the angular momenta of particles.

$$\mathbf{L} = \sum \ell_i = \sum \mathbf{r}_i \times \mathbf{P}_i$$

Simple form for a rigid body rotating about a fixed axis.

$$L_z = \sum \ell_z = \sum m_i R_i^2 \omega = I\omega$$

Angular momentum is the rotational analog of linear momentum.

Example 12.2

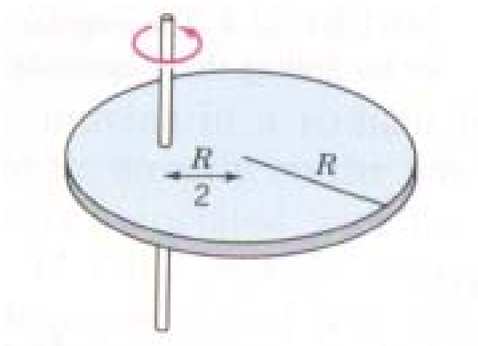
A disk of mass M and radius R is rotating at angular velocity ω about an axis perpendicular to its plane at a distance R/2 from the center, as shown in Fig.12.8. What is its angular momentum? The moment of inertia of a disk about the central axis is $mR^2/2$.

Solution:

First find out the moment of inertia by applying parallel axis theorem.

$I=MR^2/2+M(R/2)^2=3/4MR^2$.

The angular momentum is $L=I\omega=3/4MR^2\,\omega$.



12.3 Rotational Dynamics

Torque and angular momentum are rotational analogs of force and momentum. Since force is the time derivative of the momentum, torque and angular momentum can also be derived by the same technique.

$$\begin{aligned}\frac{d\ell}{dt} &= \mathbf{r}\times\frac{d\mathbf{P}}{dt}+\frac{d\mathbf{r}}{dt}\times\mathbf{P}\\ &= \mathbf{r}\times\mathbf{F}+\cancel{\mathbf{v}\times m\mathbf{v}}\\ &= \boldsymbol{\tau}\end{aligned}$$

For a single particle, the torque acting on a particle is equal to the time rate of change of its angular momentum.

$$\boldsymbol{\tau}=\frac{d\ell}{dt}$$

Rotational Dynamics for System of Particles

In the translational motion only the external force have to be considered; the internal forces between the particles cancel in pairs. In the rotational motion, a similar cancellation occurs with internal torques.

$$\boldsymbol{\tau}_{\text{ext}} = \frac{d\mathbf{L}}{dt}$$

This equation is valid only when both the torque and angular momentum are measured (i) with respect to the same origin in an inertial frame, or (ii) relative to the center of mass of the system---even if this point is acceleration.

Rigid body rotating about a fixed axis: $\boldsymbol{\tau} = I\boldsymbol{\alpha}$

Example 12.4: Solution 1

Two blocks with masses *m*1 and *m*2 are connected by a rope that passes over a pulley of radius *R* and mass *M*; see figure below. Find out the linear acceleration of the blocks. There is no friction.

Solution 1:

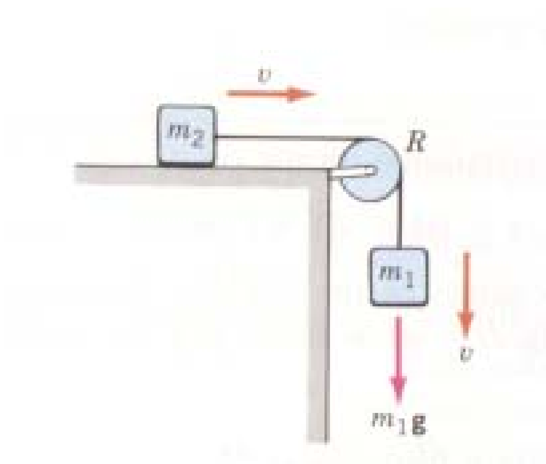
Total angular momentum:

$L=(m_1+m_2)R^2\omega+(M/2)R^2\omega$

Torque: $\tau=M_1 gR=\mathrm{d}L/\mathrm{d}t$

$(m_1+m_2+M/2)R(R\alpha)=m_1 gR$

$R\alpha=\underline{a=m_1 g/(m_1+m_2+M/2)}$ #



Example 12.4: Solution 2

Two blocks with masses *m*1 and *m*2 are connected by a rope that passes over a pulley of radius *R* and mass *M*; see figure below. Find out the linear acceleration of the blocks. There is no friction.

Solution 2:

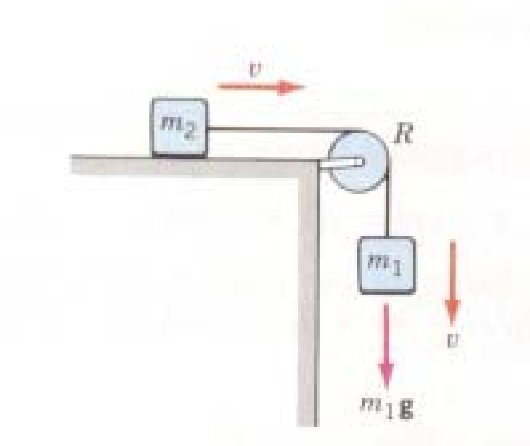
Mechanical energy conservation:

$m_1gh=(m_1+m_2)(at)^2/2+(M/4)(at)^2$

Distance: $h=(at)^2/2a$

$(m_1+m_2+M/2)a=m_1g$

$\underline{a=m_1g/(m_1+m_2+M/2)}$ #



12.4 Conservation of Angular Momentum

The principle of the **conservation of angular momentum**: If the net external torque on a system is zero, the total angular momentum is constant in magnitude and direction.

$$\boxed{\text{If } \boldsymbol{\tau}_{\text{EXT}} = 0, \quad \text{then } \mathbf{L} = \text{constant}}$$

Applying it to the special case of rigid bodies rotating about fixed axes, in which case the angular momentum is L=Iω. The initial and final angular momentum are equal.

$$\text{Rigid body} \quad I_f\omega_f = I_i\omega_i$$

Example 12.5:

A disk (lower) of moment of inertia 4 kg·m^2 is spinning freely at 3 rad/s. A second disk (upper) of moment of inertia 2 kg·m^2 slides down a spindle and they rotate together. (a) What is the angular velocity of the combination? (b) What is the change in kinetic energy of the system?

Solution:

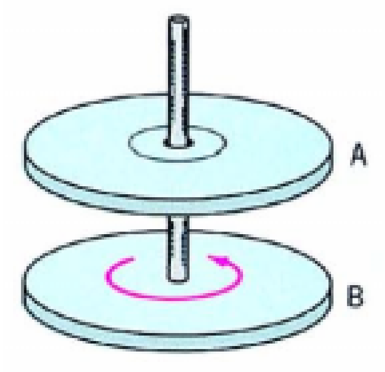
(a) Angular momentum conserved

$(6 \text{ kg}\cdot\text{m}^2)\ \omega_f = (4 \text{ kg}\cdot\text{m}^2)(3 \text{ rad/s})$

(b)

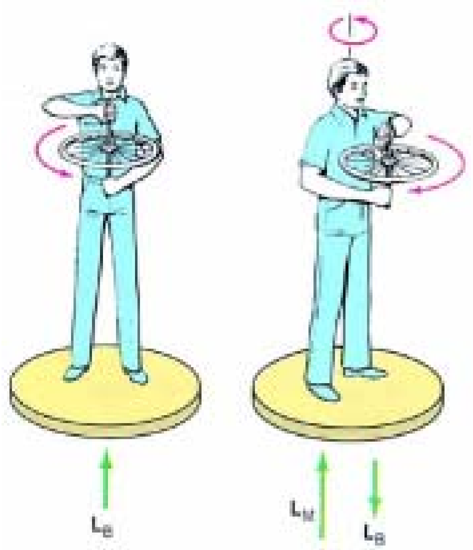
$$\Delta K = K_f - K_i = \tfrac{1}{2} I_f \omega_f^2 - \tfrac{1}{2} I_i \omega_i^2$$

$$= \tfrac{1}{2} 6 \cdot 2^2 - \tfrac{1}{2} 4 \cdot 3^2 = -6 \text{ J}$$



Example 12.7:

A man stands on a stationary platform with a spinning bicycle wheel in his hands, as in Fig. 12.14. The moment of the inertia of the man plus platform is I_M=4 kg·m^2, and for the bicycle wheel it is I_B=1 kg·m^2. The angular velocity of the wheel is 10 rad/s counterclockwise as viewed from above. Explain what occurs when the man turn the wheel upside down. This system is isolated in the sense that there are no external torques acting



Solution: See DVD example (D7 rotating stool)

Example 12.8**:

According to Kepler's second law of planetary motion, the line joining the sun to a planet sweeps out equal area in equal time intervals. Show that this is a consequence of the conservation of angular momentum.

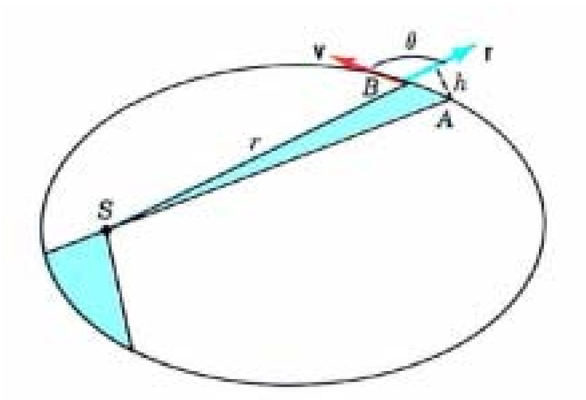
Solution:

$$\frac{\Delta A}{\Delta t} = \frac{1}{2} rh = \frac{1}{2} rv \sin\theta$$

Angular momentum

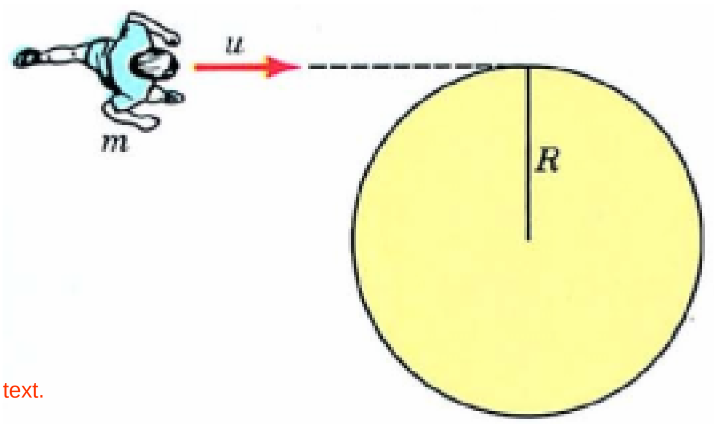
$$\ell = \mathrm{rpsin}\theta = \mathrm{m(rvsin}\theta)$$

$$\frac{\Delta A}{\Delta t} = \frac{\ell}{2m} = const.$$



Example 12.9:

A man of mass *m*=80 kg runs at a speed *u*=4 m/s along the tangent to a disk-shaped platform of mass *M*=160 kg and radius 2 m. The platform is initially at rest but can rotate freely about and axis through its center. (a) Find the angular velocity of the platform after the man jumps on. (b) He then walks to the center, find the new angular momentum. Treat the man as a point particle.



* Find out a typo in the text.

12.5 Conditions for Static Equilibrium

The subject of **statics** is concerned with the forces and torques that act on bodies at rest.

When a=0, the body is in **translational equilibrium**.

$$\sum \mathbf{F} = 0$$

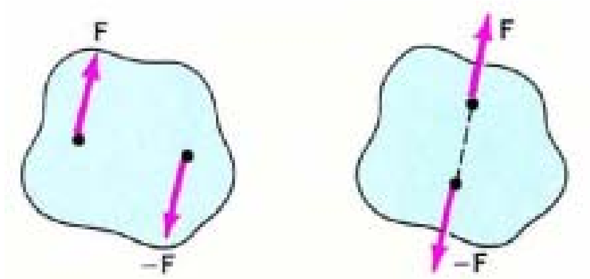
When α=0, the body is in **rotational equilibrium**.

$$\sum \tau = 0$$

The special case in which the body is *at rest* is referred to as **static equilibrium**.

What is **dynamic equilibrium**?

12.5 Conditions for Static Equilibrium: examples

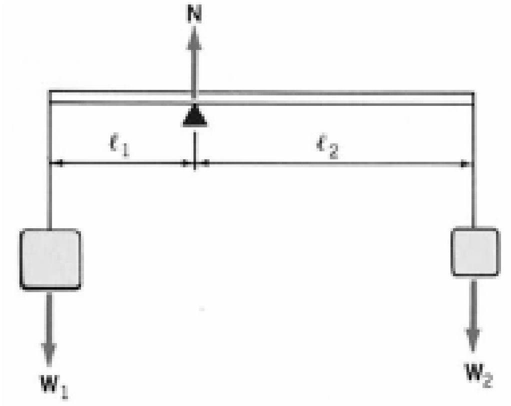


non-equilibrium equilibrium

12.6 Center of Gravity (I)

The **center of gravity** (CG) of a body is the point about which the net gravitational torque is zero.

How to determine the center of gravity? See example below,



Find the pivot position where the net gravitational torque is zero.

$$W_1\ell_1 = W_2\ell_2$$

12.6 Center of Gravity (II)

Just as the center of mass is the point at which the mass of a system appears to be concentrated, the total weight of a system may be taken to act at the center of gravity.

$$\sum \tau_i = \sum w_i x_i$$

The torque due to the total weight acting at the CG is

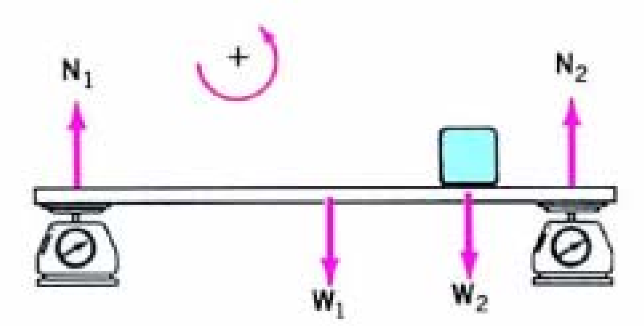
$$\sum \tau_i = (\sum w_i) x_{CG}, \quad x_{CG} = \frac{\sum w_i x_i}{\sum w_i}$$

What is the difference between **center of mass** and **center of gravity**?

$$x_{CG} = \frac{\sum m_i g_i x_i}{\sum m_i g_i} = \frac{\sum m_i x_i}{\sum m_i} = x_{CM}, \quad \text{if } g_i \text{ is constant.}$$

Example 12.10:

A uniform rod of weight W_1=35 N is supported at its ends as shown in Fig. 12.20. A block of weight W_2=10 N is placed one-quarter of the distance from one end. What are the forces exerted by the supports?



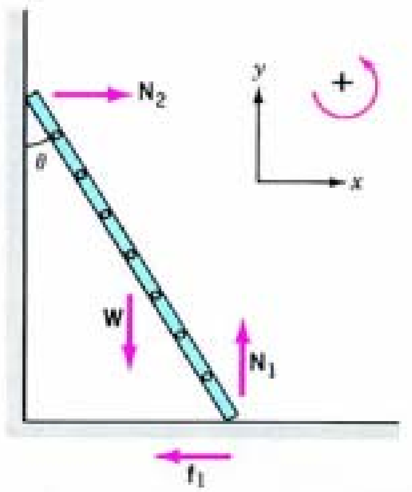
Solution: Since this system is in static equilibrium, the torque is zero to any pivot.

N1=W1/2+W2/4=20 N

N2=W1/2+W2*3/4=25 N

Example 12.12:

A ladder of length *L* and weight *W* rests on a rough floor and against a frictionless wall, as shown in Fig. 12.22. The coefficient of static friction at the floor is μ_s=0.6. (a) Find the maximum angle to the wall such that the ladder does not slip, (b) the force exerted by the wall at this θ .



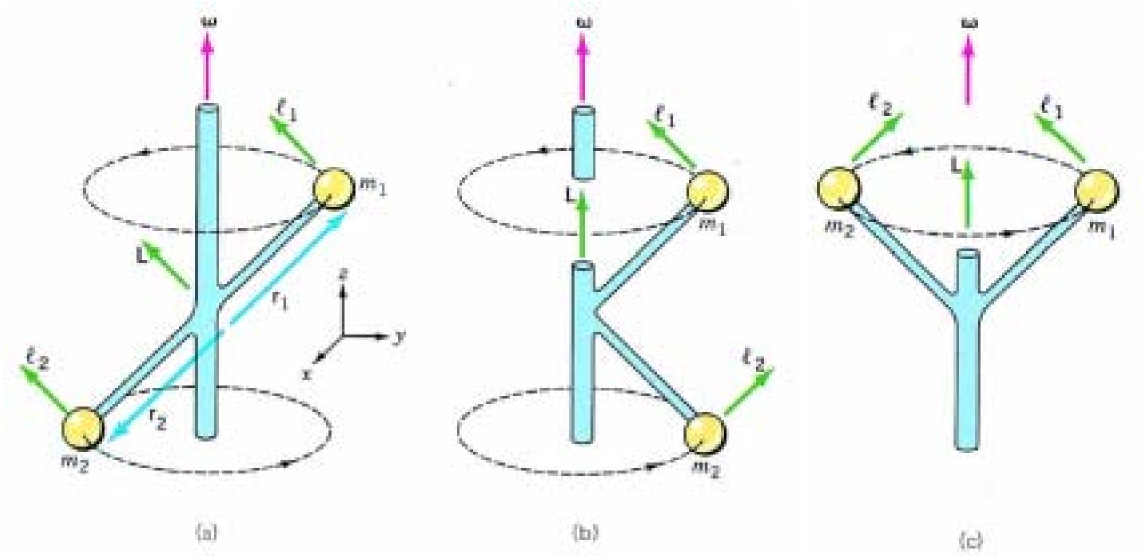
Solution:

Force N1=W, N2=f1=0.6·N1=0.6W

Torque N2·sin(90+ θ)=W/2·sin(180- θ)

tan(θ)=1.2, θ =50.2 degree.

12.7 Dynamic Balance (optional)



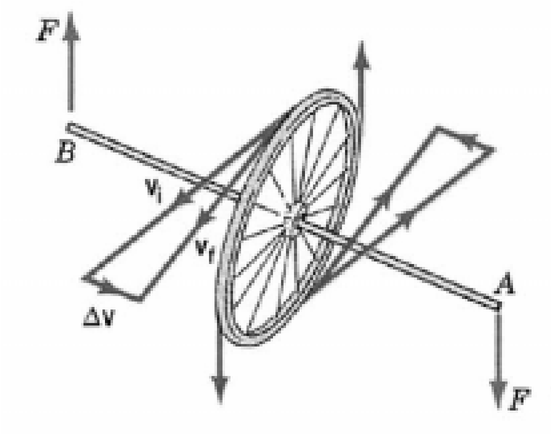
(a) dynamic imbalance

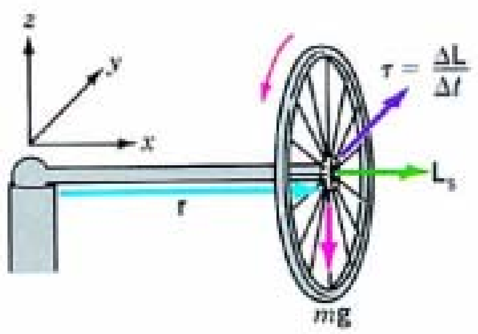
(b) static imbalance

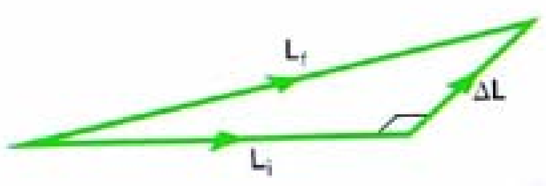
(c) both static and dynamic balance

12.9 Gyroscopic Motion (optional)

See DVD example, Disk 7







Questions

2. Why does spreading out both arms help to balance on a tightrope? Why is holding a long pole even better.

6. If you were given two spheres of the same mass and radius that appear identical, could you determine where either is a solid or a shell?

Exercises and Problems

Ch.12:

Ex.11, 12, 25

Prob. 1, 2, 3, 5, 7,10, 15