Chapter 13 Gravitation

Newton, who extended the concept of inertia to all bodies, realized that *the moon is accelerating* and is therefore *subject to a centripetal force*.

He guessed that the force that keeps the moon in its orbit has the same origin as the force that causes the apple to fall. He recalled some thirty years later:

*"I deduced that the forces that keep the planets in their orbs must be **reciprocally** as the **squares of their distances** from the centers about which they revolve, and thereby compared the force required to keep the Moon in her orb with the force of the gravity at the surface of the Earth, and found them to answer pretty nearly."*

Newton's Discoveries

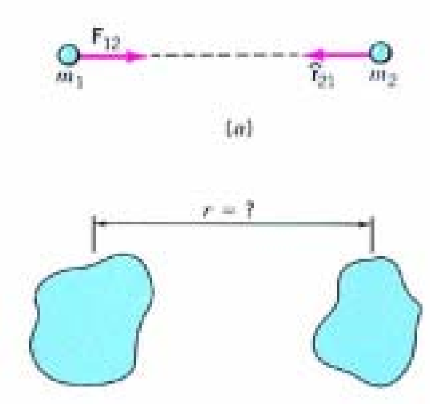
Newton used the period of the moon's orbit (27.3 days) to calculate its centripetal acceleration: a_m=1/360 m/s^2. How?

Next, he assumed that the gravitational force between two bodies varies as the inverse square of the distance between them, that is, $F \propto 1/r^2$ --- an idea that had been around since 1640.

Thus, the ration of the force on the moon to that on the apple at the surface of the earth should be R^2_E / r^2_M. He knew that $r_M \approx 60 R_E$. How do we determine R_E?

13.1 Newton's Law of Gravitation

Newton's law of universal gravitation: $\mathbf{F}_{12} = -\dfrac{Gm_1 m_2}{r^2}\hat{\mathbf{r}}_{12}$



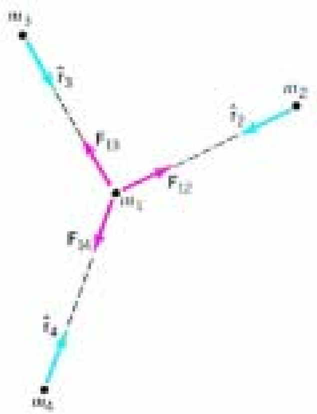
It readily apply to point masses, but how to apply this equation to bodies with arbitrary mass distributions.

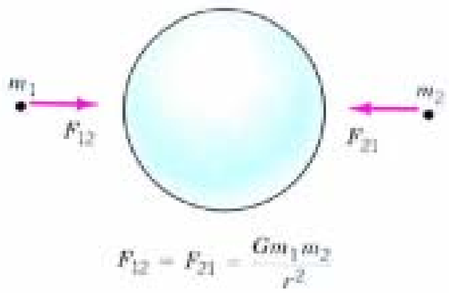
Principle of Superposition

Experiment shows that when several particles interact, the force between a given pair is *independent* of the other particles present.

Principle of superposition: $\mathbf{F}_1 = \mathbf{F}_{12} + \mathbf{F}_{13} + \cdots + \mathbf{F}_{1N}$

The net force on m1 is the vector sum of the pairwise interactions.





13.2 Gravitational and Inertia Mass

Mass has appeared in two entirely different contexts: Netwon's second law of motion and Newton's law of gravitation.

$$\mathbf{F} = m_I \mathbf{a} \qquad\qquad \mathbf{F} = m_G \frac{GM_G}{r^2}$$

What is the difference between inertia mass m_I and gravitational mass m_G?

On the free-fall case, force is expressed as $F=m_G g$.

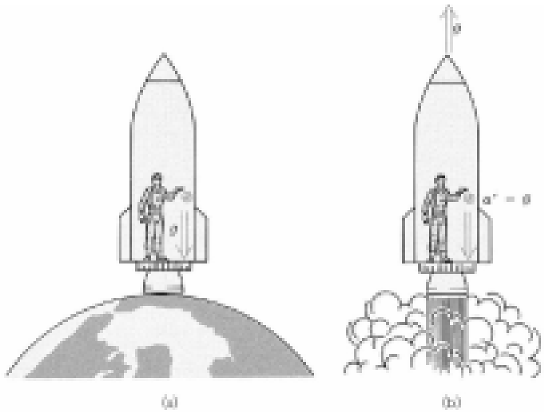
Substituting to force equation, we get $a=(m_G/m_I)g$.

Is the mass ratio varying with different materials?

What is the role of the gravitational constant G?

Principle of Equivalence

No experiment can distinguish the effects of a gravitational force from that of an inertia force in an accelerated frame.



The latest experiments show that m_I and m_G are equivalent to within 1 part in 10^{12}.

Example: the period of a simple pendulum,

$$T = 2\pi\sqrt{m_I L / m_G g}$$

13.3 The Gravitational Field Strength

How does two particles interact directly with each other through free space?

What is the "*action at a distance*"?

The same problem of interaction without actual contact occurs with electric charges and magnets.

In 1830s Faraday developed the concept of a **"field"**, resolving the problem of action at a distance.

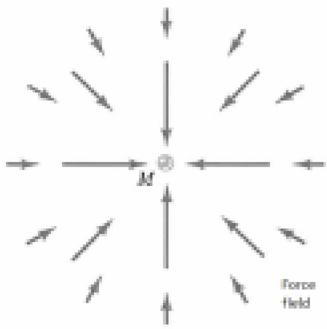
The distribution of values over a region of space is call a field. Pressure and temperature form scalar fields, whereas velocity and force give rise to vector fields.

The Gravitational Field Strength

Suppose a stationary particle of mass M . What is the force exerting on a test particle of mass m where m can be placed at different position?

$$\mathbf{F} = (-\frac{GM}{r^2}\hat{\mathbf{r}})m = \mathbf{g}m$$

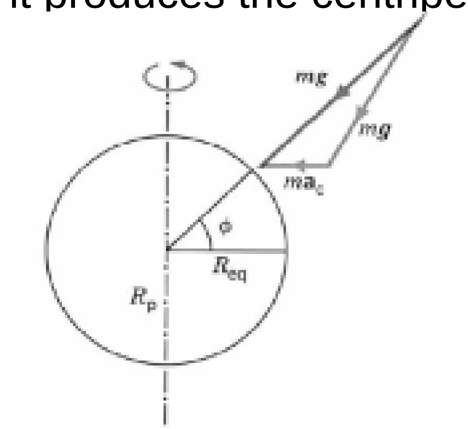
It is convenient to consider the force per unit mass **F**/m. The quantity **g**, measured in N/kg, is called the **gravitational field strength** at position *r* with respect to *M*.



Acceleration due to gravity

Although the unit N/kg reduce to m/s^2, the gravitational field strength **g** is a concept different from the acceleration due to gravity $\mathbf{g_a}$.

The gravitational force is directed to the center and serves two functions: It causes the body to fall with acceleration **g** and it produces the centripetal acceleration $\mathbf{a_c}$.



$$m\mathbf{g} = m(\mathbf{g_a} + \mathbf{a_c})$$

Gravitational Force Variation: Rotation and Non-uniform Mass Distribution

At the equator $\mathbf{a}$c=v^2/r$_E$=3.4 cm/s^2. Thus, the gravitational field strength at poles and equator is different by 3.4 cm/s^2. Measurement, however, show that the difference is 5.2 cm/s^2. Why disagree? *Non-uniform mass distribution*.

If the Earth rotates very fast, all the gravitational field strength **g** is contributed to the centripetal force **a**c and gravitational acceleration $\mathbf{g}_a$ is zero (weightless condition). How many seconds a day in such condition?

$$\omega^2 r_E = 9.8\ \mathrm{m/s^2}$$

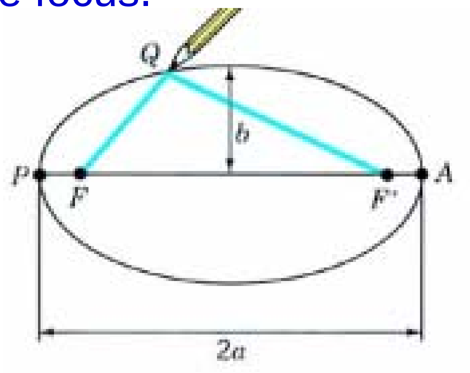
$$T = \sqrt{(6400000/9.8)} \times 2\pi \approx 5000\ \mathrm{sec}$$

13.4 Kepler's Laws of Planetary Motion

Kepler discovered three laws of planetary motion that further strengthened the idea that the earth obrits the sun rather than vise versa.

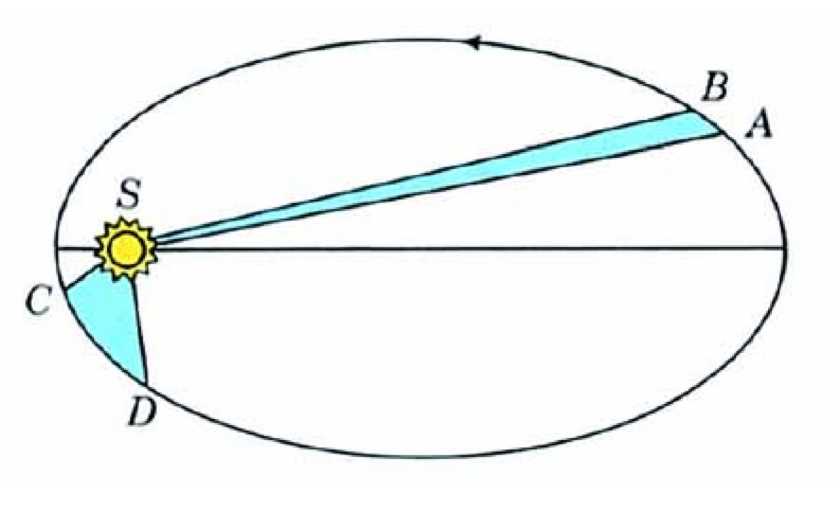
He was found through a *laborious analysis* of data left by his teacher, Tycho Brahe.

Law 1. The planets move around the sun in elliptical orbits with the sun at one focus.



Second Law: Conservation of Angular Momentum

Law 2. The line joining the sun to a planet sweeps out equal areas in equal times.



Example 12.8**:

According to Kepler's second law of planetary motion, the line joining the sun to a planet sweeps out equal area in equal time intervals. Show that this is a consequence of the conservation of angular momentum.

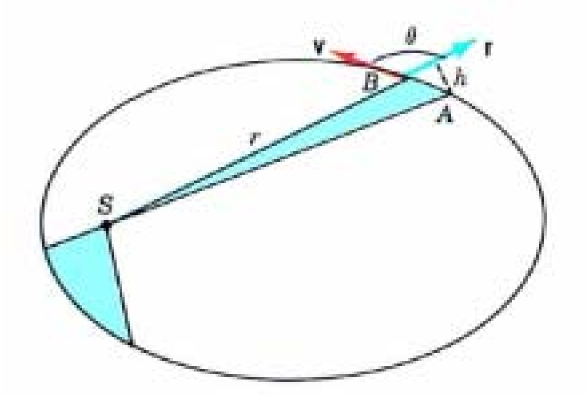
Solution:

$$\frac{\Delta A}{\Delta t} = \frac{1}{2} rh = \frac{1}{2} rv \sin\theta$$

Angular momentum

$$\ell = \mathrm{rpsin}\theta = \mathrm{m(rvsin}\theta)$$

$$\frac{\Delta A}{\Delta t} = \frac{\ell}{2m} = const.$$



Third Law:
Gravitational Force Provides Centripetal Motion

Law 3. The square of period of a planet is proportional to the cube of its mean distance from the sun.

$$\frac{v^2}{r} = \frac{GM}{r^2}, \quad v = \frac{2\pi r}{T}$$

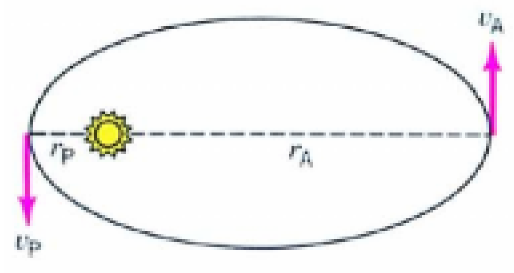
$$T^2 = \left(\frac{4\pi^2}{GM}\right) r^3$$

How to derive the third law for elliptical orbit? Extra bonus!

Energy in an Elliptical Orbit

Angular momentum conservation

$$r_P v_P = r_A v_A$$



Mechanical energy conservation

$$E = \tfrac{1}{2} m v_A^2 - GmM / r_A = \tfrac{1}{2} m v_P^2 - GmM / r_P$$

We substitute $r_P + r_A = 2a$

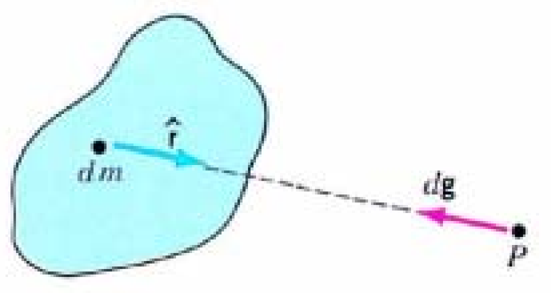
We get $$\underline{E = -\frac{GMm}{2a}}\#$$

Try to derive this result by yourself.

13.5 Continuous Distribution of Mass

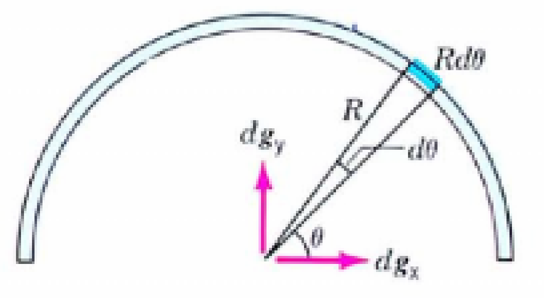
Each infinitesimal mass element contributes to the gravitational field strength is:

$$dg = \frac{Gdm}{r^2}$$



Example 13.3:

Find the field strength at the center of a thin semicircular ring of radius R and mass M, as shown in Fig. 13.16. The linear mass density is λ kg/m.



Solution:

$$dg_y = dg\sin\theta = \frac{Gdm\sin\theta}{R^2}$$

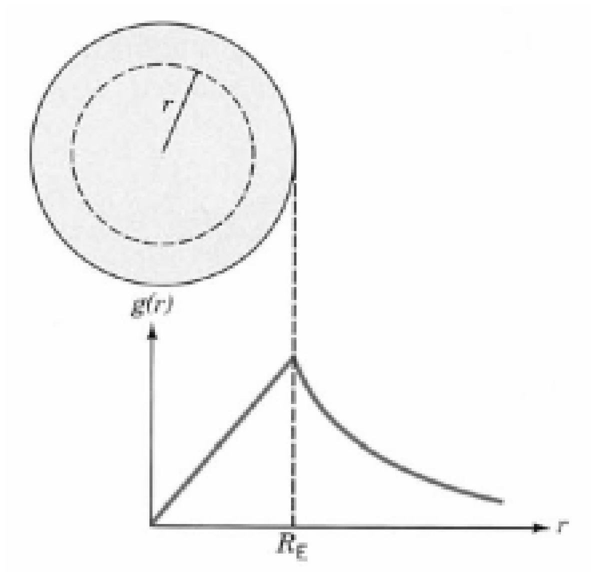
The total field strength is

$$g_y = \int_0^{\pi} \frac{G\sin\theta}{R^2} dm$$

$$= \frac{2G\lambda}{R}$$

Example 13.5:

How does the field strength vary inside a uniform solid sphere of density ρ kg/m^3 and radius R?



Solution:

Gauss law

inside $M(r) = 4\pi\rho r^3/3$

$$g(r) = \frac{GM(r)}{r^2} \propto r$$

Exercises and Problems

Ch.13:

Ex.11, 19, 27

Prob. 4, 6, 7, 9, 17