Chapter 4 Network Analysis

4.0 Introduction

▲The KVL and KCL in circuit theory are no longer valid.

▲Use the Maxwell's equations to solve all "microwave circuits:?

Field Problem → Circuit Problem

E, **H** (Vectors) → V, I (Scalars) (Equivalent quantities)

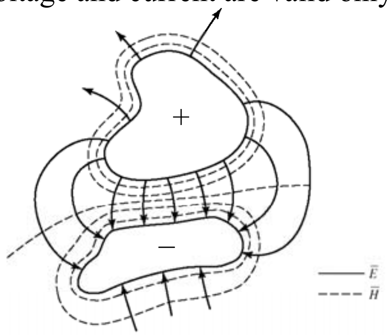
Example:

For a complicates coplanar stripline (CPS) low-pass filter, can we design it using circuit point of view ?

Agilent 8510 with head modules

4.1 Impedance, Equivalent Voltage and Current

Equivalent voltage and current are valid only for TEM lines



$$V = -\int_{-}^{+} \mathbf{E}\cdot d\overline{\ell}$$

$$I = -\oint_{C^+} \mathbf{H}\cdot d\overline{\ell}$$

$$Z_0 = \frac{V}{I}$$

For non-TEM lines, equivalent voltage and current are:

(a)Not unique but useful for an engineering approach,

(b)For certain mode only,

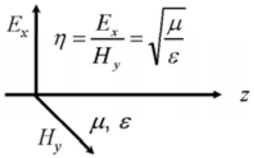
(c) VI^* = power flow of the mode, and

(d)The characteristic impedance is defined as $Z_0 = V/I$

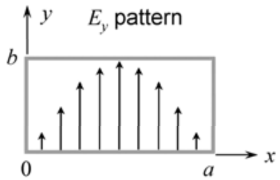
The Concept of Impedances

An important link between *EM field* and *circuit theory*

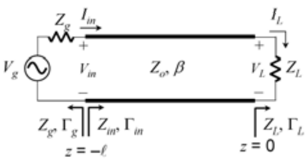
(1) $\eta = \sqrt{\mu/\varepsilon}$ = Intrinsic impedance of a medium. also the wave impedance of a plane wave.



(2) Z_W = Wave impedance (TE, TM,or TEM wave) a characteristic for a particular mode, e.g.the TE_{10} mode.



(3) Z_O = Characteristic impedance, the ratio of V and I for a traveling wave on a transmission line (I_0^-=0, $V_0^- = 0$, $V = V_0^+$, $I = I_0^+$)



Example: Application of Waveguide Impedance

Find the reflection of a TE_{10} wave incident on the interface from $z < 0$. Let $a = 22.86\ mm, b = 10.16\ mm$, and $f = 10$ GHz. $z < 0, \varepsilon_{r1} = 1;\ z > 0, \varepsilon_{r2} = 2.54$

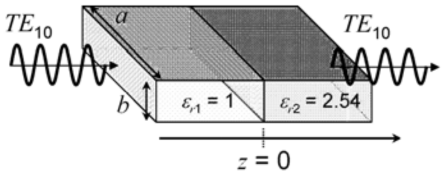
$$Sol: \beta_{air} = \sqrt{k_0^2 - \left(\frac{\pi}{a}\right)^2} = 158.0 m^{-1}$$

$$\beta_{die} = \sqrt{\varepsilon_r k_0^2 - \left(\frac{\pi}{a}\right)^2} = 304.1 m^{-1}$$

$$Z_{Wair} = \frac{k_0\eta_0}{\beta_{air}} = \frac{209.4\times 377}{158} = 500\Omega$$

$$Z_{Wdie} = \frac{k_d\eta_d}{\beta_{die}} = \frac{209.4\times 377}{304.1} = 259.6\Omega$$

$$\Gamma = \frac{Z_{Wdie} - Z_{Wair}}{Z_{Wdie} + Z_{Wair}} = \frac{259.6 - 500}{259.6 + 500} = -0.316$$



Note: The calculation of Γ is valid only when the two waveguides have identical cross section dimensions.

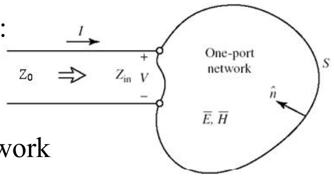
One-Port Network

▲Complex power delivered to the network:

$$P=\frac{1}{2}\oiint \underline{E}\times\underline{H}^{*}\bullet ds=P_{\ell}+2j\omega(W_m-W_e)$$

P_{ℓ} = average power dissipated by the network

$W_m,\ W_e=$ stored magnetic and electric energy



▲ Input impedance

$$Z_{\text{in}}=\frac{V}{I}=\frac{VI^{*}}{|I|^{2}}=\frac{2P}{|I|^{2}}=\frac{P_{\ell}+2j\omega(W_m-W_e)}{\frac{1}{2}|I|^{2}}$$

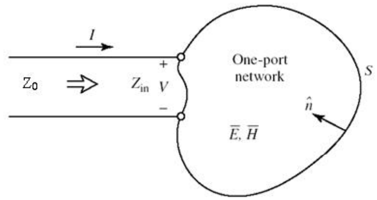
$$=R+jX,\quad X=\frac{4\omega(W_m-W_e)}{|I|^{2}}\quad \begin{cases}X>0,\ W_m>W_e, & \text{inductive load}\\ X<0,\ W_m<W_e, & \text{capactive load}\end{cases}$$

Even and Odd Properties of $Z_{\text{in}}(\omega)$ and $\Gamma(\omega)$

$$v(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}V(\omega)e^{i\omega t}d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}V(-\omega)e^{-i\omega t}d\omega$$

$$v^{*}(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}V^{*}(\omega)e^{-i\omega t}d\omega$$

Since $v(t)$ is real $(v(t)=v^{*}(t))$, $V^{*}(\omega)=V(-\omega)$



$$\text{Re}[V(\omega)]=\text{Re}[V(-\omega)],\ \ \text{Im}[V(\omega)]=-\text{Im}[V(-\omega)]$$

$$\text{Re}[I(\omega)]=\text{Re}[I(-\omega)],\ \ \text{Im}[I(\omega)]=-\text{Im}[I(-\omega)]$$

$$Z_{in}(\omega)=\frac{V(\omega)}{I(\omega)}$$

$$Z_{in}(-\omega)=\frac{V(-\omega)}{I(-\omega)}=\frac{V^{*}(\omega)}{I^{*}(\omega)}=Z_{in}^{\ *}(\omega)$$

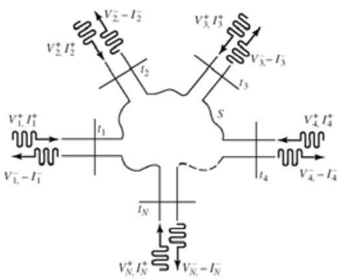
$$\Gamma(\omega)=\frac{Z_{in}(\omega)-Z_0}{Z_{in}(\omega)+Z_0}=\frac{R(\omega)-Z_0+jX(\omega)}{R(\omega)+Z_0+jX(\omega)}$$

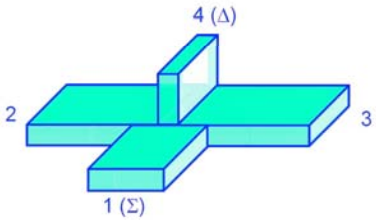
$$\Gamma(-\omega)=\frac{Z_{in}(-\omega)-Z_0}{Z_{in}(-\omega)+Z_0}=\frac{R(-\omega)-Z_0+jX(-\omega)}{R(-\omega)+Z_0+jX(-\omega)}$$

$$=\frac{R(\omega)-Z_0-jX(\omega)}{R(\omega)+Z_0-jX(\omega)}=\Gamma^{*}(\omega)$$

$$\Gamma(-\omega)=\Gamma^{*}(\omega)$$

N-Port Microwave Network



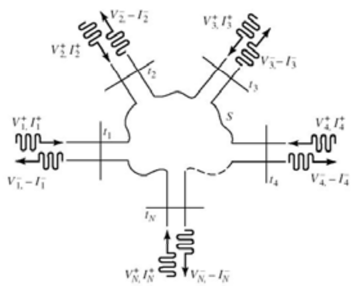


(1) Each ports has $V_k^{+},V_k^{-},I_k^{+},I_k^{-},k=1,2,...,N,$ defined at t_k plane.

(2) There could be many modes near each transmission line-network junction.

(3) Each t_k plane is chosen at a plane with a suitable distance away from the junction ensuring no higher order mode at t_k.

4.2 Z-Matrix of an N-port Network



$V_k=V_k^{+}+V_k^{-},\ k=1,2,...,N$ at t_k plane

$I_k=I_k^{+}+I_k^{-}$

$V_k=$ total voltage at port k.

$I_k=$ total current at port k.

$$\begin{bmatrix}V_1\\V_2\\\vdots\\V_N\end{bmatrix}=\begin{bmatrix}Z_{11} & Z_{12} & \cdots & Z_{1N}\\ Z_{21} & & & \vdots\\ \vdots & & & \vdots\\ Z_{N1} & \cdots & \cdots & Z_{NN}\end{bmatrix}\begin{bmatrix}I_1\\I_2\\\vdots\\I_N\end{bmatrix},$$

For an arbitary N-port linear network: $[V]=[Z][I]$

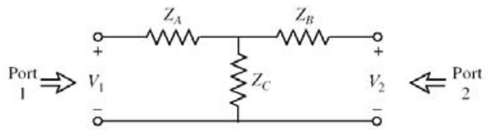
$[Z]=$ Impedance matrix

$$Z_{ij}=\left.\frac{V_i}{I_j}\right|_{I_k=0,\ k\neq j}$$

= transfer impedance between ports i and j when all other ports are open circuited.

Example 4.3 Z_{ij} of a two-port

Find Z_{ij} of the following two-port network:



$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

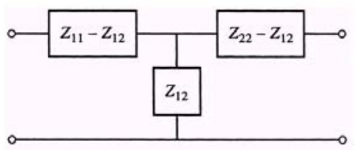
$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = Z_A + Z_C,$$

$$Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} = \frac{V_2}{I_2}\frac{Z_C}{Z_B + Z_C} = Z_C$$

$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = Z_C = Z_{12},$$

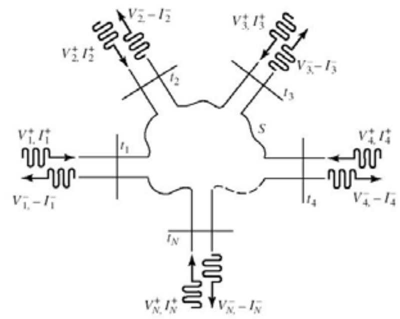
$$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} = Z_B + Z_C$$

▲ In other words, a two-port specified by $[Z]$ can be representes by



Y-Matrix of an N-Port Network

$$\begin{bmatrix} I_1 \\ I_2 \\ \cdot \\ \cdot \\ \cdot \\ I_N \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \dots Y_{1N} \\ Y_{21} & Y_{22} \dots Y_{2N} \\ \dots\dots & \dots\dots \\ \dots\dots & \dots\dots \\ \dots\dots & \dots\dots \\ Y_{N1} & Y_{N2} \dots Y_{NN} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \cdot \\ \cdot \\ \cdot \\ V_N \end{bmatrix}$$



$[I] = [Y][V]$

$[Y] =$ Admittance matrix $= [Z]^{-1}$

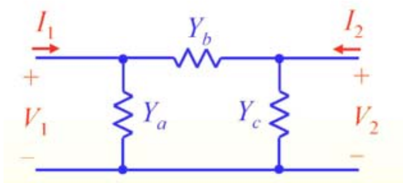
$$Y_{ij} = \left.\frac{I_i}{V_j}\right|_{V_k=0,\ k\neq j}$$

= transfer admittance between ports i and j

when all other ports are short-circuited.

Example Y_{ij} of a Two-Port Network

Find Y_{ij} of the following two-port network:



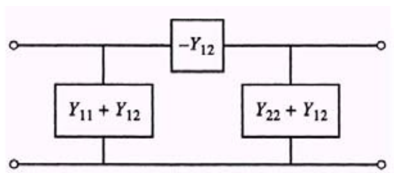
$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} = Y_a + Y_b,$$

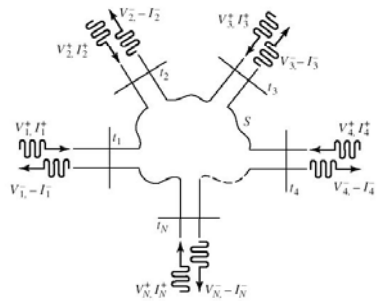
$$Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0} = Y_c + Y_b$$

$$Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} = -Y_b$$

▲ In other words, a two-port specified by $[Y]$ can be representes by



Reciprocal Network



$$\begin{bmatrix} I_1 \\ I_2 \\ \cdot \\ \cdot \\ \cdot \\ I_N \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \dots Y_{1N} \\ Y_{21} & Y_{22} \dots Y_{2N} \\ \dots\dots & \dots\dots \\ \dots\dots & \dots\dots \\ \dots\dots & \dots\dots \\ Y_{N1} & Y_{N2} \dots Y_{NN} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \cdot \\ \cdot \\ \cdot \\ V_N \end{bmatrix}$$

$$\begin{bmatrix} V_1 \\ V_2 \\ \cdot \\ \cdot \\ \cdot \\ V_N \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \dots Z_{1N} \\ Z_{21} & Z_{22} \dots Z_{2N} \\ \dots\dots & \dots\dots \\ \dots\dots & \dots\dots \\ \dots\dots & \dots\dots \\ Z_{N1} & Z_{N2} \dots Z_{NN} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ \cdot \\ \cdot \\ \cdot \\ I_N \end{bmatrix}$$

If there is no *active device or anisotropic materials*, like BJT, FET, ferrite and plasma, inside the network, one can use the reciprocity theorem to prove that

$Z_{ij} = Z_{ji}$ and $Y_{ij} = Y_{ji}$, i.e. $[Z]$ and $[Y]$ are *symmetric* matrices.

Lossless Network

$$P_{av} = \frac{1}{2}[V]^t[I]^* = \frac{1}{2}([Z][I])^t[I]^* = \frac{1}{2}\sum_{m=1}^{N}\sum_{n=1}^{N} I_m Z_{mn} I_n^*$$

For $N=2$, $P_{av} = (Z_{11}I_1I_1^* + Z_{12}I_1I_2^* + Z_{21}I_2I_1^* + Z_{22}I_2I_2^*)/2$

$$\begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_N \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} & \cdots & Z_{1N} \\ Z_{21} & & & \vdots \\ \vdots & & & \vdots \\ Z_{N1} & \cdots & \cdots & Z_{NN} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_N \end{bmatrix},$$

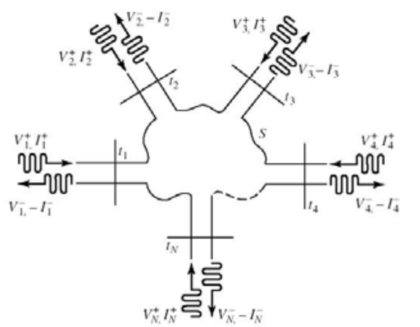
- ▹ Note that the property of $[Z]$ is neither dependent on $[I]$ nor on $[V]$
- ▹ The net real power delivered to the network is zero. Thus, $\text{Re}[P_{av}]=0$ for all possible combinations.

1. $I_n \neq 0,\ I_m = 0,\ m \neq n,\ \text{Re}\{I_n Z_{nn} I_n^*\} = |I_n|^2 \text{Re}\{Z_{nn}\} = 0 \Rightarrow \text{Re}\{Z_{nn}\} = 0$
2. $I_m \neq 0,\ I_n \neq 0,\ I_k = 0,\ k \neq m$ or n,

   $\text{Re}\left[(I_m I_n^* + I_m^* I_n) Z_{mn}\right] = 0 \Rightarrow \text{Re}\{Z_{mn}\} = 0.$

   $\Rightarrow$ Lossless network $\text{Re}\{Z_{mn}\}=0,\ \forall m,n$

4.3 The Scattering Parameter Matrix



V_n^+ and V_n^- are the amplitudes of incident and reflected voltage waves, respectively, on port n.

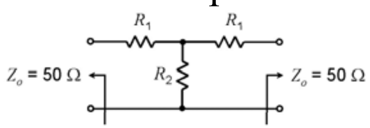
$$\begin{bmatrix} V_1^- \\ V_2^- \\ \cdot \\ \cdot \\ \cdot \\ V_N^- \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & \ldots S_{1N} \\ S_{21} & S_{22} & \ldots S_{2N} \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ S_{N1} & S_{N2} & \ldots S_{NN} \end{bmatrix} \begin{bmatrix} V_1^+ \\ V_2^+ \\ \cdot \\ \cdot \\ \cdot \\ V_N^+ \end{bmatrix}$$

$$S_{ij} = \left.\frac{V_i^-}{V_j^+}\right|_{V_k^+=0,\ k\neq j}$$

S_{ij} is determined by driving port j with an incident voltage wave V_j^+, and measuring the reflected wave V_i^-, coming out from port i.

▲ When S_{ij} is evaluated, the incident voltage waves on all ports expects port j are set to zero, which means that *all other ports should be terminated in matched load* to avoid reflections.

Example 4.4 Evaluation of *S*-parameters



Find the S-parameters of the two-port. If the circuit is a 3-dB attenuator, what are the values of R_1 and R_2?

$$Z_{in}^{(1)} = R_1 + R_2 // (Z_0 + R_1)$$

$$= R_1 + \frac{R_2(Z_0+R_1)}{Z_0+R_1+R_2} \equiv Z_0$$

Condition #1: $S_{11} = S_{22} = 0$

$$(Z_0 - R_1)(Z_0 + R_1 + R_2) = R_2(Z_0 + R_1)$$

$$Z_0^2 - R_1(R_1 + R_2) = R_1R_2$$

$$Z_0^2 = R_1(R_1 + 2R_2)$$

Condition #2: $S_{21} = \left.\frac{V_2^-}{V_1^+}\right|_{V_2^+=0} = \left.\frac{V_2}{V_1}\right|_{V_2^+=0,\ V_1^-=0}$

$$= \frac{R_2//(Z_0+R_1)}{R_1 + R_2//(Z_0+R_1)} \frac{Z_0}{Z_0+R_1} \equiv \frac{1}{\sqrt{2}} = S_{12}$$

$$\frac{R_2//(Z_0+R_1)}{Z_0}\frac{Z_0}{Z_0+R_1} = \frac{R_2(Z_0+R_1)}{Z_0+R_1+R_2}\frac{1}{Z_0+R_1} \equiv \frac{1}{\sqrt{2}}$$

$$Z_0 + R_1 + R_2 = \sqrt{2}R_2 \Rightarrow Z_0 + R_1 = (\sqrt{2}-1)R_2 \equiv xR_2$$

$$R_1(R_1 + 2R_2) = x^2R_2^2 + R_1^2 - 2xR_1R_2$$

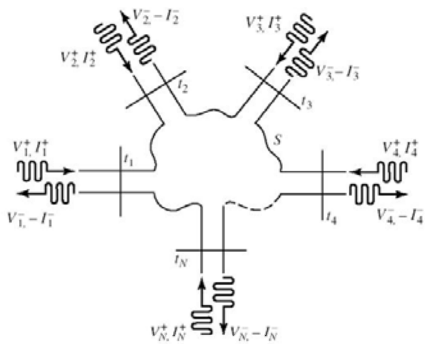
$$x^2R_2 - 2(x+1)R_1 = 0$$

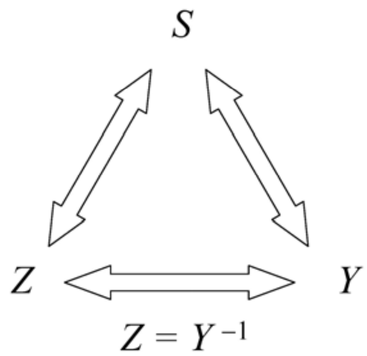
$$R_1 = \frac{x^2}{2(x+1)}R_2 \Rightarrow Z_0 + \frac{x^2}{2(x+1)}R_2 = xR_2$$

$$R_2 = \frac{2(x+1)}{x(x+2)}Z_0 = 2\sqrt{2}Z_0 = 141.42\Omega$$

$$R_1 = \frac{x^2}{2(x+1)}R_2 = \frac{x}{x+2}Z_0 = \frac{\sqrt{2}-1}{\sqrt{2}+1}Z_0 = 8.579\Omega$$

Conversion Between Network Parameters



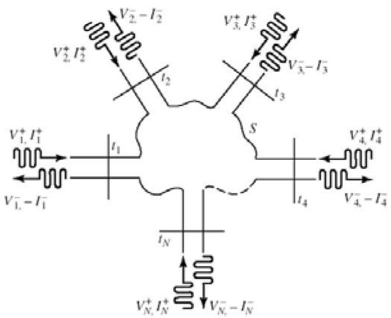


Determine $[S]$ from $[Z]$ and $[Y]$ Matrices

Z_{0n}: characteristic impedance of port n.

All the ports are identical

and $Z_{0n} = 1$ for convenience.



$V_n = V_n^+ + V_n^-$

$I_n = I_n^+ - I_n^- = V_n^+ - V_n^-$

$[Z][I] = [Z][V^+] - [Z][V^-] = [V]$

$[V] = [V^+] + [V^-]$

which can be written as

$([Z]+[U])[V^-] = ([Z]-[U])[V^+]$

$[S] \equiv ([Z]+[U])^{-1}([Z]-[U])$

where $[U]$ = identity matrix

Network Parameters for *Reciprocal* & *Lossless* Networks

(1) $[Y]$, $[Z]$ are *symmetric* matrices for *reciprocal* networks and a *purely imaginary* for *lossless* networks.

(2) $[S]$ is a *symmetric* matrix for *reciprocal* networks and a *unitary* matrix for *lossless* networks.

▲What is a unitary matrix ? If $[S]^T[S]^* = [U]$, where $[S] = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$,

i.e. $[S]^T[S]^* = \begin{bmatrix} S_{11} & S_{21} \\ S_{12} & S_{22} \end{bmatrix}\begin{bmatrix} S_{11}^* & S_{12}^* \\ S_{21}^* & S_{22}^* \end{bmatrix} = \begin{bmatrix} |S_{11}|^2+|S_{21}|^2 & S_{11}S_{12}^*+S_{21}S_{22}^* \\ S_{11}^*S_{12}+S_{21}^*S_{22} & |S_{12}|^2+|S_{22}|^2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

▲$|S_{11}|^2+|S_{21}|^2 = |S_{12}|^2+|S_{22}|^2 = 1$ indicates that the energy is conserved.

▲$S_{11}^*S_{12} + S_{22}S_{21}^* = 0$ means the different columns of $[S]$ are orthogonal.

Example 4.5 Properties of *S*-Parameters

Determine if the network is reciprocal or lossless.

What is the reflection coefficient at port 1 if port 2 is shor-circuited?

$$[S] = \begin{bmatrix} 0.1 & j0.8 \\ j0.8 & 0.1 \end{bmatrix}$$

Sol : (1) $[S]^T = [S]$ symmetric matrix ⇔ reciprocal network,

(2) $[S]^T[S]^* \neq [U]$ ⇔ It is not a lossless network.

(3) If port 2 is short-circuited,

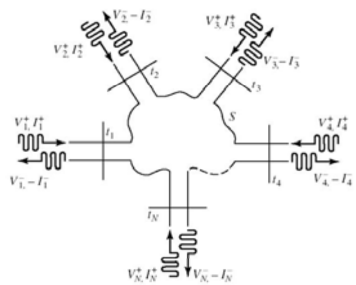
$V_2^- = S_{21}V_1^+ + S_{22}V_2^+ = -V_2^+ \Rightarrow V_2^+ = -S_{21}/(1+S_{22})V_1^+$

$V_1^- = S_{11}V_1^+ + S_{12}V_2^+ = [S_{11} - S_{12}S_{21}/(1+S_{22})]V_1^+$

$$\Gamma = \frac{V_1^-}{V_1^+} = S_{11} - \frac{S_{12}S_{21}}{1+S_{22}} = 0.1 - \frac{-0.64}{1+0.1} \cong 0.682e^{j0°}$$

Shift in Reference Planes

Reference planes should be defined before a network is characterized. The planes can be shifted.



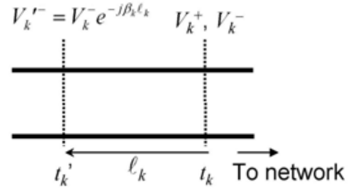
$[V^-] = [S][V^+]$ and $[V'^-] = [S'][V'^+]$

$V_n'^+ = V_n^+ e^{+j\beta_n \ell_n}$, $V_n'^- = V_n^- e^{-j\beta_n \ell_n}$,

$$[Q] = \begin{bmatrix} e^{+j\beta_1\ell_1} & & 0 \\ & e^{+j\beta_2\ell_2} & \\ 0 & & \ddots \; e^{+j\beta_n\ell_n} \end{bmatrix}$$

$V_k'^+ = V_k^+ e^{+j\beta_k \ell_k}$

$V_k'^- = V_k^- e^{-j\beta_k \ell_k}$



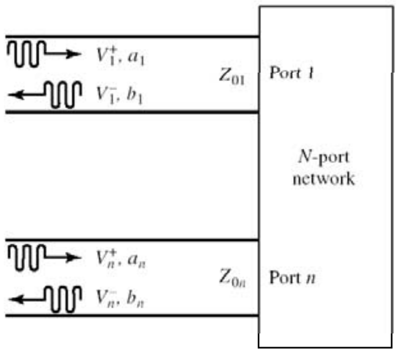
$[V^-] = [Q][V'^-]$ and $[V^+] = [Q^{-1}][V'^+]$

$[Q][V'^-] = [S][Q^{-1}][V'^+]$

$[V'^-] = [Q^{-1}][S][Q^{-1}][V'^+]$

$[S'] = [Q^{-1}][S][Q^{-1}]$

Generalized Scattering Parameters



$$a_n = \frac{V_n^+}{\sqrt{Z_{0n}}}$$

$$b_n = \frac{V_n^-}{\sqrt{Z_{0n}}}$$

$$V_n = V_n^+ + V_n^- = \sqrt{Z_{0n}}(a_n + b_n)$$

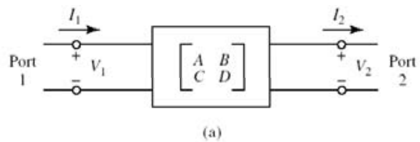
$$I_n = \frac{V_n^+ - V_n^-}{Z_{0n}} = \frac{(a_n - b_n)}{\sqrt{Z_{0n}}}$$

$$[b] = [S][a]$$

$$S_{ij} = \left.\frac{b_i}{a_j}\right|_{a_k=0,k\neq j} = \left.\frac{V_i^-\sqrt{Z_{oi}}}{V_j^+\sqrt{Z_{oj}}}\right|_{V_k^-=0,k\neq j}$$

If $[S]$ is expressed in terms of V_n^+ and V_n^-, the result depends on the *characteristic impedance* of port *n*, Z_{0n}.

4.4 The Transmission (*T* or *ABCD*) Matrix

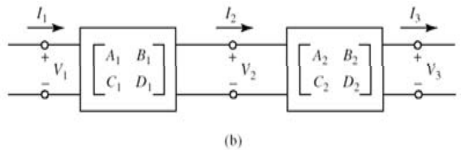


$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}$$

(1)Definition of the *ABCD* matrix is based on the total port voltage and current.

(2)The direction of I_2 is different from those of $[Y]$ and $[Z]$ matrices.

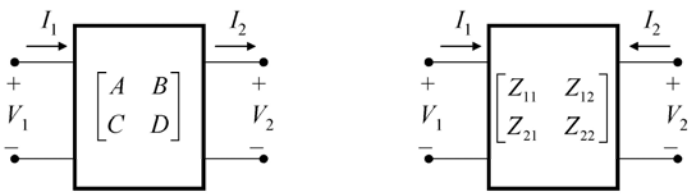
For a cascade of two two-port networks:



$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} V_3 \\ I_3 \end{bmatrix}$$

The result can be applied to a cascade of *N* two-port networks.

Relation between [*Z*] and *ABCD* Matrices



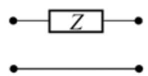
$$\begin{cases} V_2 = Z_{21}I_1 - Z_{22}I_2 \\ V_1 = Z_{11}I_1 - Z_{12}I_2 \end{cases} \Rightarrow \begin{cases} I_1 = (V_2 + Z_{22}I_2)/Z_{21} \\ V_1 = Z_{11}(V_2 + Z_{22}I_2)/Z_{21} - Z_{12}I_2 \end{cases}$$

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} \frac{Z_{11}}{Z_{21}} & \frac{Z_{11}Z_{22}}{Z_{21}} - Z_{12} \\ \frac{1}{Z_{21}} & \frac{Z_{22}}{Z_{21}} \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} \frac{Z_{11}}{Z_{21}} & \frac{\Delta_Z}{Z_{21}} \\ \frac{1}{Z_{21}} & \frac{Z_{22}}{Z_{21}} \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} \equiv \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}$$

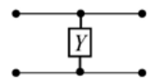
$$\Delta_Z = Z_{11}Z_{22} - Z_{12}Z_{21}$$

If the network is reciprocal, $Z_{12} = Z_{21}$, $AD - BC = \frac{Z_{11}Z_{22}}{Z_{21}^2} - \frac{Z_{11}Z_{22} - Z_{12}Z_{21}}{Z_{21}^2} = 1$

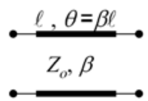
The *ABCD* Parameters of Some Useful Two-Ports



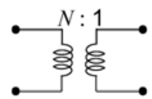
$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & Z \\ 0 & 1 \end{bmatrix}$$



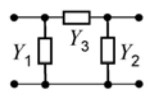
$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ Y & 1 \end{bmatrix}$$



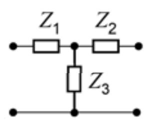
$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \cos\theta & jZ_0\sin\theta \\ jY_0\sin\theta & \cos\theta \end{bmatrix}$$



$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} N & 0 \\ 0 & \frac{1}{N} \end{bmatrix}$$



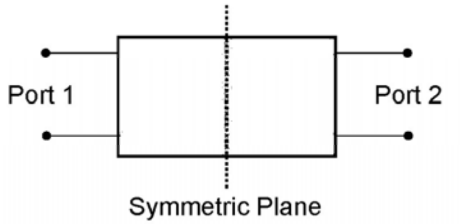
$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 + \frac{Y_2}{Y_3} & \frac{1}{Y_3} \\ Y_1 + Y_2 + \frac{Y_1Y_2}{Y_3} & 1 + \frac{Y_1}{Y_3} \end{bmatrix}$$



$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 + \frac{Z_1}{Z_3} & Z_1 + Z_2 + \frac{Z_1Z_2}{Z_3} \\ \frac{1}{Z_3} & 1 + \frac{Y_2}{Y_3} \end{bmatrix}$$

Try to derive them by yourself.

Some Properties of Symmetric Networks

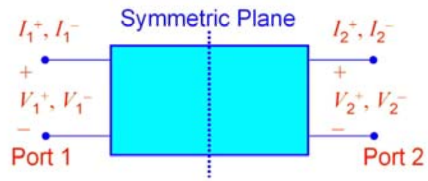


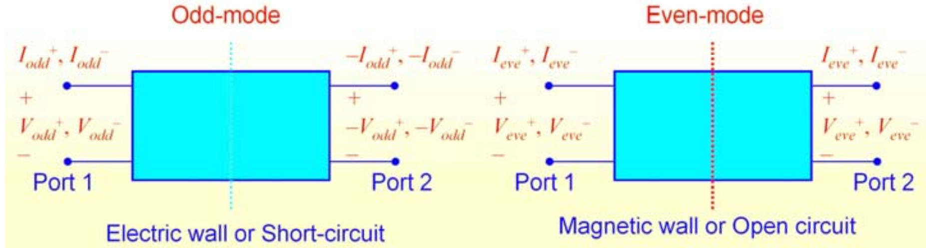
$$[Z]=\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12} & Z_{11} \end{bmatrix}$$

$$[Y]=\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12} & Y_{11} \end{bmatrix}$$

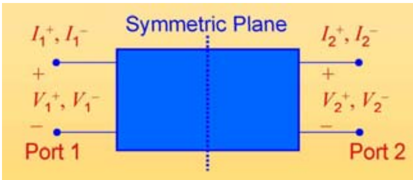
$$[S]=\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{12} & S_{11} \end{bmatrix}$$

Some Properties of Symmetric Networks





Some Properties of Symmetric Networks



$$\begin{bmatrix} V_1^+ \\ V_1^- \end{bmatrix} = \begin{bmatrix} V_{eve}^+ \\ V_{eve}^- \end{bmatrix} + \begin{bmatrix} V_{odd}^+ \\ V_{odd}^- \end{bmatrix}, \quad \begin{bmatrix} V_2^+ \\ V_2^- \end{bmatrix} = \begin{bmatrix} V_{eve}^+ \\ V_{eve}^- \end{bmatrix} - \begin{bmatrix} V_{odd}^+ \\ V_{odd}^- \end{bmatrix}$$

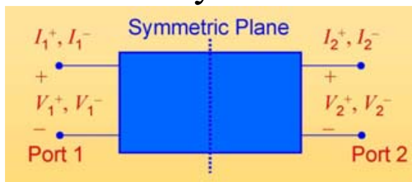
$$\begin{bmatrix} I_1^+ \\ I_1^- \end{bmatrix} = \begin{bmatrix} I_{eve}^+ \\ I_{eve}^- \end{bmatrix} + \begin{bmatrix} I_{odd}^+ \\ I_{odd}^- \end{bmatrix}, \quad \begin{bmatrix} I_2^+ \\ I_2^- \end{bmatrix} = \begin{bmatrix} I_{eve}^+ \\ I_{eve}^- \end{bmatrix} - \begin{bmatrix} I_{odd}^+ \\ I_{odd}^- \end{bmatrix}$$

$$\begin{bmatrix} V_1^+ \\ I_1^- \end{bmatrix} = \begin{bmatrix} V_{eve}^+ \\ I_{eve}^- \end{bmatrix} + \begin{bmatrix} V_{odd}^+ \\ I_{odd}^- \end{bmatrix}, \quad \begin{bmatrix} V_2^+ \\ I_2^- \end{bmatrix} = \begin{bmatrix} V_{eve}^+ \\ I_{eve}^- \end{bmatrix} - \begin{bmatrix} V_{odd}^+ \\ I_{odd}^- \end{bmatrix}$$

$$\begin{bmatrix} V_1^+ \\ V_2^+ \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} V_{eve}^+ \\ V_{odd}^+ \end{bmatrix} \Rightarrow \begin{bmatrix} V_{eve}^+ \\ V_{odd}^+ \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} V_1^+ \\ V_2^+ \end{bmatrix}$$

$$\begin{bmatrix} V_1^- \\ V_2^- \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} V_{eve}^- \\ V_{odd}^- \end{bmatrix} \Rightarrow \begin{bmatrix} V_{eve}^- \\ V_{odd}^- \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} V_1^- \\ V_2^- \end{bmatrix}$$

S-Parameter of Symmetric Networks



$$\begin{bmatrix} V_1^- \\ V_2^- \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{12} & S_{11} \end{bmatrix} \begin{bmatrix} V_1^+ \\ V_2^+ \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} V_{eve}^- \\ V_{odd}^- \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{12} & S_{11} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} V_{eve}^+ \\ V_{odd}^+ \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} V_{eve}^- \\ V_{odd}^- \end{bmatrix} &= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ S_{12} & S_{11} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} V_{eve}^+ \\ V_{odd}^+ \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} S_{11}+S_{12} & S_{11}-S_{12} \\ S_{12}+S_{11} & S_{12}-S_{11} \end{bmatrix} \begin{bmatrix} V_{eve}^+ \\ V_{odd}^+ \end{bmatrix} \\ &= \begin{bmatrix} S_{12}+S_{11} & 0 \\ 0 & S_{11}-S_{12} \end{bmatrix} \begin{bmatrix} V_{eve}^+ \\ V_{odd}^+ \end{bmatrix} \equiv \begin{bmatrix} \Gamma_{eve} & 0 \\ 0 & \Gamma_{odd} \end{bmatrix} \begin{bmatrix} V_{eve}^+ \\ V_{odd}^+ \end{bmatrix} \end{aligned}$$

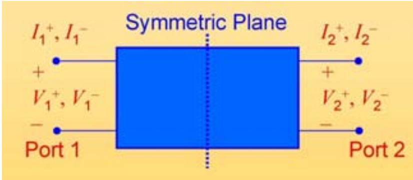
$$\Gamma_{eve} = S_{11} + S_{12}$$

$$\Gamma_{odd} = S_{11} - S_{12}$$

$$\Rightarrow S_{11} = \left(\Gamma_{eve} + \Gamma_{odd}\right)/2$$

$$S_{12} = \left(\Gamma_{eve} - \Gamma_{odd}\right)/2$$

Z-Matrix of Symmetric Networks



$$\begin{bmatrix}V_1\\V_2\end{bmatrix}=\begin{bmatrix}Z_{11}&Z_{12}\\Z_{12}&Z_{11}\end{bmatrix}\begin{bmatrix}I_1\\I_2\end{bmatrix}\Rightarrow\begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{bmatrix}V_{eve}\\V_{odd}\end{bmatrix}=\begin{bmatrix}Z_{11}&Z_{12}\\Z_{12}&Z_{11}\end{bmatrix}\begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{bmatrix}I_{eve}\\I_{odd}\end{bmatrix}$$

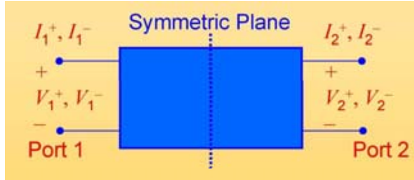
$$\begin{bmatrix}V_{eve}\\V_{odd}\end{bmatrix}=\frac{1}{2}\begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{bmatrix}Z_{11}&Z_{12}\\Z_{12}&Z_{11}\end{bmatrix}\begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{bmatrix}I_{eve}\\I_{odd}\end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{bmatrix}Z_{11}+Z_{12}&Z_{11}-Z_{12}\\Z_{12}+Z_{11}&Z_{12}-Z_{11}\end{bmatrix}\begin{bmatrix}I_{eve}\\I_{odd}\end{bmatrix}$$

$$=\begin{bmatrix}Z_{12}+Z_{11}&0\\0&Z_{11}-Z_{12}\end{bmatrix}\begin{bmatrix}I_{eve}\\I_{odd}\end{bmatrix}\equiv\begin{bmatrix}Z_{eve}&0\\0&Z_{odd}\end{bmatrix}\begin{bmatrix}I_{eve}\\I_{odd}\end{bmatrix}$$

$$Z_{eve}=Z_{11}+Z_{12}$$
$$Z_{odd}=Z_{11}-Z_{12}$$
$$\Rightarrow\quad Z_{11}=(Z_{eve}+Z_{odd})/2$$
$$Z_{12}=(Z_{eve}-Z_{odd})/2$$

Y-Matrix of Symmetric Networks



$$\begin{bmatrix}I_1\\I_2\end{bmatrix}=\begin{bmatrix}Y_{11}&Y_{12}\\Y_{12}&Y_{11}\end{bmatrix}\begin{bmatrix}V_1\\V_2\end{bmatrix}\Rightarrow\begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{bmatrix}I_{eve}\\I_{odd}\end{bmatrix}=\begin{bmatrix}Y_{11}&Y_{12}\\Y_{12}&Y_{11}\end{bmatrix}\begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{bmatrix}V_{eve}\\V_{odd}\end{bmatrix}$$

$$\begin{bmatrix}I_{eve}\\I_{odd}\end{bmatrix}=\frac{1}{2}\begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{bmatrix}Y_{11}&Y_{12}\\Y_{12}&Y_{11}\end{bmatrix}\begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{bmatrix}V_{eve}\\V_{odd}\end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{bmatrix}Y_{11}+Y_{12}&Y_{11}-Y_{12}\\Y_{12}+Y_{11}&Y_{12}-Y_{11}\end{bmatrix}\begin{bmatrix}V_{eve}\\V_{odd}\end{bmatrix}$$

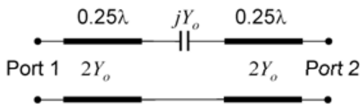
$$=\begin{bmatrix}Y_{12}+Y_{11}&0\\0&Y_{11}-Y_{12}\end{bmatrix}\begin{bmatrix}V_{eve}\\V_{odd}\end{bmatrix}\equiv\begin{bmatrix}Y_{eve}&0\\0&Y_{odd}\end{bmatrix}\begin{bmatrix}V_{eve}\\V_{odd}\end{bmatrix}$$

$$Y_{eve}=Y_{11}+Y_{12}$$
$$Y_{odd}=Y_{11}-Y_{12}$$
$$\Rightarrow\quad Y_{11}=(Y_{eve}+Y_{odd})/2$$
$$Y_{12}=(Y_{eve}-Y_{odd})/2$$

Try to derive them by yourself.

Example

The characteristic admittance of the two $\lambda/4$-section is $2Y_0$. Find the S-parameters of the two-port with reference impedance $Z_0=Y_0^{-1}$. Based on the S-matrix, show that the circuit is lossless.



Find the input reflection coefficients Γ_{ine} and Γ_{ino} when the symmetric plane is placed with open and short circuits, respectively. Using the impedance transformation property, we have $Z_{ine}=0$ and $Z_{ino}=jZ_0/2$, and $\Gamma_{ine}=-1$ and $\Gamma_{ino}=(j-2)/(j+2)$.

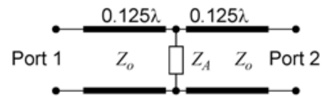
$$S_{11}=\frac{1}{2}(\Gamma_{ine}+\Gamma_{ino})=\frac{1}{2}\left(-1+\frac{j-2}{j+2}\right)=-\frac{2}{j+2}=S_{22}$$

$$S_{21}=\frac{1}{2}(\Gamma_{ine}-\Gamma_{ino})=\frac{1}{2}\left(-1-\frac{j-2}{j+2}\right)=-\frac{j}{j+2}=S_{12}$$

One can validate that the circuit is lossless since $[S][S]^{T*}=$ identity matrix.

Example

Let $Z_A=2Z_0$. Find its Z matrix and Γ at port1 (reference impedance $=Z_0$) if port2 is short-circuited.



The Z-martix can be obtained via the ABCD matrices in p. 24:

$$\begin{bmatrix}Z_{11}&Z_{12}\\Z_{21}&Z_{22}\end{bmatrix}=\frac{Z_0}{1+j4}\begin{bmatrix}j&4\\4&j\end{bmatrix}$$

Applying the even-odd analysis,

$$\begin{bmatrix}V_1\\V_2\end{bmatrix}=\begin{bmatrix}V_{eve}+V_{odd}\\V_{eve}-V_{odd}\end{bmatrix}=\begin{bmatrix}Z_{in}^{eve}&Z_{in}^{odd}\\Z_{in}^{eve}&-Z_{in}^{odd}\end{bmatrix}\begin{bmatrix}I_{eve}\\I_{odd}\end{bmatrix}$$

$$=\begin{bmatrix}Z_{in}^{eve}&Z_{in}^{odd}\\Z_{in}^{eve}&-Z_{in}^{odd}\end{bmatrix}\begin{bmatrix}\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&-\frac{1}{2}\end{bmatrix}\begin{bmatrix}I_1\\I_2\end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix}Z_{in}^{eve}+Z_{in}^{odd}&Z_{in}^{eve}-Z_{in}^{odd}\\Z_{in}^{eve}-Z_{in}^{odd}&Z_{in}^{eve}+Z_{in}^{odd}\end{bmatrix}\begin{bmatrix}I_1\\I_2\end{bmatrix}$$

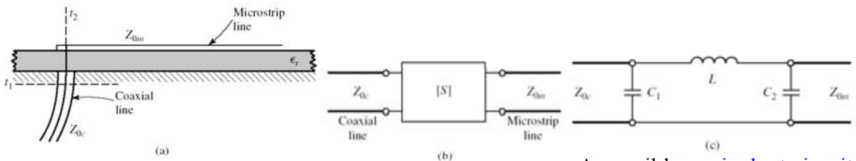
$$Z_{in}^{eve}=Z_0\frac{2Z_A+jZ_o\tan 45^\circ}{Z_0+j2Z_A\tan 45^\circ}=Z_0\frac{4+j}{1+j4}$$

$$Z_{in}^{odd}=jZ_o\tan 45^\circ=jZ_o$$

$$Z_{11}=Z_{22}=\frac{Z_{in}^{eve}+Z_{in}^{odd}}{2}=Z_0\frac{j}{1+j4}$$

$$Z_{12}=Z_{21}=\frac{Z_{in}^{eve}-Z_{in}^{odd}}{2}=Z_0\frac{4}{1+j4}$$

Equivalent Circuit for Two-Port Networks

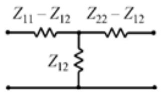


A coaxial-to-microstrip transition

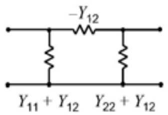
Transition represented by a black box.

A possible equivalent circuit for a transition

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \Leftrightarrow$$



$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \Leftrightarrow$$

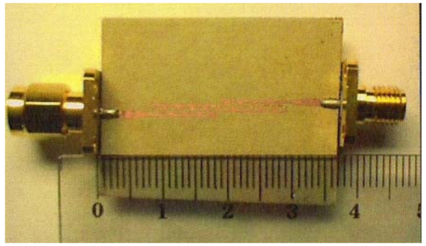


(1) If the network is *reciprocal*, there are six degrees of freedom, so the equivalent circuit has six independent parameters.

(2) A *nonreciprocal* network cannot be represented by a passive equivalent circuit using reciprocal elements.

Theory of TRL Network Analyzer Calibration

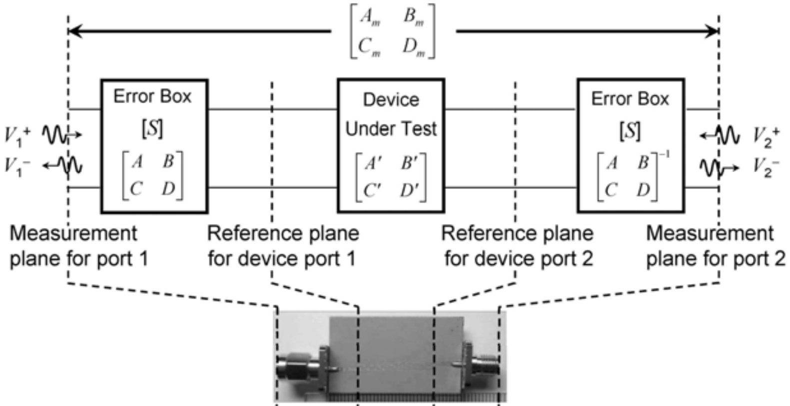
Why calibration is necessary in a microwave measurement?



(a) The S-parameters include magnitude and phase measurement.

(b) Different circuits may have different reference planes, transitions, connectors, housing structures, and/or frequency bands.

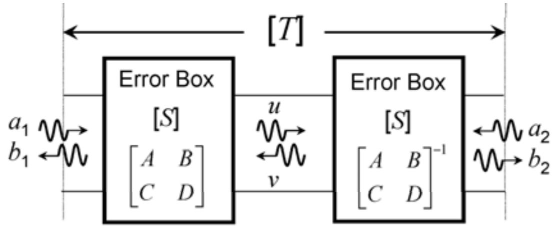
The TRL Calibration



Δ Error Box : Losses and phase delays caused by the effects of connectors, cables, and/or transitions.

Δ A calibration procedure is used to characterize the S-parameters of the error box before measurement of DUT.

Thru---Remove the DUT and Let the S-parameters be $[T]$



$S_{21} = S_{12}$ (Reciprocal). The error box are symmetric and identical.

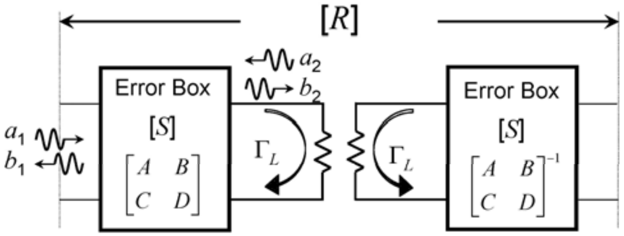
$$u = S_{21}a_1 + S_{22}v = S_{21}a_1 + S_{22}(S_{22}v + S_{21}a_2) = S_{21}\frac{a_1 + S_{22}a_2}{1 - S_{22}^2}$$

$$v = S_{22}u + S_{21}a_2 = S_{21}S_{22}\frac{a_1 + S_{22}a_2}{1 - S_{22}^2} + S_{21}a_2 = S_{21}\frac{S_{21}a_1 + a_2}{1 - S_{22}^2}$$

$$b_1 = S_{11}a_1 + S_{12}v = S_{11}a_1 + S_{12}(S_{21}\frac{S_{21}a_1 + a_2}{1 - S_{22}^2}) \equiv T_{11}a_1 + T_{12}a_2$$

$$T_{11} = S_{11} + \frac{S_{12}S_{21}S_{22}}{1 - S_{22}^2} = S_{11} + \frac{S_{21}^2 S_{22}}{1 - S_{22}^2} = T_{22}, T_{12} = \frac{S_{12}S_{21}}{1 - S_{22}^2} = \frac{S_{21}^2}{1 - S_{22}^2} = T_{21}$$

Reflect---Terminate the Error Box to a Known Load



The two measurement ports are decoupled, $R_{12} = R_{21} = 0$

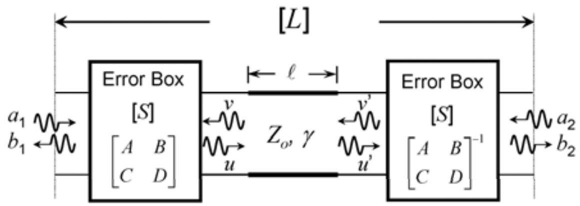
$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}_{(Error\ box)}, \quad \Gamma_L = \frac{a_2}{b_2}$$

$$b_2 = S_{21}a_1 + S_{22}a_2 = S_{21}a_1 + S_{22}\Gamma_L b_2 = \frac{S_{21}}{1 - S_{22}\Gamma_L} a_1$$

$$b_1 = S_{11}a_1 + S_{12}a_2 = S_{11}a_1 + S_{12}\Gamma_L b_2 = \left(S_{11} + \frac{S_{12}S_{21}\Gamma_L}{1 - S_{22}\Gamma_L} \right) a_1 \equiv R_{11}a_1$$

$$R_{11} = S_{11} + \frac{S_{12}S_{21}\Gamma_L}{1 - S_{22}\Gamma_L} = R_{22}$$

Line---Remove the DUT and Insert a Section Z_0 Line



$$\begin{bmatrix} b_1 \\ u \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ v \end{bmatrix}$$

$$\begin{bmatrix} v e^{\gamma\ell} \\ b_2 \end{bmatrix} = \begin{bmatrix} S_{22} & S_{21} \\ S_{12} & S_{11} \end{bmatrix} \begin{bmatrix} u e^{-\gamma\ell} \\ a_2 \end{bmatrix}$$

$$ve^{\gamma\ell} = S_{22}ue^{-\gamma\ell} + S_{21}a_2 = S_{22}(S_{21}a_1 + S_{22}v)e^{-\gamma\ell} + S_{21}a_2$$

$$v = S_{22}(S_{21}a_1 + S_{22}v)e^{-2\gamma\ell} + S_{21}a_2e^{-\gamma\ell} = e^{-\gamma\ell}\frac{S_{22}S_{21}a_1e^{-\gamma\ell} + S_{21}a_2}{1 - S_{22}^2e^{-2\gamma\ell}}$$

$$b_1 = S_{11}a_1 + S_{12}v = S_{11}a_1 + S_{12}\left(e^{-\gamma\ell}\frac{S_{22}S_{21}a_1e^{-\gamma\ell} + S_{21}a_2}{1 - S_{22}^2e^{-2\gamma\ell}} \right) \equiv L_{11}a_1 + L_{12}a_2$$

$$L_{11} = S_{11} + \frac{S_{12}^2S_{22}e^{-2\gamma\ell}}{1 - S_{22}^2e^{-2\gamma\ell}} = L_{22}, L_{12} = \frac{S_{12}^2e^{-\gamma\ell}}{1 - S_{22}^2e^{-2\gamma\ell}} = L_{21}$$

TRL Calibration (cont'd)

$$\left.\begin{array}{ll} T_{11} = S_{11} + \dfrac{S_{22}S_{12}^2}{1 - S_{22}^2} & L_{11} = S_{11} + \dfrac{S_{12}^2S_{22}e^{-2\gamma\ell}}{1 - S_{22}^2e^{-2\gamma\ell}} \\ T_{12} = \dfrac{S_{12}^2}{1 - S_{22}^2} & \\ R_{11} = S_{11} + \dfrac{S_{12}^2\Gamma_L}{1 - S_{22}^2\Gamma_L} & L_{12} = \dfrac{S_{12}^2e^{-\gamma\ell}}{1 - S_{22}^2e^{-2\gamma\ell}} \end{array}\right\} \text{Solve } \underbrace{S_{11}, S_{12}, S_{22}, \Gamma_L, and\ e^{-\gamma\ell}}_{\text{5 unknowns}}$$

After some simple algebraic manipulations, one can obtain

$$e^{-\gamma\ell} = \frac{L_{12}^2 + T_{12}^2 - (T_{11} - L_{11})^2 \pm \sqrt{\left[L_{12}^2 + T_{12}^2 - (T_{11} - L_{11})^2 \right]^2 - 4L_{12}^2T_{12}^2}}{2L_{12}^2T_{12}^2}$$

$$S_{22} = \frac{T_{11} - L_{11}}{T_{12} - L_{12}e^{-\gamma\ell}}, S_{11} = T_{11} - S_{22}T_{12}, S_{12}^2 = T_{12}\left(1 - S_{22}^2\right)$$

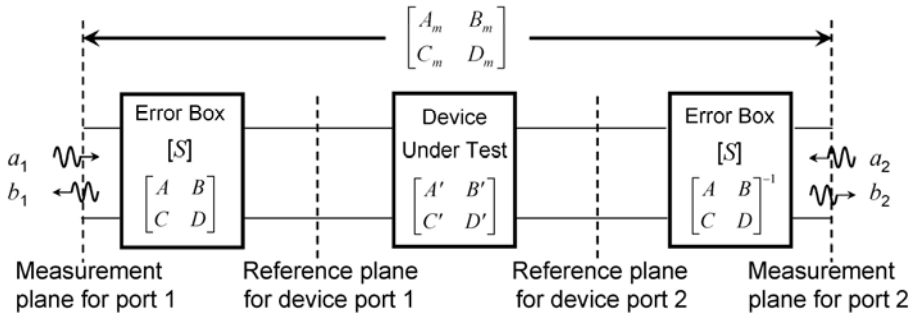
$$\Gamma_L = \frac{R_{11} - S_{11}}{S_{12}^2 + S_{22}(R_{11} - S_{11})}$$

TRL Calibration (cont'd)

(1) The S-parameters for the error boxes are known.

(2) The S-parameters for DUT can then be obtained as

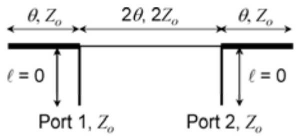
$$\begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix}_{\text{DUT}} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}_{\text{Error Box}}^{-1} \begin{bmatrix} A_m & B_m \\ C_m & D_m \end{bmatrix}_{\text{Measured}} \begin{bmatrix} A & B \\ C & D \end{bmatrix}_{\text{Error box}}$$



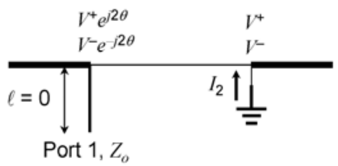
Optional

Example (Midterm 2004, part 1)

Find the Y-parameters of the network and the input admittances seen at port 1 when port 2 is terminated in $Z_2 = 0.5Z_0$, $Z_2 = \infty$. The reference impedance is Z_0.



If can be known that $Y_{11} = Y_{22}, Y_{12} = Y_{21}$. When $V_2 = 0$, $V^- = -V^+$



$$(1) Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} = jY_0 \tan\theta - j\frac{Y_0}{2}\cot 2\theta = Y_{22}$$

$$(2) V_{11} = V^+ e^{j2\theta} + V^- e^{-j2\theta} = 2jV^+ \sin 2\theta$$

$$I_2 = -\left(\frac{V^+}{2Z_0} - \frac{V^-}{2Z_0}\right) = -\frac{V^+}{Z_0}$$

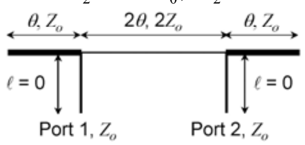
$$Y_{12} = Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} = j\frac{Y_0}{2}\csc 2\theta$$

Y_{12} cannot see the open stub?

Optional

Example (Midterm 2004 part 2)

Find the Y-parameters of the network and the input admittances seen at port 1 when port 2 is terminated in $Z_2 = 0.5Z_0$, $Z_2 = \infty$. The reference impedance is Z_0.



$$Y_{eve} = jY_0 \tan\theta + j\frac{Y_0}{2}\tan\theta$$

$$Y_{odd} = jY_0 \tan\theta - j\frac{Y_0}{2}\cot 2\theta$$

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} V_{eve} \\ I_{eve} \end{bmatrix} + \begin{bmatrix} V_{odd} \\ I_{odd} \end{bmatrix} = \begin{bmatrix} 1 \\ Y_{eve} \end{bmatrix} V_{eve} + \begin{bmatrix} 1 \\ Y_{odd} \end{bmatrix} V_{odd}$$

$$\begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} V_{eve} \\ I_{eve} \end{bmatrix} - \begin{bmatrix} V_{odd} \\ I_{odd} \end{bmatrix} = \begin{bmatrix} 1 \\ Y_{eve} \end{bmatrix} V_{eve} - \begin{bmatrix} 1 \\ Y_{odd} \end{bmatrix} V_{odd}$$

$$V_2 = 0 \Rightarrow V_1 = 2V_{eve} = 2V_{odd}$$

$$I_1 = Y_{eve}V_{eve} + Y_{odd}V_{odd} = (Y_{eve} + Y_{odd})V_{eve}$$

$$I_2 = Y_{eve}V_{eve} - Y_{odd}V_{odd} = (Y_{eve} - Y_{odd})V_{eve}$$

$$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} = \frac{Y_{eve} + Y_{odd}}{2}$$

$$= jY_0 \tan\theta + j\frac{Y_0}{4}(\tan\theta - \cot\theta)$$

$$= jY_0 \tan\theta - j\frac{Y_0}{2}\cot 2\theta = Y_{22}$$

$$Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} = \frac{Y_{eve} - Y_{odd}}{2} = j\frac{Y_0}{4}(\tan\theta + \cot\theta)$$

$$= j\frac{Y_0}{2}\csc 2\theta = Y_{12}$$

Part II

Excitation of a TE_{mn} Waveguide Mode (Pozar 4.7&4.8)

- This study presents a methodology of exciting a pure circular TE_{mn} mode using cascaded Y-type power dividers.
- The dividers partition the input signal into several parts which are then coupled to a circular waveguide through apertures. The coupling apertures induce magnetic dipoles. With proper arrangement of the magnetic dipoles on the circumference of the circular waveguide, they then jointly excite the desired mode.
- The coupling strength and the mode purity are calculated using the reciprocity theorem and the magnetic current sources.
- Three mode converters, TE_{21}, TE_{01}, and TE_{41}, were designed, built, and tested at *W*-band. Back-to-back transmission measurements exhibit excellent agreement to the results of computer simulations when the conductor loss is taken into consideration. The measured transmissions are high and the bandwidths are broad.
- These Y-type converters are structurally simple but the machining errors are critical. The factors affecting the performance will be discussed in detail.

T. H. Chang*, C. S. Li, C. N. Wu, and C. F. Yu, "Exciting circular TEmn modes at low terahertz region", Appl. Phys. Lett. 93, 111503 (2008).

Part II

Applications

- Exciting a specific waveguide mode plays a key role in many applications, such as the gyrotron traveling-wave tube (gyro-TWT) and the gyrotron backward-wave oscillator (gyro-BWO).
- In the gyro-TWT, the mode converter launches a wave of a specific mode into the interaction structure to interact with the electron beam; while, in the gyro-BWO, the mode converter extracts the wave power at the upstream end.
- In addition to the gyrotron applications, the mode converters can be used in microwave/plasma systems, radar/antennas systems, and rotary joints.

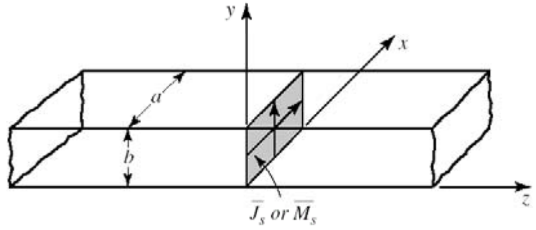
Part II

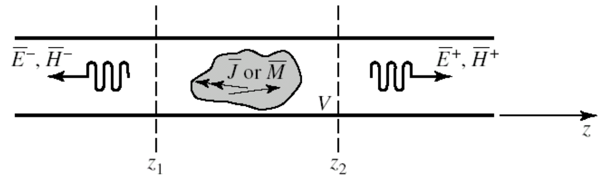
Techniques to Excite a Specific Waveguide Mode

- By coupling method, they can be classified into two types: serpentine/corrugated structure and sidewall coupling structure.
- Taking gyrotron as an example, the general requirements for the mode converter/launcher include high converting efficiency, high mode purity, broad bandwidth, high-power capability, and short converting length.
- Y-type converters are superior over other converters in the gyrotron applications.

Part II

Current Sheets and Arbitrary Current Source





Part II

Properties and Characteristics of the Waveguide Modes

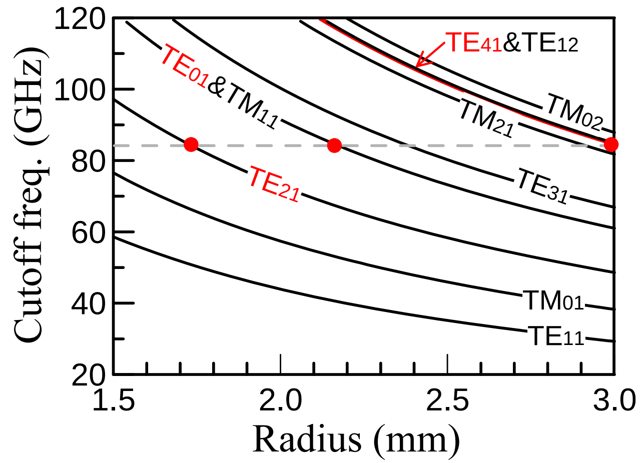
Table 1 summarizes the desired modes and their corresponding coupling structures, radii, and parasitic modes.

Desired mode	TE_{21}	TE_{01}	TE_{41}
Coupling structure	Dual-feed	Quad-feed	Quad-feed
Waveguide radius	1.74 mm	2.15 mm	3.00 mm
Parasitic modes	$TE_{11,A}$, $TE_{11,B}$	$TE_{11,A}$, $TE_{11,B}$	$TE_{11,A}$, $TE_{11,B}$
	TM_{01}	$TE_{21,A}$, $TE_{21,B}$	$TE_{21,A}$, $TE_{21,B}$
		TM_{01},	TE_{01}
		$TM_{11,A}$, $TM_{11,B}$	$TE_{31,A}$, $TE_{31,B}$
			$TE_{12,A}$, $TE_{12,B}$
			TM_{01},
			$TM_{11,A}$, $TM_{11,B}$
			$TM_{21,A}$, $TM_{21,B}$

Part II

Difficulties to Excite a Higher-Order Mode

Cutoff frequency vs waveguide radius. For the *W*-band operation, the desired waveguide radii are 1.74, 2.15, and 3.00 mm, for TE_{21}, TE_{01}, and TE_{41}, respectively.

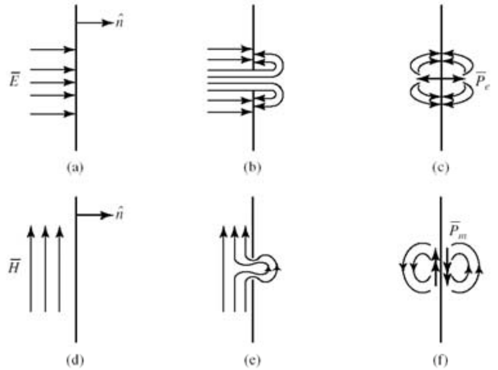


Mode Synthesizing

Excitation of a given mode can be achieved using properly arranged electric or magnetic current sources. These current sources come from the probe feeds, the loop feeds, or the coupling apertures. The sidewall coupling apertures, capable of high power operation, thus are the best choice for present study. Mode excitation from an arbitrary electric and magnetic current source can be found in Pozar's Chap.4. Here we present a complete approach to exciting a cylindrical TE_{mn} mode using multiple magnetic current sources.

- Induce current sources
- Synthesize the desired mode
- Analyze mode purity

Electric/Magnetic Dipoles and Currents



$$\bar{P}_e = \alpha_e \varepsilon_0 E_n \hat{n} \delta(\bar{x} - \bar{x}_0)$$
$$\bar{P}_m = -\alpha_m \bar{H}_t \delta(\bar{x} - \bar{x}_0)$$

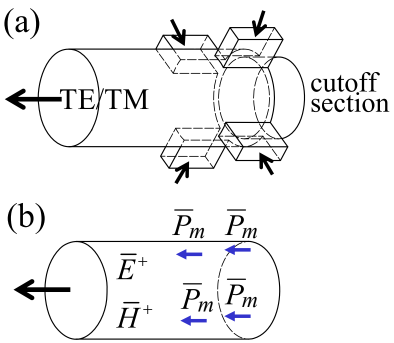
TABLE 4.3 Electric and Magnetic Polarizations

Aperture Shape	α_e	α_m
Round hole	$\frac{2r_0^3}{3}$	$\frac{4r_0^3}{3}$
Rectangular slot ($\bar{H}$ across slot)	$\frac{\pi \ell d^2}{16}$	$\frac{\pi \ell d^2}{16}$

Induce Current Sources

The idea of the sidewall coupling structure is to equip the waveguide with some dipoles.

Figure shows the geometry of the sidewall coupling structure under study.



The coupling aperture induce equivalent electric and magnetic dipole moments. The electric dipole moment is proportional to the normal electric field while the magnetic dipole moment is proportional to the tangential magnetic field

$$\bar{P}_e = \alpha_e \varepsilon_0 E_n \hat{n} \delta(\bar{x} - \bar{x}_0)$$
$$\bar{P}_m = -\alpha_m \bar{H}_t \delta(\bar{x} - \bar{x}_0)$$

where $\bar{x}$ and $\bar{x}_0$ are positions of the observer and the aperture; α_e and α_m are constants that depend on the size and shape of the aperture, and ε_0 is the permittivity.

Equivalent Polarization Currents

The electric and magnetic dipole moments are

$$\bar{P}_e = \alpha_e \varepsilon_0 E_n \hat{n} \delta(\bar{x} - \bar{x}_0)$$
$$\bar{P}_m = -\alpha_m \bar{H}_t \delta(\bar{x} - \bar{x}_0)$$

The equivalent electric and magnetic polarization currents are

$$\bar{J} = \sum j\omega \bar{P}_e = 0$$
$$\bar{M} = \sum j\omega\mu_0 \bar{P}_m = -j\omega\mu_0 \alpha_m H_z \hat{\mathbf{z}} \sum_i \delta(\bar{x} - \bar{x}_i)$$

where the wave is assumed to be sinusoidal time-dependent ($e^{j\omega t}$) $\bar{x}_i$ indicates the position of the ith coupling hole and H_z is the amplitude of the tangential magnetic field.

Part II

Synthesize the desired mode:
the reciprocity theorem (I), Pozar 1.9

The current sources just induced might excite desired mode as well as parasitic modes. In this section we provide a method to show how the desired mode is synthesized and how the parasitic modes are suppressed.

A useful theorem in electromagnetism is to be mentioned---the reciprocity theorem.

Two sets of current sources: $(\bar{J}_1, \bar{M}_1)$ and $(\bar{J}_2, \bar{M}_2)$

Their corresponding fields: $(\bar{E}_1, \bar{H}_1)$ and $(\bar{E}_2, \bar{H}_2)$

Assume $\bar{M}_2$ to be the only nonzero current source. Then, the reciprocity theorem reads:

$$\oint_S (\bar{E}_1 \times \bar{H}_1 - \bar{E}_2 \times \bar{H}_2)\cdot ds = \int_V (\bar{M}_2 \cdot \bar{H}_1) dv$$

Part II

Synthesize the desired mode:
the reciprocity theorem (II)

Since $\bar{J}_1 = \bar{M}_1 = 0$, the corresponding $\bar{E}_1$ and $\bar{H}_1$ are assumed to be certain circular TE_{mn} waveguide mode.

$\bar{E}_2$ and $\bar{H}_2$ are the electric and magnetic fields due to $\bar{M}_2$.

The resulting electric and magnetic fields in the cylindrical waveguide can be expressed as:

$$\bar{E}_2^+ = \sum_n A_n^+ \left(\bar{e}_n + \hat{\mathbf{z}} e_{zn}\right)\cdot e^{-j\beta_n z}$$

$$\bar{H}_2^+ = \sum_n A_n^+ \left(\bar{h}_n + \hat{\mathbf{z}} h_{zn}\right)\cdot e^{-j\beta_n z}$$

where $\left(\bar{e}_n + \hat{\mathbf{z}} e_{zn}\right)$ and $\left(\bar{h}_n + \hat{\mathbf{z}} h_{zn}\right)$ are the normalized fields of the nth mode and A_n^+ represents the amplitude of the nth mode.

Part II

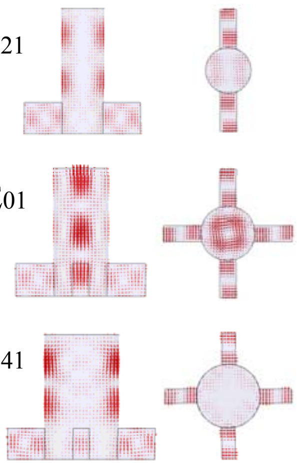
Synthesize the desired mode:
How many dipoles are needed?

Side view Top view

(a) TE21

(b) TE01

(c) TE41



Side view of the magnetic field and top view of the electric field for (a) TE_{21} mode, (b) TE_{01} mode, and (c) TE_{41} mode. The field profiles are calculated using HFSS. Two couplings can excite TE_{21} modes, while four couplings with proper phase control can excite either TE_{01} or TE_{41} mode.

Part II

Synthesize the desired mode:
Demonstration

How the magnetic dipoles are formulated? The case for the TE_{01} mode is demonstrated.

$$\bar{P}_m = -\alpha_m H_z \hat{\mathbf{z}}\delta(\rho-\rho_0)\delta(z-z_0)\cdot\left[\delta(\phi-0)+\delta(\phi-\frac{\pi}{2})+\delta(\phi-\pi)+\delta(\phi-\frac{3\pi}{2})\right]$$

where H_z is the tangential H field (along z direction) and α_m is a constant depending on the geometry of the hole. Four magnetic dipoles evenly distributed in the circumference are assumed.

With the lengthy calculation, we would find the solution for the amplitude of the nth mode:

$$A_n^+ = \frac{1}{P_n}\int_V (\hat{\mathbf{z}}h_{zn})\cdot j\omega\mu_0 \bar{P}_m e^{j\beta_n z} dV = -\frac{1}{P_n}\alpha_m H_{z0} j\omega\mu_0 \cdot\left[h_{zn}(0)+h_{zn}(\frac{\pi}{2})+h_{zn}(\pi)+h_{zn}(\frac{3\pi}{2})\right]$$

where $P_n = 2\oint_{S_0} ((\bar{e}_n \times \bar{h}_n)\cdot\hat{\mathbf{z}})ds$ is a normalization constant proportional to the power flow of the nth mode and $h_{zn}(\phi) = (A\sin m\phi + B\cos m\phi)J_m(p'_{mn})$.

Analyze mode purity

Using the above mentioned approaches, we could synthesize the desired mode. However, some unwanted modes may inevitably be generated and result in a serious mode-competition problem in a gyrotron experiment. Therefore, mode purity is an important issue in the design of a mode converter.

- TE_{21} mode

$$\overline{P}_m = -\alpha_m H_z \delta(\rho-\rho_0)\delta(z-z_0)\hat{\mathbf{z}}\cdot[\delta(\phi-0)+\delta(\phi-\pi)]$$

$$A^+_{\mathrm{TE21,B}} = -\frac{1}{P_{21}}\alpha_m H_{z0} j\omega\mu_0 \cdot 2BJ_m(p'_{mn})$$

$$A^+_{\mathrm{TE21,A}} = 0,\ A^+_{\mathrm{TE11,A}} = A^+_{\mathrm{TE11,B}} = 0,\ A^+_{\mathrm{TM01}} = 0$$

The arrangement of dual inputs only excites the linear polarization of the TE_{21} mode, e.g. , and luckily, it eliminates non two-fold symmetric modes. So the major parasitic mode TE_{11} mode cannot be excited.

Analyze mode purity

The field pattern of TE_{01} mode is azimuthally symmetric. We can choose dual feeds or quad feeds. Quad-feed structure is sufficient to eliminate these two unwanted modes. The magnetic dipoles can be expressed as:

- TE_{01} mode

$$\overline{P}_m = -\alpha_m H_z \delta(\rho-\rho_0)\delta(z-z_0)\hat{\mathbf{z}}\cdot[\delta(\phi-0)+\delta(\phi-\frac{\pi}{2})+\delta(\phi-\pi)+\delta(\phi-\frac{3\pi}{2})]$$

$$A^+_{\mathrm{TE01}} = \frac{1}{P_n}\int_V (\hat{z}h_{zn})\cdot j\omega\mu_0 \overline{P}_m e^{j\beta_n z} dv = -\frac{4B}{P_n}\alpha_m H_{z0} j\omega\mu_0 J_0(p'_{01})$$

$$A^+_{\mathrm{TE11,A}} = A^+_{\mathrm{TE11,B}} = 0,\ A^+_{\mathrm{TE21,\,A}} = A^+_{\mathrm{TE21,\,B}} = 0$$

$$A^+_{\mathrm{TM01}} = 0,\ A^+_{\mathrm{TM11,A}} = A^+_{\mathrm{TM11,B}} = 0$$

The quad-feed arrangement avoids exciting the parasitic modes. So the mode purity would be very

Analyze mode purity

The field pattern of TE_{41} mode suggests a quad-feed structure. The magnetic dipoles can be expressed as:

- TE_{41} mode

$$\overline{P}_m = -\alpha_m H_z \delta(\rho-\rho_0)\delta(z-z_0)\hat{\mathbf{z}}\cdot[\delta(\phi-0)+\delta(\phi-\frac{\pi}{2})+\delta(\phi-\pi)+\delta(\phi-\frac{3\pi}{2})]$$

$$A^+_{\mathrm{TE41,B}} = -\frac{4B}{P_{41}}\alpha_m H_{z0} j\omega\mu_0 J_4(p'_{41})$$

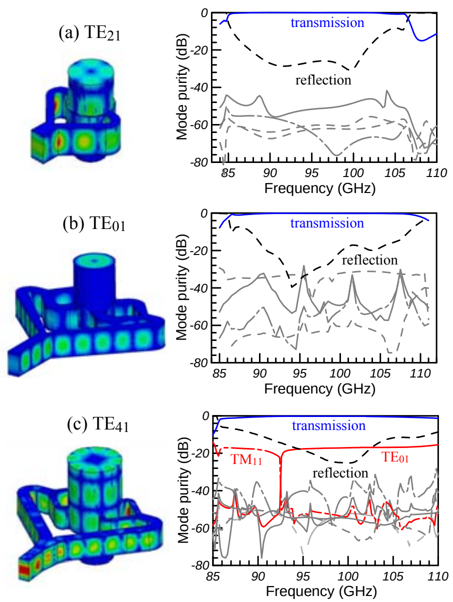
$$A^+_{\mathrm{TE01}} = -\frac{4B}{P_{01}}\alpha_m H_{z0} j\omega\mu_0 J_0(p'_{01}) \neq 0$$

$$P_{41}^{total} = \frac{Z_0 k_0 \beta_{41}\pi}{2k_{c,41}^4 \varepsilon_{04}}({p'_{41}}^2 - 4^2)J_4^2(p'_{41})16\cdot(\alpha_m H_{z0} j\omega\mu_0)^2 J_4^2(p'_{41})$$

$$P_{01}^{total} = \frac{Z_0 k_0 \beta_{01}\pi}{2k_{c,01}^4 \varepsilon_{00}}({p'_{01}}^2 - 0^2)J_0^2(p'_{01})16\cdot(\alpha_m H_{z0} j\omega\mu_0)^2 J_0^2(p'_{01})$$

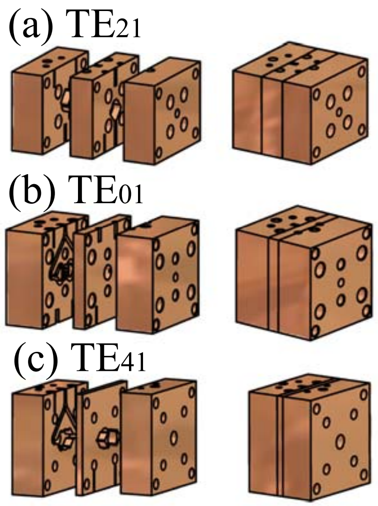
The power ratio is: $\frac{P_{01}^{total}}{P_{41}^{total}} = \frac{\beta_{41}{p'_{01}}^4\varepsilon_{04}}{\beta_{01}{p'_{41}}^4\varepsilon_{00}}\frac{({p'_{01}}^2-0^2)}{({p'_{41}}^2-4^2)}\frac{J_0^4(p'_{01})}{J_4^4(p'_{41})}$

Simulation Results Using HFSS



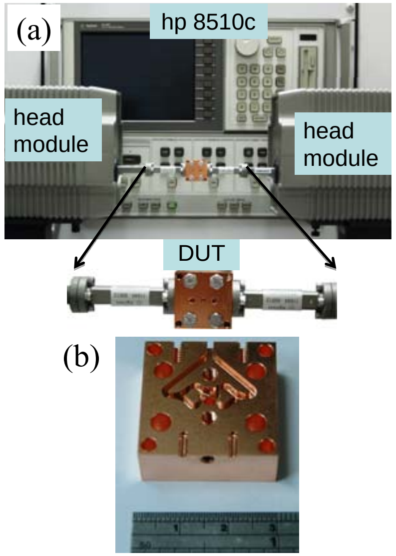
Calculated field pattern and the mode purity for (a) TE_{21} mode, (b) TE_{01} mode, and (c) TE_{41} mode. The transmissions of the desired modes are shown in blue lines and the reflections are shown in dashed lines. The minor and major parasitic modes are shown in gray and red, respectively.

Design and Fabrication



Two identical converters joined back-to-back for the three modes of interest. Each set consists of three pieces made of oxygen-free high-conductivity copper.

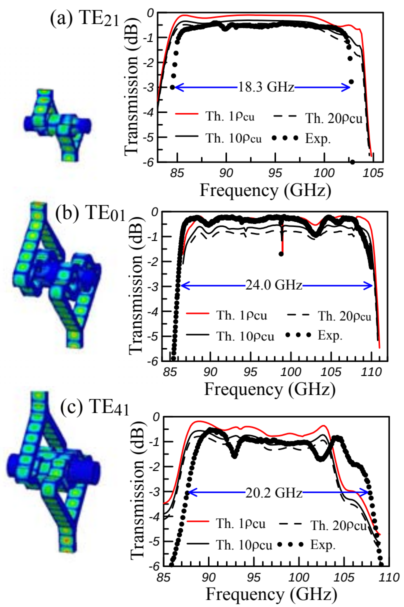
Experimental Setup



(a) Photo of the experimental setup for directly measuring the back-to-back transmission. The test set is connected to the head modules which are calibrated. The test set is enlarged.

(b) Slotted plate of TE_{01} converter as an example.

Simulated and Measured Results



Field pattern and transmission for (a) TE_{21}, (b) TE_{01}, and (c) TE_{41}, respectively. The field patterns are HFSS's simulation results. The solid dots represent the measured results and the lines are the simulations. Three different resistivity of the copper are displayed.

Conclusion

Using Y-type power divider to excite pure TE_{mn} modes was reported. Three mode converters were designed, fabricated, and tested. These converters feature a high back-to-back converting efficiency, high mode purity, broad bandwidth, and compact converting section. Such a converter is suitable for a variety of applications, especially the gyrotrons to generate low-terahertz radiation [34]. At higher frequency, like terahertz region, the micro-fabrication technique is need, which is currently under investigation. The authors would like to thank the technical support of Mr. C. Lee of Ansoft, Taiwan Branch.

T. H. Chang, C. H. Li, C. N. Wu, and C. F. Yu, " Generating pure circular TE_{mn} modes using Y-type power dividers", IEEE Trans. Microwave Theory Tech. 58, 1543 (2010).

The End of Chap. 4