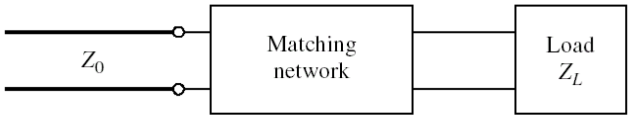
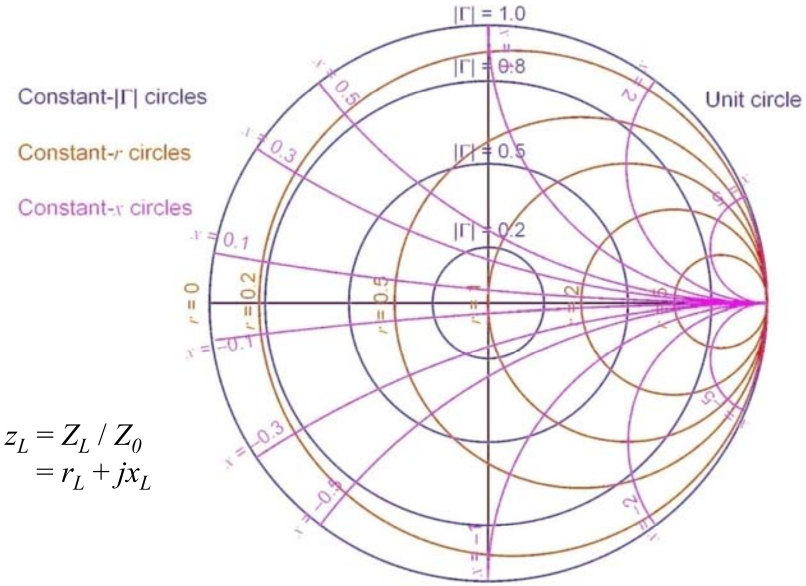
# Chapter 5 Impedance Matching and Tuning




- The matching network is *ideally lossless*, and is usually designed so that the impedance seen looking into the matching network is $Z_0$.
- Only if Re[$Z_L$] ≠ 0, a matching network can always be found.
- The quarter-wave impedance transformer is for read load.
- Important factors in selecting matching networks:
  (1)complexity, (2)bandwidth, (3)implementation, (4)adjustability.



# Review of Smith Chart

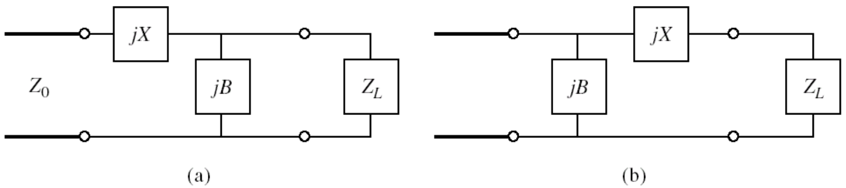



$z_L = Z_L / Z_0$
$= r_L + jx_L$



# 5.1 Lumped Element Matching (L-networks)

*L*-type or *L*-network matching is the simplest matching network.




(1) $z_L = Z_L/Z_0$ is inside the $r$=1 circle ($r_L > 1$).

(2) $Z_L$ is shunt with $jB$, then series with $jX$.

(1) $z_L = Z_L/Z_0$ is outside the $r$=1 circle ($r_L < 1$).

(2) $Z_L$ is series with $jB$, then shunt with $jX$.



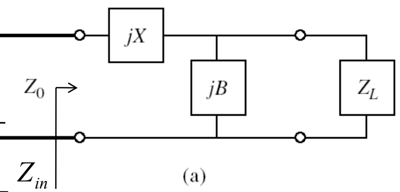
# Analytical Solutions ($r_L$>1)

- When $r_L > 1$, let $Z_L = R_L + jX_L$ and $z_L = Z_L / Z_0 = r_L + jx_L$

$$Z_{in} = jX + \cfrac{1}{jB + \cfrac{1}{R_L + jX_L}} \equiv Z_0$$

$$\begin{cases} B(XR_L - X_L Z_0) = R_L - Z_0 \\ X(1 - BX_L) = BZ_0 R_L - X_L \end{cases}$$

$$\Rightarrow \begin{cases} B = \dfrac{X_L \pm \sqrt{\dfrac{R_L}{Z_0}}\sqrt{R_L^2 + X_L^2 - Z_0 R_L}}{R_L^2 + X_L^2} \\ X = \dfrac{1}{B} + \dfrac{X_L Z_0}{R_L} - \dfrac{Z_0}{BR_L} \end{cases}$$




- Note that when $r_L > 1$, the square root in $B$ has real results.

# Analytical Solutions ($r_L$<1)

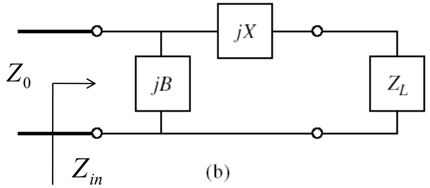
- When $r_L < 1$, let $Z_L = R_L + jX_L$ and $z_L = Z_L / Z_0 = r_L + jx_L$

$$\frac{1}{Z_{in}} = jB + \frac{1}{R_L + j(X + X_L)} \equiv \frac{1}{Z_0}$$

$$BZ_0(X + X_L) = Z_0 - R_L$$

$$(X + X_L) = BZ_0 R_L$$

$$\Rightarrow \begin{cases} B = \pm \dfrac{\sqrt{(Z_0 - R_L)/R_L}}{Z_0} \\ X = -X_L \pm \sqrt{R_L(Z_0 - R_L)} \end{cases}$$

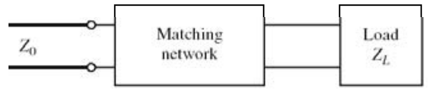



- Two analytical solutions are physically realizable.
- One of the two solutions may result in smaller reactive elements, and may have better matching, bandwidth, or better *SWR* on the line.



# Example 5.1 *L*-Section Impedance Matching

$Z_L = 200 - j100\ \Omega$, $Z_0 = 100\ \Omega$,
$f = 500$ MHz, $z_L = Z_L/Z_0 = 2 - j1$,
$r_L = 2 > 1$ inside the $r = 1$ circle.




Exact solutions:

$$B = \frac{X_L \pm \sqrt{\frac{R_L}{Z_0}}\sqrt{R_L^2 + X_L^2 - R_L Z_0}}{R_L^2 + X_L^2}$$

$$= \frac{-100 \pm \sqrt{2}\sqrt{200^2 + 100^2 - 100 \times 200}}{200^2 + 100^2} = \begin{cases} 2.899 \times 10^{-3}\,\Omega^{-1} \\ -6.899 \times 10^{-3}\,\Omega^{-1} \end{cases}$$

$$X = \frac{1}{B} + \frac{X_L Z_0}{R_L} - \frac{Z_0}{BR_L} = \begin{cases} 122.47\Omega \\ -122.47\Omega \end{cases}$$

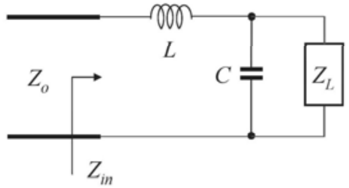


# Example 5.1 *L*-Section Impedance Matching

The two solutions:

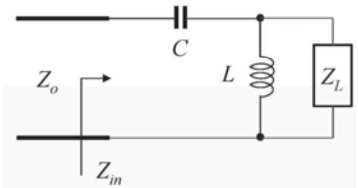
(1) $C = \dfrac{B}{\omega} = \dfrac{2.899 \times 10^{-3}}{2\pi \times 500 \times 10^6} = 0.9228$ pF

$L = \dfrac{X}{\omega} = \dfrac{122.47}{2\pi \times 500 \times 10^6} = 38.98$ nH




(2) $C = -\dfrac{1}{X\omega} = \dfrac{1}{122.47 \times 2\pi \times 500 \times 10^6}$
$= 2.599$ pF

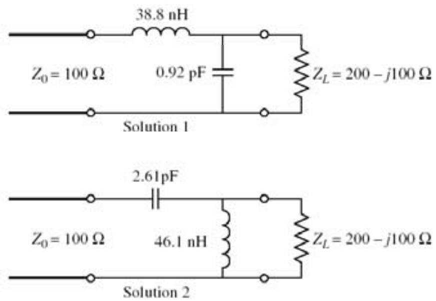
$L = -\dfrac{1}{B\omega} = \dfrac{1}{6.899 \times 10^{-3} \times 2\pi \times 500 \times 10^6}$
$= 46.14$ nH



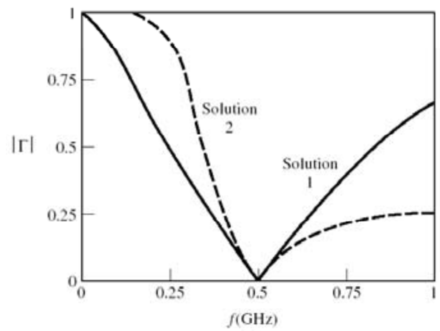



# Example 5.1 *L*-Section Impedance Matching

The two solutions:

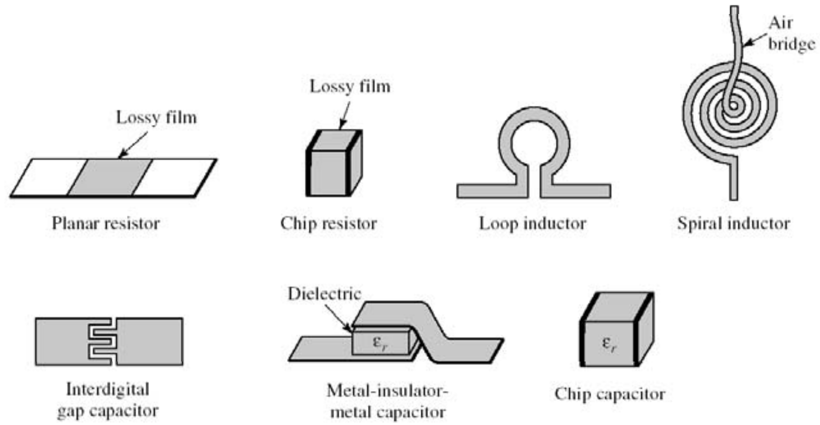



(b)




(c)

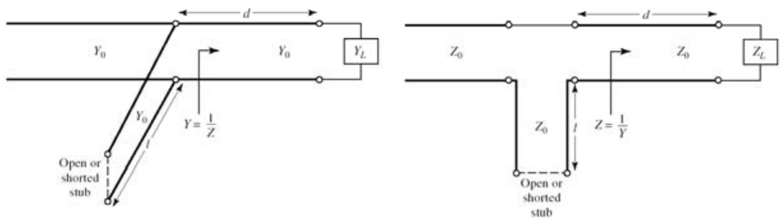
# Lumped Elements for Microwave Integrated Circuits (MICs)






# 5.2 Single-Stub Tuning

Shunt-stub tuning                Series-stub tuning




- No lumped element is required. Convenient for MIC fabrication.
- Characteristic impedances of the lines and the stub can be different.
- Keep the matching stub as close as possible to the load.
- Solution procedure for *shunt-stub* and *series-stub tunings*.
  (1) Locate $z_L$ ($y_L$) on the Smith Chart.
  (2) Move along the const-Γ circle with *2d/λ wavelengths* to reach the $g = 1$ ($r = 1$) circle.
  (3) Shunt with a susceptance or series with a reactance to cancel the imaginary part.



# Shunt-Stub Tuning Analytical Solution

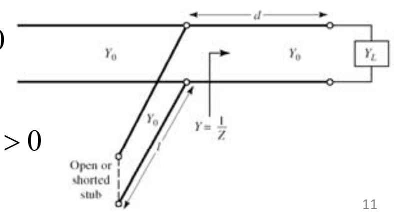
$$Z_{in}\Big|_{no\ stub} = Z_0\frac{Z_L + jZ_0 t}{Z_0 + jZ_L t}\Bigg|_{t=\tan\beta d} = \frac{1}{Y_0}\frac{R_L + j(X_L + Z_0 t)}{Z_0 - X_L t + jR_L t} \equiv \frac{1}{G_{in} + jB_{in}}$$

$$G_{in} = \frac{R_L(1+t^2)}{R_L^2 + (X_L + Z_0 t)^2}, \quad B_{in} = Y_0\frac{R_L^2 t - (Z_0 - X_L t)(Z_0 t + X_L)}{R_L^2 + (X_L + Z_0 t)^2}$$

$$G_{in} = Y_0 \Rightarrow Z_0 R_L(1+t^2) = R_L^2 + (X_L + Z_0 t)^2 \quad \text{real part}$$

If $R_L = Z_0$, and $t = \tan\beta d = -\dfrac{X_L}{2Z_0}$

$$\Rightarrow \frac{d}{\lambda} = \begin{cases} \dfrac{1}{2\pi}\tan^{-1}\left(-\dfrac{X_L}{2Z_0}\right), & X_L < 0 \\ \dfrac{1}{2\pi}\left[\pi + \tan^{-1}\left(-\dfrac{X_L}{2Z_0}\right)\right], & X_L > 0 \end{cases}$$



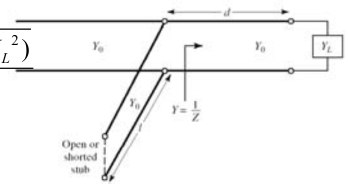



# Shunt-Stub Tuning Analytical Solution

If $R_L \neq Z_0$,

$$t = \frac{X_L Z_0 \pm \sqrt{(X_L Z_0)^2 - Z_0(R_L - Z_0)(R_L Z_0 - R_L^2 - X_L^2)}}{Z_0(R_L - Z_0)}$$

$$= \frac{X_L \pm \sqrt{(\frac{R_L}{Z_0})^2[(R_L - Z_0)^2 + X_L^2]}}{R_L - Z_0}$$




here $t = \tan\beta l$

Let the stub susceptance $B_{stub}$ (or $B_s$) = $-B_{in}$,

(a) open-stub: $Z_{stub} = -jZ_0/t$ and $Y_{stub} = +jY_0 t = jB_{stub} \equiv jB_s$

$$\frac{l_o}{\lambda} = \frac{1}{2\pi}\tan^{-1}(\frac{B_s}{Y_0}) = -\frac{1}{2\pi}\tan^{-1}\left(\frac{B_{in}}{Y_0}\right)$$

(b) Short-stub: $Z_{stub} = jZ_0 t$ and $Y_{stub} = -jY_0/t = jB_{stub} \equiv jB_s$

$$\frac{l_s}{\lambda} = -\frac{1}{2\pi}\tan^{-1}(\frac{Y_0}{B_s}) = \frac{1}{2\pi}\tan^{-1}\left(\frac{Y_0}{B_{in}}\right)$$

If $l_o < 0$ or $l_s < 0$, $\lambda/2$ must be added to have a realistic result.

# Example 5.2 Shunt Single-Stub Tuning

Match $Z_L = 15 + j10\ \Omega$ to a 50-$\Omega$ line. Use a shunt open-stub.

Sol: $R_L \neq Z_0$,

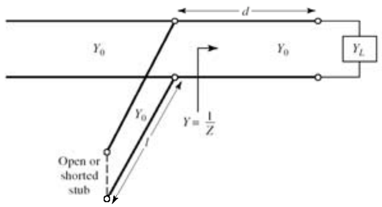
$$t = \frac{X_L \pm \sqrt{(\frac{R_L}{Z_0})^2\left[(R_L - Z_0)^2 + X_L{}^2\right]}}{R_L - Z_0} = -\frac{10 \pm \sqrt{397.5}}{35}$$

$$t_1 = -\frac{10+\sqrt{397.5}}{35} \Rightarrow \frac{d}{\lambda} = \frac{1}{2\pi}(\pi + \tan^{-1} t_1) = 0.387$$

$$\Rightarrow B_{in} = \ldots \Rightarrow \frac{l_o}{\lambda} = -\frac{1}{2\pi}\tan^{-1}\left(\frac{B_{in}}{Y_0}\right)$$

$$t_2 = -\frac{10-\sqrt{397.5}}{35} \Rightarrow \frac{d}{\lambda} = \frac{1}{2\pi}\tan^{-1} t_2 = 0.044$$

$$\Rightarrow B_{in} = \ldots \Rightarrow \frac{l_o}{\lambda} = -\frac{1}{2\pi}\tan^{-1}\left(\frac{B_{in}}{Y_0}\right)$$

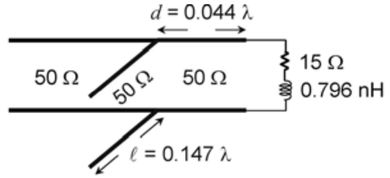




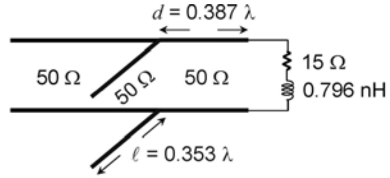

# Example 5.2 Shunt Single-Stub Tuning (Cont'd)

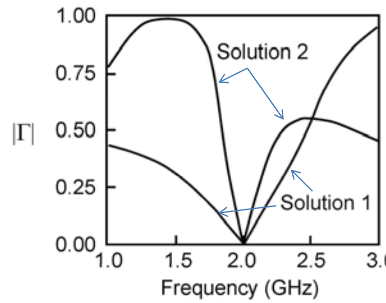
Use a shunt open-stub to match $Z_L = 15 + j10\ \Omega$ line.

Solution 1:




Solution 2:

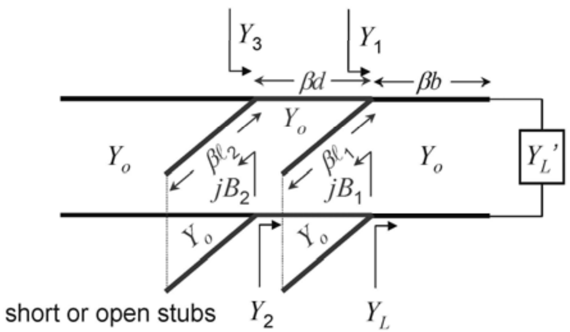





# 5.3 Double-Stub Tuning – Analytical Solution

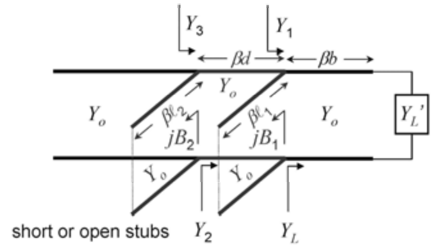
- In single-stub tuning, the length $d$ is not tunable.
- If this causes difficulty in circuit implementation, use double-stub tuning.




$$Y_L = Y_0 \frac{Y_L' + jY_0 \tan\beta b}{Y_0 + jY_L' \tan\beta b} \equiv G_L + jB_L$$



# 5.3 Double-Stub Tuning – Analytical Solution




$$Y_1 = G_L + j(B_L + B_1)$$

$$Y_2 = Y_0 \frac{G_L + j(B_L + B_1 + Y_0 t)}{Y_0 + jt(G_L + jB_L + jB_1)}, \quad t = \tan\beta d$$

$$\mathrm{Re}[Y_2] = Y_0, \quad \mathrm{Im}[Y_2] = -B_2$$

$$G_L{}^2 - Y_0 \frac{1+t^2}{t^2} G_L + \frac{(Y_0 - B_L t - B_1 t)^2}{t^2} = 0$$

$$G_L = Y_0 \frac{1+t^2}{t^2}\left\{1 \pm \sqrt{1 - \frac{4t^2(Y_0 - B_L t - B_1 t)^2}{Y_0(1+t^2)^2}}\right\}$$

$$0 \le 4t^2(Y_0 - B_L t - B_1 t)^2 \le Y_0{}^2(1+t^2)^2$$

$$0 \le G_L \le Y_0 \frac{1+t^2}{2t^2} = \frac{Y_0}{\sin^2\beta d}$$

$G_L$ has an upper limit.

Given $d$, one can obtain:

$$B_1 = -B_L + \frac{Y_0 + \sqrt{(1+t^2)G_L Y_0 - G_L{}^2 t^2}}{t}$$

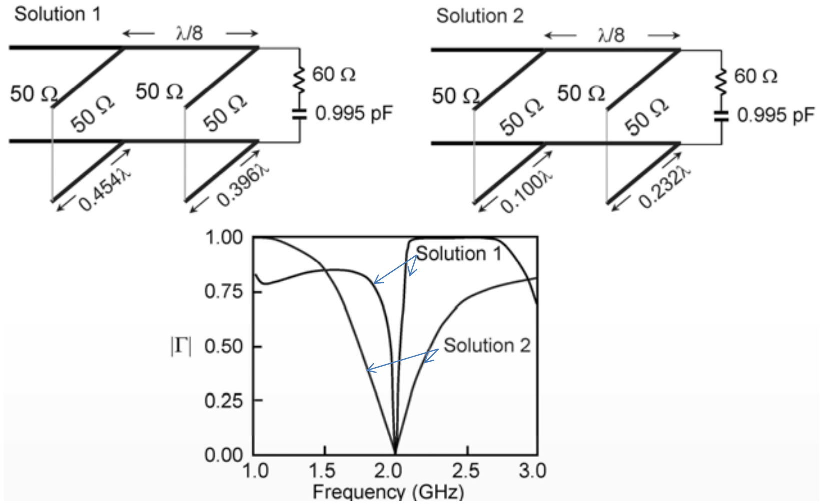
$$B_1 = Y_0 \frac{G_L \pm \sqrt{(1+t^2)G_L Y_0 - G_L{}^2 t^2}}{t}$$

$$\frac{\ell_0}{\lambda} = \frac{1}{2\pi}\tan^{-1}\left(\frac{B}{Y_0}\right), \quad B = B_1, B_2$$

$$\frac{\ell_s}{\lambda} = \frac{1}{2\pi}\tan^{-1}\left(\frac{Y_0}{B}\right), \quad B = B_1, B_2$$

# Example 5.4 Double-Stub Tuning - Performance

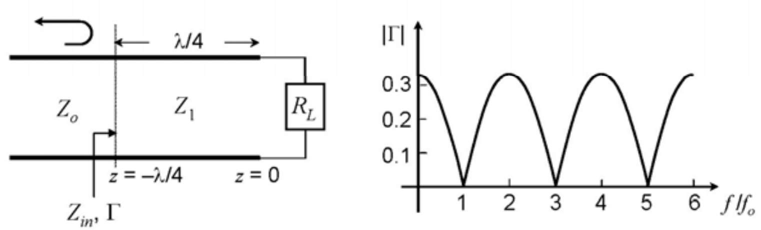
Use double-stub matching scheme to match $Z_L = 60 - j80\Omega$ at 2.0 GHz to a 50-Ω line.






# 5.4 The Quarter-Wave Transformer

- A single-section transformer may suffice for a *narrow-band* impedance matching.
- Single-section quarter-wave impedance matching $\lambda = \lambda_0 / 4$ at the desired frequency. (See Chap 2)



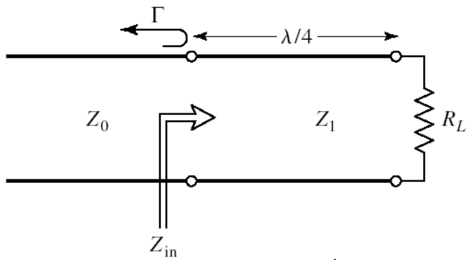

- Multisection quarter-wave transformer designs can be synthesized to yield *optimum matching* characteristics over a broad bandwith.



Review

# 2.5 The Quarter-Wave Transformer

A quarter-wave transformer is an impedance matching circuit



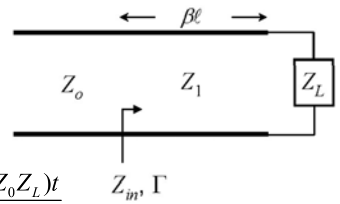

$$Z_{in} = Z_1 \left. \frac{R_L + jZ_1 \tan \beta l}{Z_1 + jR_L \tan \beta l} \right|_{\beta l = \frac{\pi}{2}} = \frac{Z_1^2}{R_L}$$

$$\Gamma_{in} = \frac{Z_{in} - Z_0}{Z_{in} + Z_0}$$

If $\Gamma_{in} = 0$ is required, $Z_{in} = Z_0$, then $Z_1 = \sqrt{Z_0 R_L}$



# Estimate the Bandwidth of a Single-Section Impedance Transformer




$$Z_1 = \sqrt{Z_0 Z_L}$$

$$Z_{in} = Z_1 \frac{Z_L + jZ_1 t}{Z_1 + jZ_L t}, \quad t = \tan \beta \ell = \tan \theta$$

$$\Gamma = \frac{Z_{in} - Z_0}{Z_{in} + Z_0} = \frac{Z_1 \dfrac{Z_L + jZ_1 t}{Z_1 + jZ_L t} - Z_0}{Z_1 \dfrac{Z_L + jZ_1 t}{Z_1 + jZ_L t} + Z_0} = \frac{Z_1(Z_L - Z_0) + j(Z_1^2 - Z_0 Z_L)t}{Z_1(Z_L + Z_0) + j(Z_1^2 + Z_0 Z_L)t}$$

$$= \frac{Z_L - Z_0}{Z_L + Z_0 + j2t\sqrt{Z_0 Z_L}} \Rightarrow |\Gamma| = \frac{|Z_L - Z_0|}{\sqrt{(Z_L + Z_0)^2 + 4t^2 Z_0 Z_L}} = \frac{1}{\sqrt{1 + \dfrac{4Z_L Z_0}{(Z_L - Z_0)^2} \sec^2 \theta}}$$

When $\theta \approx \dfrac{\pi}{2}$, $\quad |\Gamma| \approx \dfrac{|Z_L - Z_0|}{2\sqrt{Z_L Z_0}} |\cos \theta|$

Set a max $|\Gamma| = \Gamma_m$, or $\Gamma_m = \dfrac{|Z_L - Z_0|}{2\sqrt{Z_L Z_0}} |\cos \theta_m|$

# Bandwidth of the Matching Transformer

$$\Delta\theta = 2(\pi/2 - \theta_m)$$

$$\theta_m: \quad \frac{1}{\Gamma_m^2} = 1 + \frac{4Z_0Z_L}{(Z_0 - Z_L)^2}\sec^2\theta_m$$

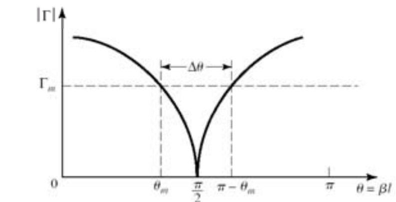
$$\theta \equiv \frac{\pi}{2}\frac{f}{f_0}, \quad \theta_m = \frac{\pi}{2}\frac{f_m}{f_0}, \quad f_m = \frac{2\theta_m}{\pi} f_0$$

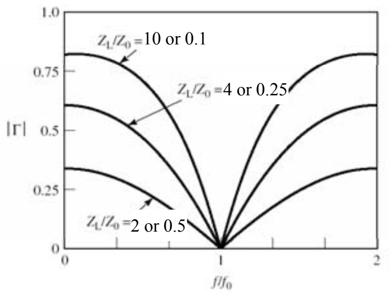
$$\cos\theta_m = \frac{\Gamma_m}{\sqrt{1-\Gamma_m^2}}\frac{2\sqrt{Z_0Z_L}}{|Z_0 - Z_L|}$$

Fractional bandwidth :

$$\frac{\Delta f}{f_0} = \frac{2(f_0 - f_m)}{f_0} = 2 - \frac{4\theta_m}{\pi}$$

$$= 2 - \frac{4}{\pi}\left(\cos^{-1}\left[\frac{\Gamma_m}{\sqrt{1-\Gamma_m^2}}\frac{2\sqrt{Z_0Z_L}}{|Z_0 - Z_L|}\right]\right)$$






# Example 5.5
# Single-Section Quarter-Wave Transformer Bandwidth

$Z_L = 10\ \Omega$, $Z_0 = 100\ \Omega$, $SWR = 1.2$

Sol:

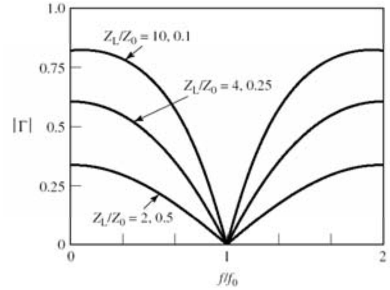
$$Z_1 = \sqrt{Z_0Z_L} = 31.6\Omega$$

$$\Gamma_m = \frac{SWR - 1}{SWR + 1} = 0.1$$

$$SWR = \frac{V_{max}}{V_{min}} = \frac{V_0^+ + V_0^-}{V_0^+ - V_0^-} = \frac{1+\Gamma}{1-\Gamma}$$

$$\frac{\Delta f}{f_0} = 2 - \frac{4}{\pi}\left(\cos^{-1}\left[\frac{\Gamma_m}{\sqrt{1-\Gamma_m^2}}\frac{2\sqrt{Z_0Z_L}}{|Z_0 - Z_L|}\right]\right)$$

$$= 2 - \frac{4}{\pi}\left(\cos^{-1}\left[\frac{0.1}{\sqrt{0.99}}\frac{2\times 31.6}{90}\right]\right) = 2 - \frac{6}{\pi} \approx 9\%$$

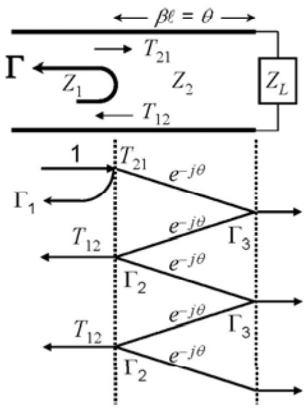





# 5.5 Theory of Small Reflections

- For applications requiring more bandwidth than that a single quarter-wave section can provide, multi-section transformers can be used.

(1) Single-Section Transformer




$$\Gamma_1 = \frac{Z_2 - Z_1}{Z_2 + Z_1} = -\Gamma_2$$

$$\Gamma_3 = \frac{Z_L - Z_2}{Z_L + Z_2}$$

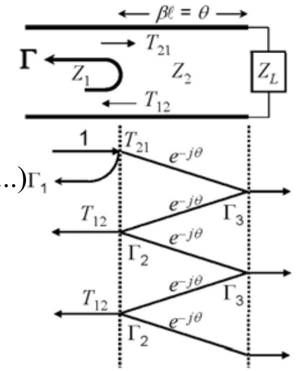
$$T_{21} = 1 + \Gamma_1 = \frac{2Z_2}{Z_2 + Z_1}$$

$$T_{12} = 1 + \Gamma_2 = \frac{2Z_1}{Z_2 + Z_1}$$



# 5.5 Theory of Small Reflections

(1) Multi-Reflections




$\Gamma$

$$= \Gamma_1 + T_{12}T_{21}\Gamma_3 e^{-j2\theta} + T_{12}T_{21}\Gamma_3^2\Gamma_2 e^{-j4\theta} + \ldots$$

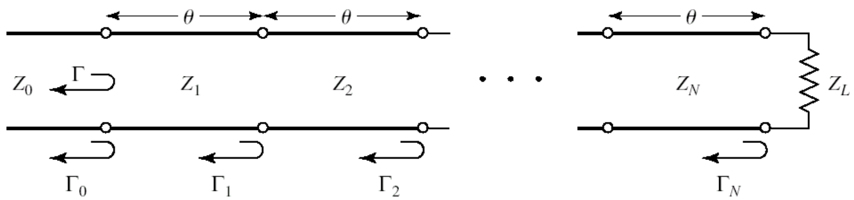
$$= \Gamma_1 + T_{12}T_{21}\Gamma_3 e^{-j2\theta}(1 + \Gamma_3\Gamma_2 e^{-j2\theta} + (\Gamma_3\Gamma_2)^2 e^{-j4\theta} + \ldots)\Gamma_1$$

$$= \Gamma_1 + T_{12}T_{21}\Gamma_3 e^{-j2\theta}\sum_{n=0}^{\infty}(\Gamma_3\Gamma_2)^n e^{-j2n\theta}$$

$$= \Gamma_1 + \frac{T_{12}T_{21}\Gamma_3 e^{-j2\theta}}{1 - \Gamma_3\Gamma_2 e^{-j2\theta}} = \frac{\Gamma_1 + \Gamma_3 e^{-j2\theta}}{1 + \Gamma_1\Gamma_3 e^{-j2\theta}}$$

- If $|\Gamma_1\Gamma_3|$ is small, $\Gamma \approx \Gamma_1 + \Gamma_3 e^{-2j\theta}$
- $\Gamma \approx$ the reflection from the initial discontinuity between $Z_1$ and $Z_2$ + the first reflection from the discontinuity between $Z_2$ and $Z_L$

# Multisection Transformer

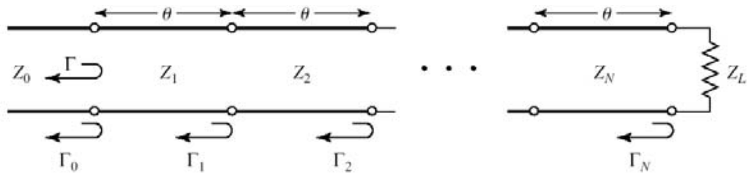



- $N$ commensurate (equal-length) sections of transmission lines. Let the total reflection be Γ.
- Assume that all $Z_n$ increase or decrease monotonically, $Z_L$ is real, and "*The theory of small reflections*" holds.
- It can be validated that

$$\Gamma_0 = \frac{Z_1 - Z_0}{Z_1 + Z_0}, \quad \Gamma_1 = \frac{Z_2 - Z_1}{Z_2 + Z_1}, \quad \Gamma_2 = \frac{Z_3 - Z_2}{Z_3 + Z_2}, \quad \ldots \quad \Gamma_N = \frac{Z_L - Z_N}{Z_L + Z_N}$$



# Multisection Transformer




$$\Gamma \approx \Gamma_0 + \Gamma_1 e^{-j2\theta} + \Gamma_2 e^{-j4\theta} + \ldots + \Gamma_N e^{-j2N\theta}$$

$$= e^{-jN\theta}[\Gamma_0(e^{jN\theta} + e^{-jN\theta}) + \Gamma_1(e^{j(N-2)\theta} + e^{-j(N-2)\theta}) + \ldots] \quad \text{(if symmetry)}$$

$$= 2e^{-jN\theta}\begin{cases}\left[\Gamma_0 \cos N\theta + \Gamma_1 \cos(N-2)\theta + \ldots + \Gamma_{N/2}\right], & N = even \\ \left[\Gamma_0 \cos N\theta + \Gamma_1 \cos(N-2)\theta + \ldots + \Gamma_{(N-1)/2} \cos\theta\right], & N = odd\end{cases}$$

- We can synthesize any desired $|\Gamma|$ response as a function of frequency or $\theta$, by properly choosing the $\Gamma_n$'s and enough number of sections $N$.
- The most commonly used passband responses are:
  (1) Binomial or maximally flat response, and
  (2) Chebyshev or equal-ripple response.



# How to Find the Corresponding Reflection?

Theory of Samll Reflection:

$$\Gamma(\theta) \approx \Gamma_0 + \Gamma_1 e^{-j2\theta} + \Gamma_2 e^{-j4\theta} + \ldots + \Gamma_N e^{-j2N\theta}$$

Binomial Multi-Section Transformer:

$$\Gamma(\theta) = A(1 + e^{-2j\theta})^N$$

Chebyshev Multi-Section Transformer:

$$\Gamma(\theta) = AT_n(\sec\theta_m \cos\theta)$$



# 5.6 Binomial Multisection Matching Transformer

- Maximally flat response: the response is *as flat as possible* near the design frequency. Also called Binomial matching transformer.
- Let $\Gamma(\theta) = A(1 + e^{-2j\theta})^N \Leftarrow$ Assumed artificially

$$|\Gamma(\theta)| = 2^N |A| |\cos\theta|^N$$

$$\Gamma(0) = 2^N A = \frac{Z_L - Z_0}{Z_L + Z_0} \Rightarrow A = \frac{\Gamma(0)}{2^N}$$

$$\Gamma(\theta) = A(1 + e^{-2j\theta})^N = A\sum_{n=0}^{N} C_n^N e^{-j2n\theta}, \quad C_n^N = \frac{N!}{(N-n)!n!}$$

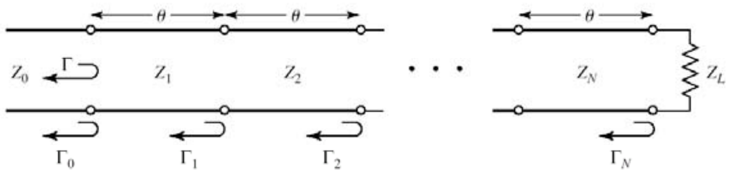
$$= \Gamma_0 + \Gamma_1 e^{-j2\theta} + \Gamma_2 e^{-j4\theta} + \ldots + \Gamma_N e^{-j2N\theta}$$

$$\Gamma_n = AC_n^N = \frac{\Gamma(0)}{2^N} C_n^N$$

$N$! ("enn factorial")
"Four factorial" is written as "4!"

Properly configure the impedances to synthesize the needed reflections $\Gamma_n$.

# Binomial Multisection Matching Transformer




$$\Gamma_n = \frac{Z_{n+1} - Z_n}{Z_{n+1} + Z_n} \approx \frac{1}{2} \ln \frac{Z_{n+1}}{Z_n}, \quad \ln x \approx 2\frac{x-1}{x+1}, \quad |x-1| << 1$$

$$\ln \frac{Z_{n+1}}{Z_n} \approx 2\Gamma_n = 2AC_n^N = \frac{1}{2^{N-1}} C_n^N \frac{Z_L - Z_0}{Z_L + Z_0} \approx \frac{1}{2^N} C_n^N \ln \frac{Z_L}{Z_0}$$

$$\frac{Z_{n+1}}{Z_n} \approx \left( \frac{Z_L}{Z_0} \right)^{C_n^N / 2^N}$$

- Exact results for $Z_{n-1} / Z_0$ for $N$ = 2 thru 6 are given in Table 5.1.



# Binomial Transformer Design Table 5.1

**TABLE 5.1 Binomial Transformer Design**

| $Z_L/Z_0$ | $N = 2$ | | $N = 3$ | | | $N = 4$ | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $Z_1/Z_0$ | $Z_2/Z_0$ | $Z_1/Z_0$ | $Z_2/Z_0$ | $Z_3/Z_0$ | $Z_1/Z_0$ | $Z_2/Z_0$ | $Z_3/Z_0$ | $Z_4/Z_0$ |
| 1.0 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 1.5 | 1.1067 | 1.3554 | 1.0520 | 1.2247 | 1.4259 | 1.0257 | 1.1351 | 1.3215 | 1.4624 |
| 2.0 | 1.1892 | 1.6818 | 1.0907 | 1.4142 | 1.8337 | 1.0444 | 1.2421 | 1.6102 | 1.9150 |
| 3.0 | 1.3161 | 2.2795 | 1.1479 | 1.7321 | 2.6135 | 1.0718 | 1.4105 | 2.1269 | 2.7990 |
| 4.0 | 1.4142 | 2.8285 | 1.1907 | 2.0000 | 3.3594 | 1.0919 | 1.5442 | 2.5903 | 3.6633 |
| 6.0 | 1.5651 | 3.8336 | 1.2544 | 2.4495 | 4.7832 | 1.1215 | 1.7553 | 3.4182 | 5.3500 |
| 8.0 | 1.6818 | 4.7568 | 1.3022 | 2.8284 | 6.1434 | 1.1436 | 1.9232 | 4.1597 | 6.9955 |
| 10.0 | 1.7783 | 5.6233 | 1.3409 | 3.1623 | 7.4577 | 1.1613 | 2.0651 | 4.8424 | 8.6110 |

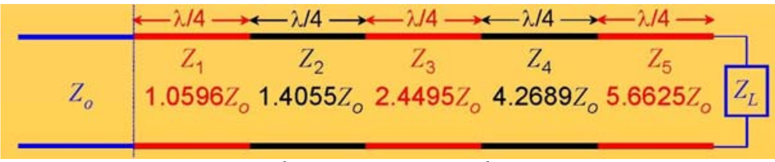
| $Z_L/Z_0$ | $N = 5$ | | | | | $N = 6$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | $Z_1/Z_0$ | $Z_2/Z_0$ | $Z_3/Z_0$ | $Z_4/Z_0$ | $Z_5/Z_0$ | $Z_1/Z_0$ | $Z_2/Z_0$ | $Z_3/Z_0$ | $Z_4/Z_0$ | $Z_5/Z_0$ | $Z_6/Z_0$ |
| 1.0 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 1.5 | 1.0128 | 1.0790 | 1.2247 | 1.3902 | 1.4810 | 1.0064 | 1.0454 | 1.1496 | 1.3048 | 1.4349 | 1.4905 |
| 2.0 | 1.0220 | 1.1391 | 1.4142 | 1.7558 | 1.9569 | 1.0110 | 1.0790 | 1.2693 | 1.5757 | 1.8536 | 1.9782 |
| 3.0 | 1.0354 | 1.2300 | 1.7321 | 2.4390 | 2.8974 | 1.0176 | 1.1288 | 1.4599 | 2.0549 | 2.6577 | 2.9481 |
| 4.0 | 1.0452 | 1.2995 | 2.0000 | 3.0781 | 3.8270 | 1.0225 | 1.1661 | 1.6129 | 2.4800 | 3.4302 | 3.9120 |
| 6.0 | 1.0596 | 1.4055 | 2.4495 | 4.2689 | 5.6625 | 1.0296 | 1.2219 | 1.8573 | 3.2305 | 4.9104 | 5.8275 |
| 8.0 | 1.0703 | 1.4870 | 2.8284 | 5.3800 | 7.4745 | 1.0349 | 1.2640 | 2.0539 | 3.8950 | 6.3291 | 7.7302 |
| 10.0 | 1.0789 | 1.5541 | 3.1623 | 6.4346 | 9.2687 | 1.0392 | 1.2982 | 2.2215 | 4.5015 | 7.7030 | 9.6228 |



# Example Binomial Transformer Design

| $Z_L/Z_0$ | $N$=5 | | | | |
|---|---|---|---|---|---|
| | $Z_1/Z_0$ | $Z_2/Z_0$ | $Z_3/Z_0$ | $Z_4/Z_0$ | $Z_5/Z_0$ |
| 1.0 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 1.5 | 1.0128 | 1.0790 | 1.2247 | 1.3902 | 1.4810 |
| 2.0 | 1.0220 | 1.1391 | 1.4142 | 1.7558 | 1.9569 |
| 3.0 | 1.0354 | 1.2300 | 1.7321 | 2.4390 | 2.8974 |
| 4.0 | 1.0452 | 1.2995 | 2.0000 | 3.0781 | 3.8270 |
| 6.0 | 1.0596 | 1.4055 | 2.4495 | 4.2689 | 5.6625 |
| 8.0 | 1.0703 | 1.4870 | 2.8284 | 5.3800 | 7.4745 |
| 10.0 | 1.0789 | 1.5541 | 3.1623 | 6.4346 | 9.2687 |

- A 5th order binomial transformer for $Z_L = 6Z_0$



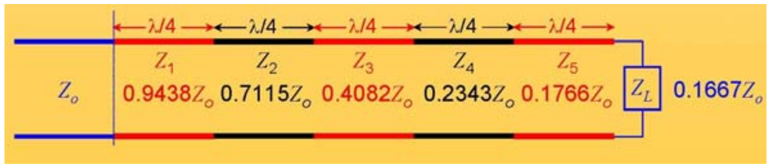

- Also note that $\dfrac{6}{5.6625} = 1.0596, \quad \dfrac{6}{4.2689} = 1.4055$



# Example Binomial Transformer Design $Z_L/Z_0 < 1$

| $Z_L/Z_0$ | N=5 | | | | |
|---|---|---|---|---|---|
| | $Z_1/Z_0$ | $Z_2/Z_0$ | $Z_3/Z_0$ | $Z_4/Z_0$ | $Z_5/Z_0$ |
| 6.0000 | 1.0596 | 1.4055 | 2.4495 | 4.2689 | 5.6625 |
| 0.1667 | 0.9438 | 0.7115 | 0.4082 | 0.2343 | 0.1766 |

- A 5th order binomial transformer for $Z_L = Z_0 / 6$



# Bandwidth of the Binomial Transformer

$\Gamma_m = 2^N |A| \cos^N \theta_m$, i.e, tolerable max $|\Gamma_m|$ over the passband.

$$\theta_m = \cos^{-1}\left(\frac{1}{2}\left(\frac{\Gamma_m}{|A|}\right)^{1/N}\right)$$

$$\frac{\Delta f}{f_0} = \frac{2(f_0 - f_m)}{f_0} = 2 - \frac{4\theta_m}{\pi} = 2 - \frac{4}{\pi}\cos^{-1}\left(\frac{1}{2}\left(\frac{\Gamma_m}{|A|}\right)^{1/N}\right)$$

Example 5.6 $N = 3$, $Z_L = 2Z_0 = 100\ \Omega$, find the BW for $\Gamma_m = 0.05$.

Sol: From Table 5.1, the required impedance $Z_n$ can be found to be

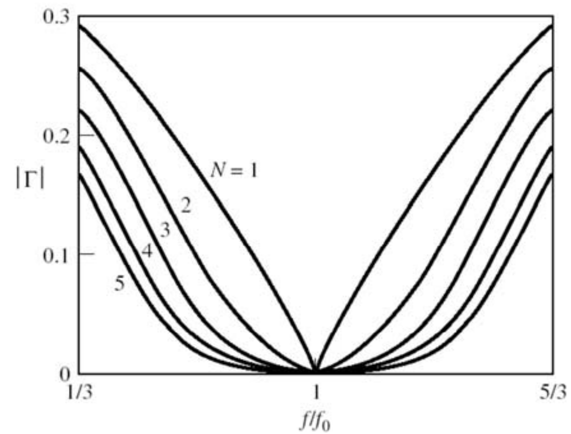
$$Z_1 = 54.5\Omega, \quad Z_2 = 70.7\Omega, \quad Z_3 = 91.7\Omega$$

$$A = 2^{-N}\Gamma(0) = \frac{1}{8}\frac{100-50}{100+50} = 0.4167$$

$$\frac{\Delta f}{f_0} = 2 - \frac{4}{\pi}\cos^{-1}\left[\frac{1}{2}\left(\frac{0.05}{0.4167}\right)^{\frac{1}{3}}\right] = 70\%$$



# Binomial Transformer's Frequency Response



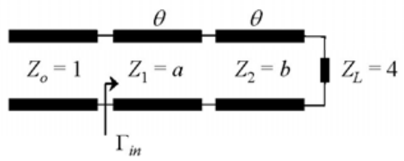

Reflection coefficient magnitude versus frequency for multisection binomial matching transformer of Ex. 5.6 with $Z_L = 2Z_0 = 100\ \Omega$ and $\Gamma_m = 0.05$.



DIY

# Example (2nd Midterm)

Design a Butterworth transformer of $N = 2$ for $Z_L = 4Z_0$. Let $\Gamma_0$, $\Gamma_1$ and $\Gamma_L$ be respectively the reflection coefficients at the $Z_0 - Z_1$, $Z_1 - Z_2$ and $Z_2 - Z_L$ junctions, and $\Gamma_0 = \Gamma_L$. (a) Based on the "theory of small reflection," find the $\Gamma_{in}$ terms of $\Gamma_0$, $\Gamma_1$ and θ. (b) If $\Gamma_{in} = 0$ and $\partial|\Gamma_{in}|/\partial f = 0$ are required when $\theta = \lambda / 4$, find $a$ and $b$.




$$|\Gamma_{in}| \approx |\Gamma_0 + \Gamma_1 e^{-j2\theta} + \Gamma_2 e^{-j4\theta}| = |2\Gamma_0 \cos 2\theta + \Gamma_1|$$

$$2\Gamma_0 \cos\left(2\times\frac{\pi}{2}\right) + \Gamma_1 = 0 \Rightarrow \Gamma_1 = 2\Gamma_0$$

$$\frac{\partial|\Gamma_{in}|}{\partial\theta} = -4\Gamma_0 \sin 2\theta = 0$$

$$\frac{a-1}{a+1} = \frac{4-b}{4+b} \Rightarrow 4a + ab - 4 - b = 4a + 4 - ab - b \Rightarrow ab = 4$$

$$\frac{b-a}{b+a} = 2\frac{a-1}{a+1} \Rightarrow ab - a^2 + b - a = 2(ab + a^2 - b - a) \Rightarrow a^3 - \frac{1}{3}a^2 + \frac{4}{3}a - 4 = 0$$

$$r = \frac{1}{6}\left(-\frac{4}{9} + 12\right) + \frac{1}{27}\frac{1}{27} = 1.9273, \quad q = \frac{4 - 1/9}{9} = \frac{35}{81} = 0.4321$$

$$\sqrt[3]{r + \sqrt{q^3 + r^2}} = 1.5707, \quad \sqrt[3]{r - \sqrt{q^3 + r^2}} = -0.2751$$

$$a = \frac{1}{9} + 1.5707 - 0.2751 = 1.4067$$

$$b = 4/a = 2.8435$$

The "exact" solution in Table 5.1 is $a$ = 1.4142 and $b$ =2.8285. The error is from the approximation used in the "theory of small reflection."



# 5.7 Chebyshev Multisection Matching Transformers

Chebyshev polynomials:

$T_0(x) = 1$

$T_1(x) = x$

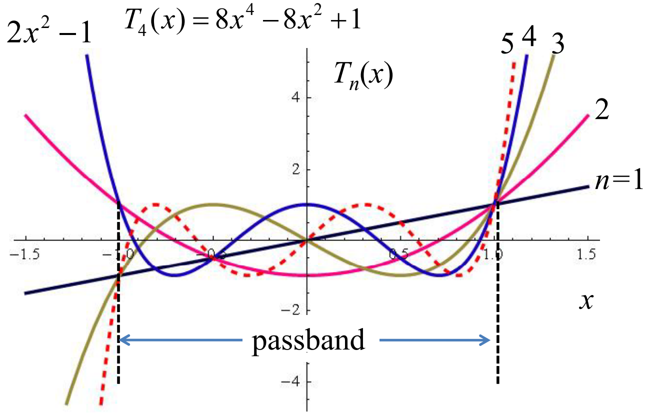
$T_2(x) = 2x^2 - 1$

$T_3(x) = 4x^3 - 3x$

$T_4(x) = 8x^4 - 8x^2 + 1$

Recurrence formula:

$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) \qquad n \geq 2$



# 5.7 Chebyshev Multisection Matching Transformers

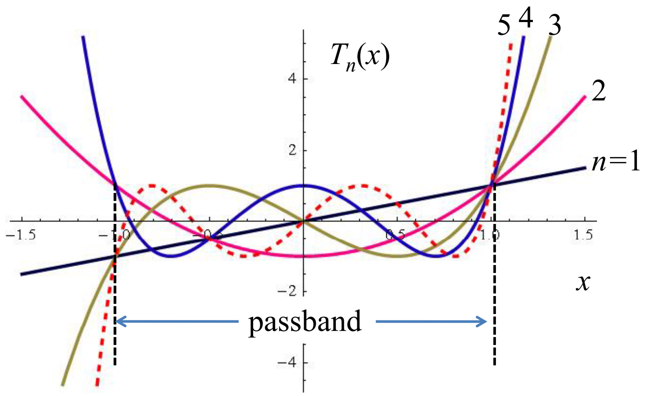
(1) $|x| \le 1, \quad |T_n(x)| \le 1$ mapped to passband

Let $x = \cos\theta$, it can be shown that $T_n(\cos\theta) = \cos n\theta$

(2) $|x| \ge 1, \quad |T_n(x)| \ge 1$ outside the passband

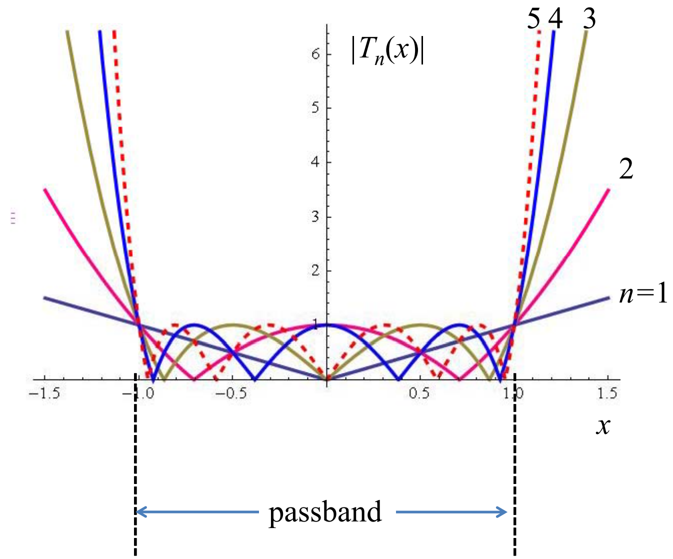
(3) In general, $T_n(x) = \cos(n\cos^{-1} x) \quad |x| \le 1$

$= \cosh(n\cosh^{-1} x) \quad |x| \ge 1$



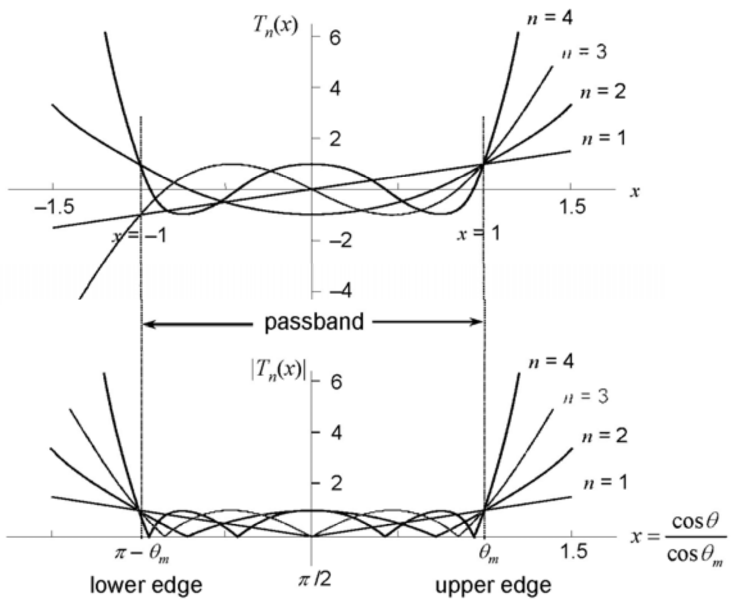



# Magnitude of Chebyshev Polynomials



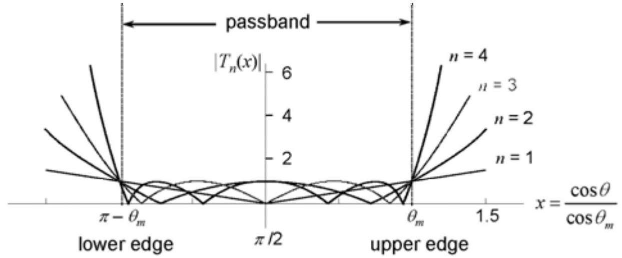



# Chebyshev Responses






# Mapping of Passband and Stopband




Define $x = \cos\theta / \cos\theta_m \Rightarrow |x| \le 1$ is mapped to passband

$\theta_m \Rightarrow x = 1,$ the upper passband edge

$\pi - \theta_m \Rightarrow x = -1,$ the lower passband edge

$$T_n(x) = \cos n[\cos^{-1}(\cos\theta / \cos\theta_m)] = T_n(\sec\theta_m \cos\theta)$$

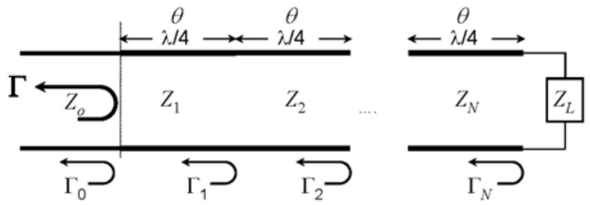
$$T_1(\sec\theta_m \cos\theta) = \sec\theta_m \cos\theta$$

$$T_2(\sec\theta_m \cos\theta) = \sec^2\theta_m(1 + \cos 2\theta) - 1$$

$$T_3(\sec\theta_m \cos\theta) = \sec^3\theta_m(\cos 3\theta + 3\cos\theta) - 3\sec\theta_m \cos\theta$$

$$T_4(\sec\theta_m \cos\theta) = \sec^4\theta_m(\cos 4\theta + 4\cos 2\theta + 3) - 4\sec^2\theta_m(1 + \cos 2\theta) + 1$$

# Design of Chebyshev Transformer




$$\Gamma(\theta) = 2e^{-jN\theta}[\Gamma_0 \cos N\theta + \Gamma_1 \cos(N-2)\theta + ... + \Gamma_n \cos(N-2n)\theta + ...]$$

$$\Gamma(0) = \frac{Z_L - Z_0}{Z_L + Z_0} = AT_N(\sec\theta_m) \Rightarrow A = \frac{1}{T_N(\sec\theta_m)}\frac{Z_L - Z_0}{Z_L + Z_0}$$

$$\Gamma_m = |A||T_N(\cos\theta_m \sec\theta_m)| = |A|$$

$$T_N(\sec\theta_m) = \frac{1}{A}\frac{Z_L - Z_0}{Z_L + Z_0} = \frac{1}{\Gamma_m}\left|\frac{Z_L - Z_0}{Z_L + Z_0}\right|, \quad \sec\theta_m = \cosh\left[\frac{1}{N}\cosh^{-1}\left(\frac{1}{\Gamma_m}\left|\frac{Z_L - Z_0}{Z_L + Z_0}\right|\right)\right]$$

$$\frac{\Delta f}{f_0} = 2 - \frac{4\theta_m}{\pi}$$



# Table 5.2 Chebyshev Transformer Design

| | $N = 2$ | | | | $N = 3$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | $\Gamma_m = 0.05$ | | $\Gamma_m = 0.20$ | | $\Gamma_m = 0.05$ | | | $\Gamma_m = 0.20$ | | |
| $Z_L/Z_0$ | $Z_1/Z_0$ | $Z_2/Z_0$ | $Z_1/Z_0$ | $Z_2/Z_0$ | $Z_1/Z_0$ | $Z_2/Z_0$ | $Z_3/Z_0$ | $Z_1/Z_0$ | $Z_2/Z_0$ | $Z_3/Z_0$ |
| 1.0 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 1.5 | 1.1347 | 1.3219 | 1.2247 | 1.2247 | 1.1029 | 1.2247 | 1.3601 | 1.2247 | 1.2247 | 1.2247 |
| 2.0 | 1.2193 | 1.6402 | 1.3161 | 1.5197 | 1.1475 | 1.4142 | 1.7429 | 1.2855 | 1.4142 | 1.5558 |
| 3.0 | 1.3494 | 2.2232 | 1.4565 | 2.0598 | 1.2171 | 1.7321 | 2.4649 | 1.3743 | 1.7321 | 2.1829 |
| 4.0 | 1.4500 | 2.7585 | 1.5651 | 2.5558 | 1.2662 | 2.0000 | 3.1591 | 1.4333 | 2.0000 | 2.7908 |
| 6.0 | 1.6047 | 3.7389 | 1.7321 | 3.4641 | 1.3383 | 2.4495 | 4.4833 | 1.5193 | 2.4495 | 3.9492 |
| 8.0 | 1.7244 | 4.6393 | 1.8612 | 4.2983 | 1.3944 | 2.8284 | 5.7372 | 1.5766 | 2.8284 | 5.0742 |
| 10.0 | 1.8233 | 5.4845 | 1.9680 | 5.0813 | 1.4385 | 3.1623 | 6.9517 | 1.6415 | 3.1623 | 6.0920 |

| | $N = 4$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | $\Gamma_m = 0.05$ | | | | $\Gamma_m = 0.20$ | | | |
| $Z_L/Z_0$ | $Z_1/Z_0$ | $Z_2/Z_0$ | $Z_3/Z_0$ | $Z_4/Z_0$ | $Z_1/Z_0$ | $Z_2/Z_0$ | $Z_3/Z_0$ | $Z_4/Z_0$ |
| 1.0 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 1.5 | 1.0892 | 1.1742 | 1.2775 | 1.3772 | 1.2247 | 1.2247 | 1.2247 | 1.2247 |
| 2.0 | 1.1201 | 1.2979 | 1.5409 | 1.7855 | 1.2727 | 1.3634 | 1.4669 | 1.5715 |
| 3.0 | 1.1586 | 1.4876 | 2.0167 | 2.5893 | 1.4879 | 1.5819 | 1.8965 | 2.0163 |
| 4.0 | 1.1906 | 1.6414 | 2.4369 | 3.3597 | 1.3692 | 1.7490 | 2.2870 | 2.9214 |
| 6.0 | 1.2290 | 1.8773 | 3.1961 | 4.8820 | 1.4415 | 2.0231 | 2.9657 | 4.1623 |
| 8.0 | 1.2583 | 2.0657 | 3.8728 | 6.3578 | 1.4914 | 2.2428 | 3.5670 | 5.3641 |
| 10.0 | 1.2832 | 2.2268 | 4.4907 | 7.7930 | 1.5163 | 2.4210 | 4.1305 | 6.5950 |



# Example 5.6

Design a Chebyshev transformer of $N$=3 for $Z_L = 2Z_0 = 100\ \Omega$ with $\Gamma_m = 0.05$.

Sol: $\Gamma(\theta) = 2e^{-j3\theta}[\Gamma_0 \cos 3\theta + \Gamma_1 \cos\theta]$

$$= Ae^{-j3\theta}T_3(\sec\theta_m \cos\theta)$$

$$= A\sec^3\theta_m(\cos 3\theta + 3\cos\theta) - 3A\sec\theta_m \cos\theta$$

$$A = \Gamma_m = 0.05$$

$$\sec\theta_m = \cosh\left[\frac{1}{N}\cosh^{-1}\left(\frac{1}{\Gamma_m}\left|\frac{Z_L - Z_0}{Z_L + Z_0}\right|\right)\right]$$

$$= \cosh\left[\frac{1}{3}\cosh^{-1}\left(\frac{20}{3}\right)\right] = 1.408$$

$$2\Gamma_0 = A\sec^3\theta_m \qquad \Rightarrow \Gamma_0 = 0.0698 = \Gamma_3$$

$$2\Gamma_1 = 3A(\sec^3\theta_m - \sec\theta_m) \Rightarrow \Gamma_1 = 0.1037 = \Gamma_2$$



# Example 5.6

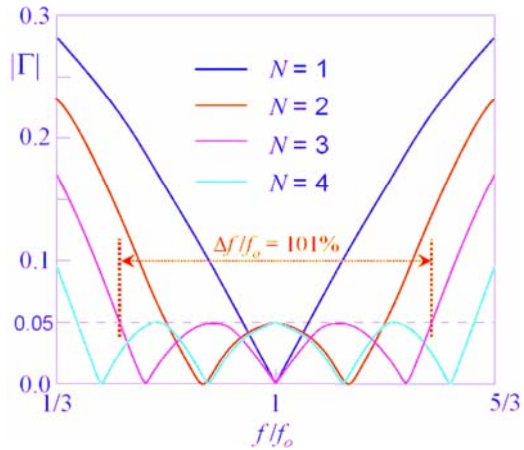
Design a Chebyshev transformer of $N$=3

$$Z_1 = Z_0\frac{1+\Gamma_0}{1-\Gamma_0} = 57.50\Omega$$

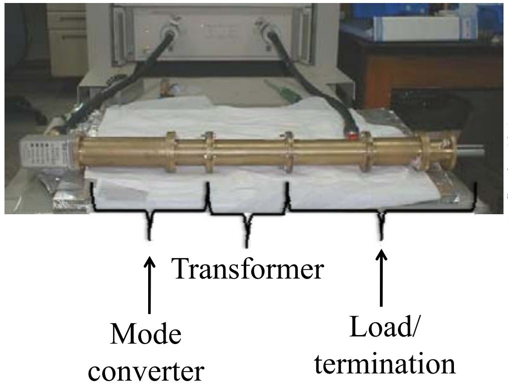
$$Z_2 = Z_1\frac{1+\Gamma_1}{1-\Gamma_1} = 70.81\Omega$$

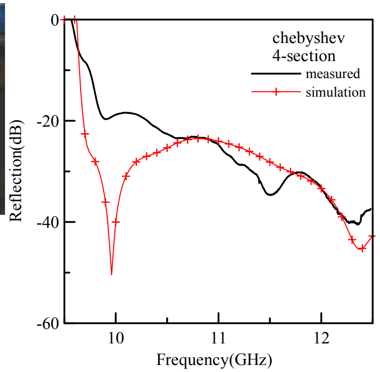
$$Z_3 = Z_2\frac{1+\Gamma_2}{1-\Gamma_2} = 87.20\Omega$$

$$\frac{\Delta f}{f_0} = 2 - \frac{4\theta_m}{\pi}\bigg|_{\theta_m = 44.7^o} = 101\%$$



## Example: “Chebyshev轉換器特性之深入探討電腦模與實驗量測” 清大物理 碩士論文 張靜宜




Chebyshev四階轉換器電腦模擬與實驗結果之比較



*The End of Chap. 5*