Chap. 9 Theory and Design of Ferrimagnetic Components

Basic Phenomena

9.1 Basic Properties of Ferrimagnetic Materials

9.2 Plane Wave Propagation in a Ferrite Medium

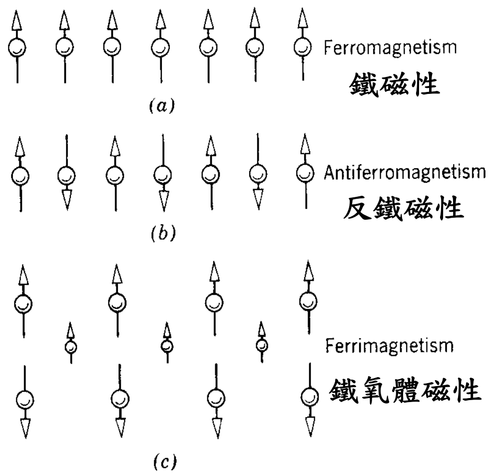
9.3 Propagation in a Ferrite-Loaded Rectangular Waveguide

Applications

9.5 Ferrite Phase Shifters

9.6 Ferrite Circulator/Isolator

Introduction to ferrite materials



The ferrites are crystals having small electric conductivity compared to ferromagnetic materials,

Thus they are useful in high-frequency situations because of the absence of significant eddy current losses.

Properties of ferrite materials (I)

Nonreciprocal electrical property: the transmission coefficient through the device is not the same for different direction of propagation.

Unequal propagation constant: The left and right circularly polarized waves have different propagation constant along the direction of external magnetic field B_0.

Anisotropic magnetic properties: The permeability of the ferrite is not a single scalar quantity, but instead is a tensor, which can be represented as a matrix.

Properties of ferrite materials (II)

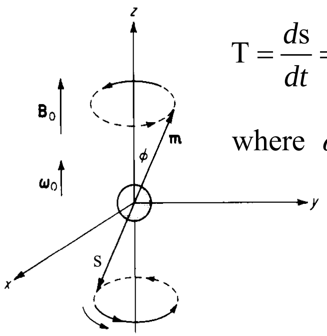
Ferrites are **ceramiclike materials** with *specific resistivities* that may be as much as 10^{14} greater than that of metals and with *dielectric constants* around 10 to 15 or greater.

Ferrites are made by sintering **a mixture of metal oxides** and have the general chemical composition $MO \cdot Fe_2O_3$, where M is a divalent metal such as Mn, Mg, Fe, Zn, Ni, Cd, etc.

Relative permeabilities of **several thousand** are common. The magnetic properties of ferrites arise mainly from the **magnetic dipole moment** associated with the **electron spin**.

Classical picture of the magnetization process --- By treating the spinning electron as a gyroscopic top.

If an electron is located in a uniform static magnetic field $\mathbf{B}_0$, a torque is given by



$$\mathrm{T} = \mathrm{m} \times \mathrm{B}_0 = -\mu_0 \gamma \mathrm{s} \times \mathrm{H}_0$$

$$\mathrm{T} = \frac{ds}{dt} = \frac{-1}{\gamma}\frac{dm}{dt} \quad \Rightarrow \quad \frac{dm}{dt} = -\mu_0 \gamma \mathrm{m} \times \mathrm{H}_0$$

where $\omega_0 = \dfrac{eB_0}{m_e}$ is called the Larmor fequency;

$s = \dfrac{\hbar}{2}$ is spin angular momentum;

$m = \dfrac{e}{m_e}\dfrac{\hbar}{2} = -\gamma s$ is magnetic dipole moment.

Quantum mechanics' viewpoint $s_z = \pm 1/2$

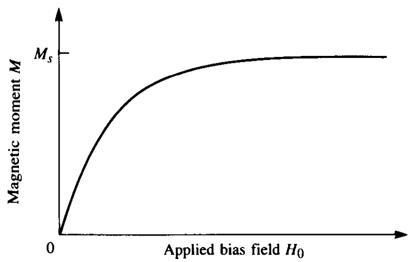
In the absence of any damping forces, the actual precession angle will be determined by the initial position of the magnetic dipole, and the dipole will precess about $\mathbf{B}_0$ at this angle indifferently (free precession).

In reality, however, the existence of damping forces will cause the magnetic dipole *to spiral in* from its initial angle until **m** is aligned with $\mathbf{B}_0$.

This explains why s_z equals ±1/2 in the Quantum Mechanics.

But where does the damping force come from?

Saturation magnetization



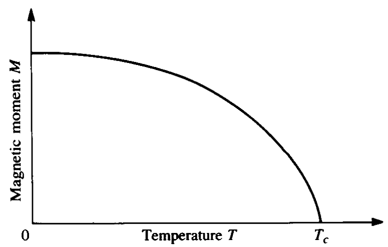
As the strength of the bias field H_0 is increased, more magnetic dipole moments will align with H_0 until all are aligned, and $\boldsymbol{M}$ reaches an upper limit.

The material is then said to be magnetically saturated, and M_s is denoted as the saturation magnetization. M_s typically ranges from $4\pi M_s$=300 to 5000 Gauss.

Below saturation, ferrite materials can be very lossy at microwave frequencies, and rf interaction is reduced.

The ferrites are usually operated in the saturated state.

Curie temperature



The saturation magnetization of a material is a strong function of temperature, decreasing as temperature increases.

This effect can be understood by noting that the vibrational energy of an atom increases with temperature, making it more difficult to align all the magnetic dipoles.

At a high enough temperature a zero net magnetization results. This temperature is called the Curie temperature, T_c.

Properties of some ferrite materials

Material	Trans-Tech Number	$4\pi M_s$ G	ΔH Oe	ϵ_r	$\tan\delta$	T_c °C	$4\pi M_r$ G
Magnesium ferrite	TT1-105	1750	225	12.2	0.00025	225	1220
Magnesium ferrite	TT1-390	2150	540	12.7	0.00025	320	1288
Magnesium ferrite	TT1-3000	3000	190	12.9	0.0005	240	2000
Nickel ferrite	TT2-101	3000	350	12.8	0.0025	585	1853
Nickel ferrite	TT2-113	500	150	9.0	0.0008	120	140
Nickel ferrite	TT2-125	2100	460	12.6	0.001	560	1426
Lithium ferrite	TT73-1700	1700	<400	16.1	0.0025	460	1139
Lithium ferrite	TT73-2200	2200	<450	15.8	0.0025	520	1474
Yttrium garnet	G-113	1780	45	15.0	0.0002	280	1277
Aluminum garnet	G-610	680	40	14.5	0.0002	185	515

Why use $4\pi M_s$? $\mathbf{B} = 4\pi\mathbf{M} + \mathbf{H} = \mu\mathbf{H}$ (Gaussian unit)

The unit of **B** is Gauss; the unit of **H** is Oersted. They have same dimension.

What does ΔH and M_r mean?

Ferrite linewidth and remanent magnetization

HIMAG & Trans-Tech特性比較

- 高科磁技 HIMAG

Item		Unit	Measuring Condition		MZ-30	MZ-2	MZ-4	MZ-5	M 07
Initial Permeability	μ i		100KHz 0.5mA	25	3000 ± 25%	2500 ± 25%	2400 ± 25%	5000 ± 25%	7000 ± 25%
Core Loss	Pc	mW/c.c.	25KHz sine wave 200mT	25	160	150			
				60	120	120			
				100	150	115			
				120	180	120			
			100KHz sine wave 200mT	25		800	720		
				60		650	550		
				100		650	480		
				120		750	600		
Relative Loss Factor	tan /μi (*10-6)		10KHz 0.5mA	25				4	6
			100KHz 0.5mA					35	45
Saturation Flux Density	Bms	mT	10KHz H= 1000A/M	25	430	450	430	400	400
Remanence	Brms	mT			150	150	150	130	130
Coercivity	Hc	Oe			0.15	0.15	0.15	0.12	0.1
Curie temperature	Tc				>190	>210	>210	>130	>120

- **Trans-Tech**

	Saturation Magnetization $4\pi M_s$	Landé g-Factor g-eff	Line Width δH oe @ -3dB	Dielectric Constant E'	Dielectric Loss Tangent Tan=E"/E'	Curie Temperature T_c (°C)	Spin Wave Line Width δH_k oe	Remanent Induction B_r (Gauss)	Coercive Force H_c (oe)	Initial Permeability μo
TTVG-800	800 ± 5%	2	<=15	13.9 ± 5%	<.0002	192	2.0	560	60%	129
TTVG-930	930 ± 5%	2	<=10	14.0 ± 5%	<.0002	188	2.0	380	40%	225
TTVG-1000	1000 ± 5%	2	<=10	14.0 ± 5%	<.0002	199	2.0	320	30%	210
TTVG-1100	1100 ± 5%	2	<=10	14.1 ± 5%	<.0002	205	2.0	600	60%	209

Anisotropic magnetic properties (I)

If $\bar{H}$ is the applied ac field, the total magnetic field is $\bar{H}_t = H_0\hat{z} + \bar{H}$,

where $|\bar{H}| << H_0$. The field produced a total magnetization on the ferrite is given by $\bar{M}_t = M_s\hat{z} + \bar{M}$

M_s is the dc saturation magnetization and $\bar{M}$ is the additional ac magnetization (in the *xy* plane) caused by applied field.

$$\left.\begin{aligned}\bar{M}_t &= M_s\hat{z} + \bar{M} \\ \bar{H}_t &= H_0\hat{z} + \bar{H}\end{aligned}\right\} \Rightarrow \frac{d\bar{\mathrm{M}}}{dt} = -\mu_0\gamma\bar{\mathrm{M}} \times \mathrm{H}_0$$

Anisotropic magnetic properties (I)

The component equations of motion:

$$\frac{dM_x}{dt} = -\mu_0\gamma M_y(H_0 + H_z) + \mu_0\gamma(M_s + M_z)H_y$$

$$\frac{dM_y}{dt} = -\mu_0\gamma M_x(H_0 + H_z) - \mu_0\gamma(M_s + M_z)H_x$$

$$\frac{dM_z}{dt} = -\mu_0\gamma M_x H_y + \mu_0\gamma M_y H_x$$

Anisotropic magnetic properties (II)

Omitting higher order terms, the equations can be reduced to

$$\left.\begin{aligned}\frac{dM_x}{dt}&=-\omega_0 M_y+\omega_m H_y,\\ \frac{dM_y}{dt}&=-(\omega_0 M_x+\omega_m H_x),\end{aligned}\right\} \Rightarrow \begin{aligned}\frac{d^2M_x}{dt^2}+\omega_0^2 M_x&=\omega_m\frac{dH_y}{dt}+\omega_0\omega_m H_x,\\ \frac{d^2M_y}{dt^2}+\omega_0^2 M_y&=-\omega_m\frac{dH_x}{dt}+\omega_0\omega_m H_y.\end{aligned}$$

$$\frac{dM_z}{dt}=0,\quad \text{where } \omega_0=\mu_0\gamma H_0 \text{ and } \omega_m=\mu_0\gamma M_s$$

If $\bar{M}$ and $\bar{H}\propto e^{j\omega t}$, the above equtions can be reduced to the phasor equations:

$$\begin{aligned}(\omega_0^2-\omega^2)M_x&=\omega_0\omega_m H_x+j\omega\omega_m H_y,\\ (\omega_0^2-\omega^2)M_y&=-j\omega\omega_m H_x+\omega_0\omega_m H_y.\end{aligned} \Rightarrow \bar{M}=[\chi]\bar{H}=\begin{bmatrix}\chi_{xx} & \chi_{xy} & 0\\ \chi_{yx} & \chi_{yy} & 0\\ 0 & 0 & 0\end{bmatrix}\bar{H},$$

where $\chi_{xx}=\chi_{yy}=\dfrac{\omega_0\omega_m}{\omega_0^2-\omega^2}$ and $\chi_{xy}=-\chi_{yx}=\dfrac{j\omega\omega_m}{\omega_0^2-\omega^2}$

Anisotropic magnetic properties (III)

To relate *B* and *H*, we have

$$\bar{B}=\mu_0(\bar{M}+\bar{H})=[\mu]\bar{H} \Rightarrow [\mu]=\mu_0([U]+[\chi])=\begin{bmatrix}\mu & j\kappa & 0\\ -j\kappa & \mu & 0\\ 0 & 0 & \mu_0\end{bmatrix}$$

$$\begin{cases}\mu=\mu_0(1+\chi_{xx})=\mu_0(1+\chi_{yy})=\mu_0\left(1+\dfrac{\omega_0\omega_m}{\omega_0^2-\omega^2}\right)\\ \kappa=-j\mu_0\chi_{xy}=j\mu_0\chi_{yx}=\mu_0\dfrac{\omega\omega_m}{\omega_0^2-\omega^2}\end{cases}$$

A material having a permeability tensor of this form is called gyrotropic.

How to apply this concept to a circularly polarized wave?

Forced precession of spinning electron (I)

If a small ac magnetic field is superimposed on the static field $\mathbf{H}_0$, the magnetic dipole moment will undergo a forced precession.

Of particular interest is the case where the ac magnetic field is circularly polarized in the plane perpendicular to $\mathbf{H}_0$.

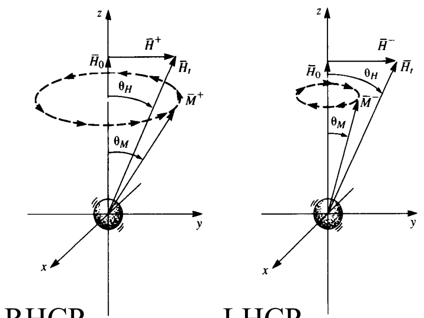
A right-hand circularly polarized wave can be expressed in phasor form as

$$\bar{H}^+=H^+(\hat{x}-j\hat{y})$$

and in time-domain form as

$$\bar{H}^+=\mathrm{Re}\{\bar{H}^+e^{j\omega t}\}=H^+(\hat{x}\cos\omega t+\hat{y}\sin\omega t)$$

Forced precession of spinning electron (II)



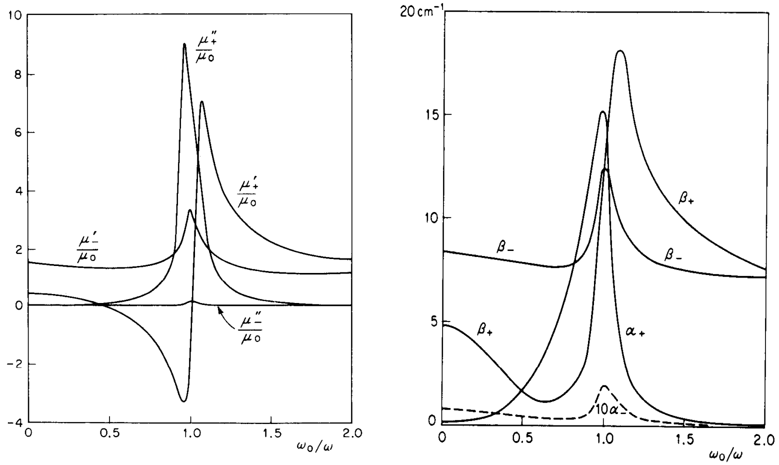
RHCP (a) LHCP (b)

propagation constant

$$\beta_\pm=\omega\sqrt{\varepsilon\mu^\pm}$$

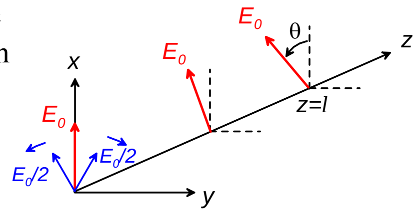
$$\left.\begin{aligned}M_x^+&=\frac{\omega_m}{\omega_0-\omega}H^+,\\ M_y^+&=\frac{-j\omega_m}{\omega_0-\omega}H^+.\end{aligned}\right\} \Rightarrow \bar{M}^+=\frac{\omega_m}{\omega_0-\omega}\bar{H}^+ \Rightarrow \begin{cases}\mu^+=\mu_0\left(1+\dfrac{\omega_m}{\omega_0-\omega}\right) \text{ RHCP}\\ \mu^-=\mu_0\left(1+\dfrac{\omega_m}{\omega_0+\omega}\right) \text{ LHCP}\end{cases}$$

Real and imaginary permeability & propagation and attenuation constant.



Faraday rotation --- a nonreciprocal effect

Consider linearly polarized electric field at $z = 0$, represented as the sum of a RHCP and a LHCP wave:



$$\bar{E}\Big|_{(z=0)} = \hat{x}E_0 = \frac{E_0}{2}(\hat{x} - j\hat{y}) + \frac{E_0}{2}(\hat{x} + j\hat{y})$$

These two polarized waves propagate with different propagation constants.

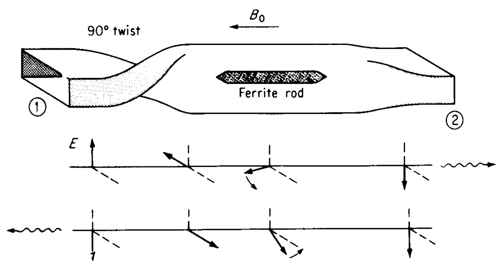
$$\bar{E}\Big|_{(z=\ell)} = \frac{E_0}{2}(\hat{x} - j\hat{y})e^{-j\beta_+\ell} + \frac{E_0}{2}(\hat{x} + j\hat{y})e^{-j\beta_-\ell}$$

$$= E_0\left[\hat{x}\cos\left(\frac{\beta_+ - \beta_-}{2}\right)\ell - \hat{y}\sin\left(\frac{\beta_+ - \beta_-}{2}\right)\ell\right]e^{-j(\beta_+ + \beta_-)\ell/2}$$

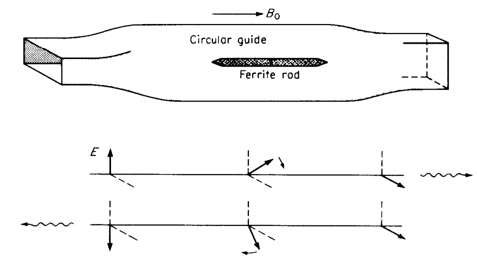
$$\theta = \tan^{-1}\frac{E_y}{E_x} = -\left(\frac{\beta_+ - \beta_-}{2}\right)\ell.$$ This effect is called *Faraday rotation*.

Microwave gyrator

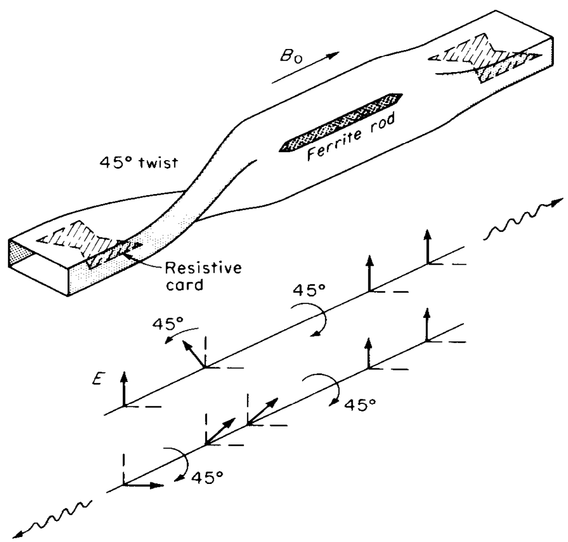
Gyrator with a twist section.



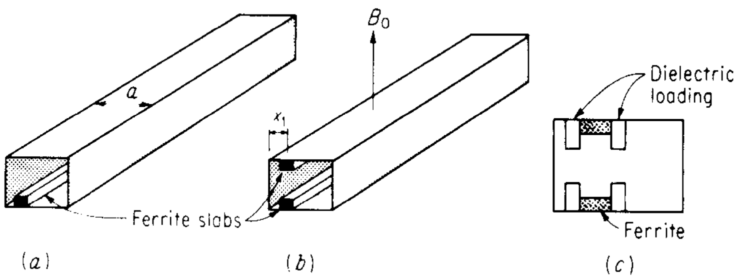
Gyrator without a twist section.



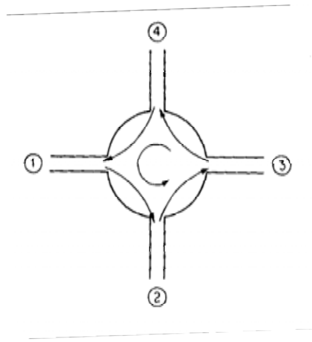
Faraday-rotation isolator

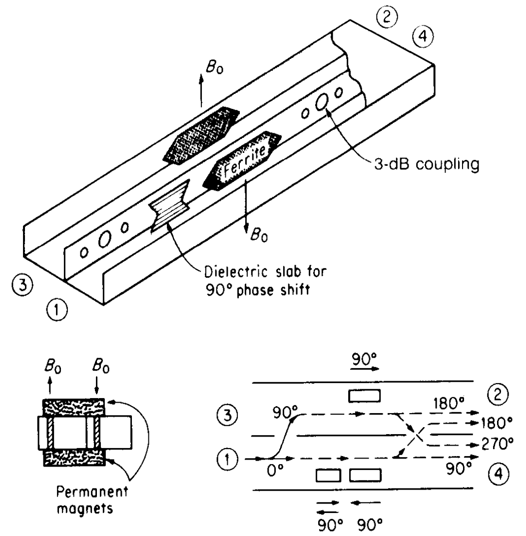


Resonance isolator

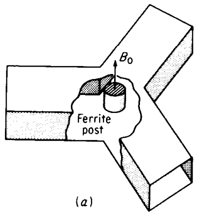


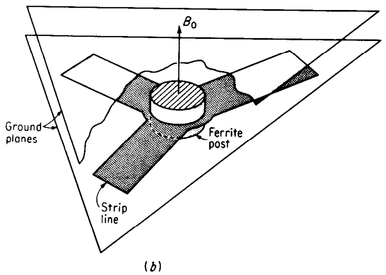
Four-port circulator

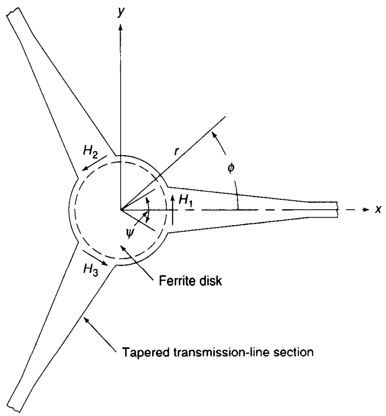




Three-port circulator



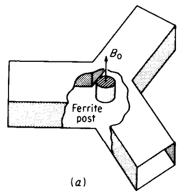


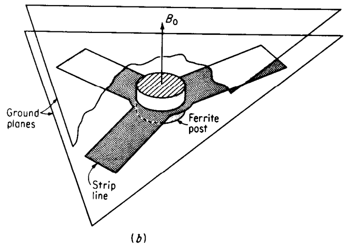


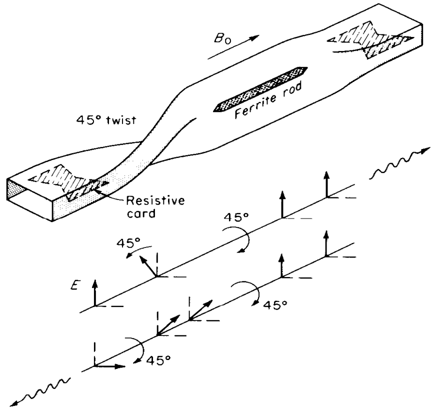
Devices

循環器
circulator

隔離器
isolator





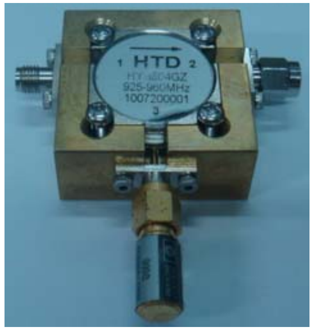


Example：Circulator (I)

Circulator

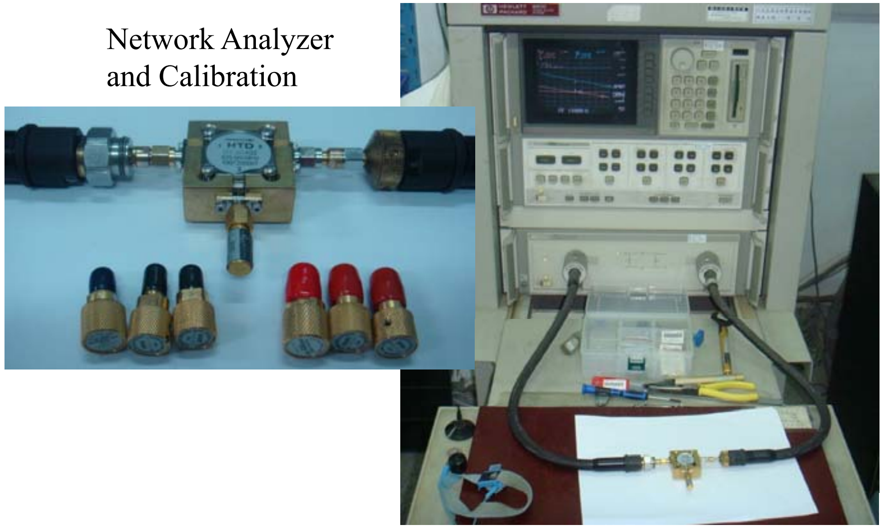
Fixture

Assembly

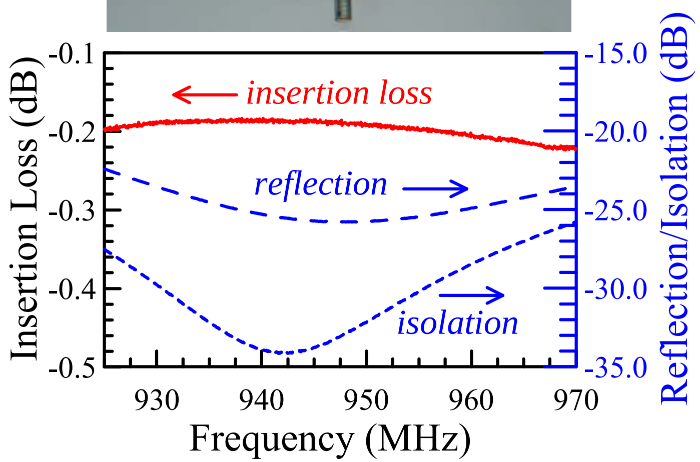


Circulator Measurement (II)

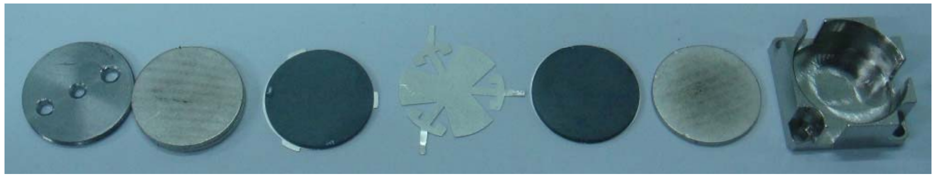
Network Analyzer
and Calibration

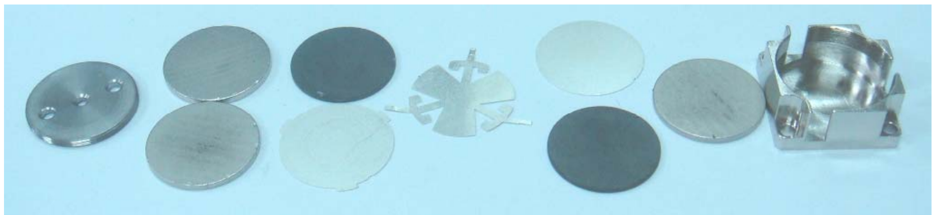


Circulator Measured Results (III)



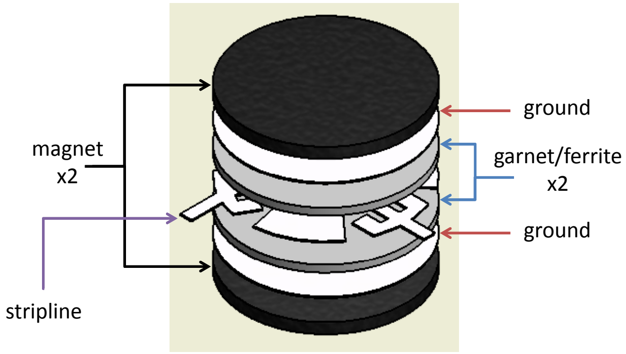
Circulator Construction (IV)



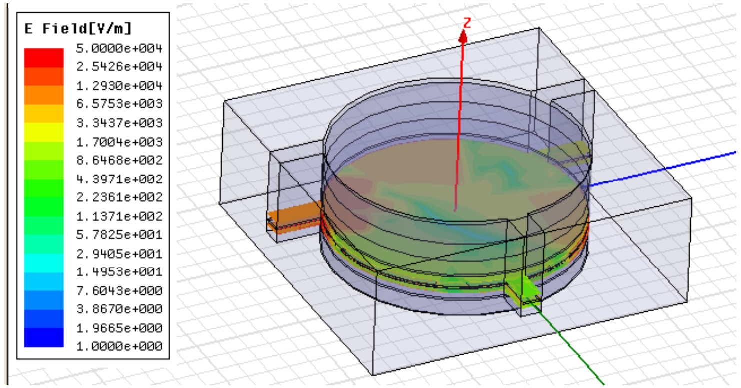


可否再簡化，i.e. cost down?
Ans:需掌握關鍵技術➔模擬分析能力。

Circulator Configuration (IV)



Circulator Simulation (IV)

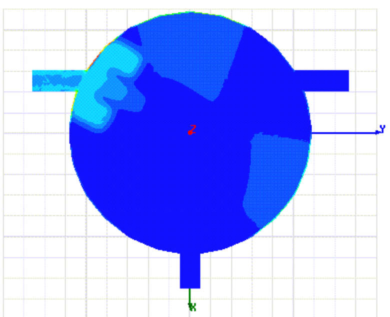


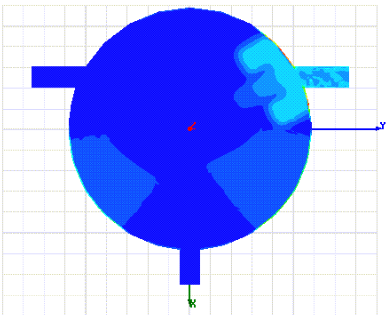
全尺寸電磁場與熱分析模擬!

Circulator Simulated Results (IV)

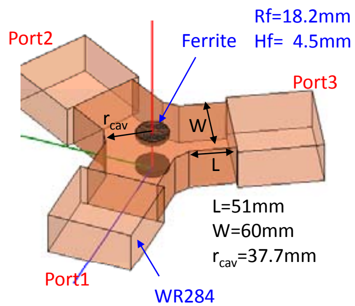
input 1
output 2

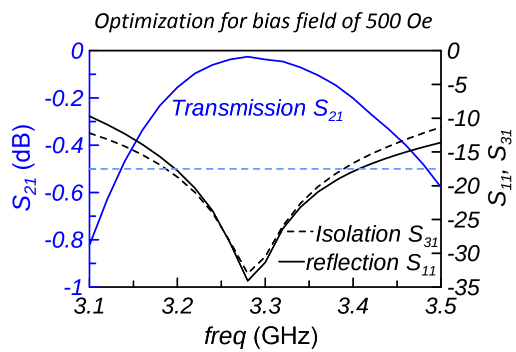
input 2
output 3





Example #2：S 頻段微波循環器 (波導高功率)





優點：low insertion loss, high isolation, and low reflection

p.s. 本實驗室陳乃慶、蔡育超、蔡鎮鴻設計。

例二所用Garnet 材料特性比較

Pacific Ceramics, Inc.

Garnets

Product	Saturation Magnetization $4\pi M_S$ (gauss)	Dielectric Constant ε'	Loss Tangent $\tan\delta_\varepsilon$ ($\times 10^{-4}$)	Resonance Linewidth ΔH @3dB (oe)	Curie Temperature T_C (°C)
Gadolinium Substituted Yttrium Iron Garnet					
64-1600	1600	15.1	≤ 1	50	280

GARNETS – Yttrium Iron Garnet

Family Type	Saturation Magnetization $4\pi M_s$ (Gauss)	Line Width ΔH (oe @ -3dB)	Dielectric Constant, ε'	Dielectric Loss Tangent $\tan\delta = \varepsilon''/\varepsilon' \times 10^4$	Curie Temperature T_c(°C), Nominal	Spin Wave Line Width ΔH_k (oe), Nominal	Remanent Induction B_r (Gauss), Nominal	Coercive Force H_c (oe) Nomina	α_s $\Delta M_s/M_s\Delta T \times 10^3$ Nominal	Comments (See page 36 for details)
YG – 1780 – 45			15.1	≤2	280	1.5	—	—	2.0	SG
YG – 1780 – 30			15.1	≤2	280	1.5	1275	0.5	2.0	NL

Ferrite Circulator for HFSS Simulation

Analytic solution by Eric Chao

優點：low insertion loss, high isolation, and low reflection

$$\nabla\cdot\vec{E}=0$$
$$\nabla\cdot(\hat{u}\cdot\vec{H})=0$$
$$\nabla\times\vec{E}=i\omega\vec{B}=i\omega\hat{u}\cdot\vec{H}$$
$$\nabla\times\vec{H}=\frac{\partial\vec{D}}{\partial t}=-i\omega\varepsilon\vec{E}$$

$$\hat{u}=\begin{pmatrix}\mu & -i\kappa & 0\\ i\kappa & \mu & 0\\ 0 & 0 & \mu_0\end{pmatrix}$$

$$\hat{u}\cdot\vec{H}=\begin{pmatrix}\mu & -i\kappa & 0\\ i\kappa & \mu & 0\\ 0 & 0 & \mu_0\end{pmatrix}\begin{pmatrix}H_\rho\\ H_\phi\\ 0\end{pmatrix}=\begin{pmatrix}\mu H_\rho-i\kappa H_\phi\\ i\kappa H_\rho+\mu H_\phi\\ 0\end{pmatrix}$$

$$\vec{E}=E_z(\rho,\phi)\hat{e}_z$$
$$\vec{H}=H_\rho(\rho,\phi)\hat{e}_\rho+H_\phi(\rho,\phi)\hat{e}_\phi$$

$$\nabla\cdot\vec{E}=\frac{\partial E_z}{\partial z}=0$$

$$\nabla\cdot(\hat{u}\cdot\vec{H})=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\left(\mu H_\rho-i\kappa H_\phi\right)\right]+\frac{1}{\rho}\frac{\partial}{\partial\phi}\left(i\kappa H_\rho+\mu H_\phi\right)=0$$

$$\nabla\times\vec{E}=i\omega\hat{u}\cdot\vec{H}\Rightarrow\hat{e}_\rho\frac{1}{\rho}\frac{\partial E_z}{\partial\phi}-\hat{e}_\phi\frac{\partial E_z}{\partial\rho}=i\omega\left[\hat{e}_\rho\left(\mu H_\rho-i\kappa H_\phi\right)+\hat{e}_\phi\left(i\kappa H_\rho+\mu H_\phi\right)\right]$$

$$\nabla\times\vec{H}=-i\omega\varepsilon\vec{E}\Rightarrow\frac{1}{\rho}\left[\frac{\partial\left(\rho H_\phi\right)}{\partial\rho}-\frac{\partial H_\rho}{\partial\phi}\right]=-i\omega\varepsilon E_z$$

$$\begin{cases}H_\rho=-\dfrac{1}{\omega\left(\mu^2-\kappa^2\right)}\left(\dfrac{i\mu}{\rho}\dfrac{\partial E_z}{\partial\phi}+\kappa\dfrac{\partial E_z}{\partial\rho}\right)\\ H_\phi=\dfrac{1}{\omega\left(\mu^2-\kappa^2\right)}\left(i\mu\dfrac{\partial E_z}{\partial\rho}-\dfrac{\kappa}{\rho}\dfrac{\partial E_z}{\partial\phi}\right)\\ \dfrac{1}{\rho}\left[\dfrac{\partial\left(\rho H_\phi\right)}{\partial\rho}-\dfrac{\partial H_\rho}{\partial\phi}\right]=-i\omega\varepsilon E_z\end{cases}\Rightarrow\frac{\partial^2E_z}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial E_z}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2E_z}{\partial\phi^2}+\omega^2\varepsilon\frac{\left(\mu^2-\kappa^2\right)}{\mu}E_z=0$$

$$\frac{\partial^2E_z}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial E_z}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2E_z}{\partial\phi^2}+\omega^2\varepsilon\frac{\left(\mu^2-\kappa^2\right)}{\mu}E_z=0$$

$$\mu_e=\frac{\left(\mu^2-\kappa^2\right)}{\mu}\ ,\ \gamma^2=\omega^2\varepsilon\frac{\left(\mu^2-\kappa^2\right)}{\mu}=\omega^2\varepsilon\mu_e$$

$$\frac{\partial^2E_z}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial E_z}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2E_z}{\partial\phi^2}+\gamma^2E_z=0$$

$$E_z=(\mathrm{A}_m e^{im\phi}+\mathrm{B}_m e^{-im\phi})\mathrm{J}_m(\gamma_{mn}\rho)$$

$$H_\rho=-\frac{1}{\omega\left(\mu^2-\kappa^2\right)}\left(\frac{i\mu}{\rho}\frac{\partial E_z}{\partial\phi}+\kappa\frac{\partial E_z}{\partial\rho}\right)$$
$$=-\frac{\kappa\gamma_{mn}}{\omega\left(\mu^2-\kappa^2\right)}\left(\left(\mathrm{J}'_m(\gamma_{mn}\rho)-\frac{m\mu}{\kappa\gamma_{mn}\rho}\mathrm{J}_m(\gamma_{mn}\rho)\right)\mathrm{A}_m e^{im\phi}+\left(\mathrm{J}'_m(\gamma_{mn}\rho)+\frac{m\mu}{\kappa\gamma_{mn}\rho}\mathrm{J}_m(\gamma_{mn}\rho)\right)\mathrm{B}_m e^{-im\phi}\right)$$

$$H_\phi=\frac{1}{\omega\left(\mu^2-\kappa^2\right)}\left(i\mu\frac{\partial E_z}{\partial\rho}-\frac{\kappa}{\rho}\frac{\partial E_z}{\partial\phi}\right)$$
$$=-\frac{i\mu\gamma_{mn}}{\omega\left(\mu^2-\kappa^2\right)}\left(\left(\mathrm{J}'_m(\gamma_{mn}\rho)-\frac{m\kappa}{\mu\gamma_{mn}\rho}\mathrm{J}_m(\gamma_{mn}\rho)\right)\mathrm{A}_m e^{im\phi}+\left(\mathrm{J}'_m(\gamma_{mn}\rho)+\frac{m\kappa}{\mu\gamma_{mn}\rho}\mathrm{J}_m(\gamma_{mn}\rho)\right)\mathrm{B}_m e^{-im\phi}\right)$$

$$E_z = (A_1 e^{i\phi} + B_1 e^{-i\phi}) J_1(\gamma\rho)$$

$$E_z(\rho = R, \phi) = \begin{cases} E_0 \ , \ \phi = 0 \\ -E_0 \ , \ \phi = \dfrac{2\pi}{3} \\ 0 \ , \ \phi = \dfrac{4\pi}{3} \end{cases}$$

$$\Rightarrow A_1 = \frac{E_0}{2J_1(\gamma R)}\left(1 + i\frac{1}{\sqrt{3}}\right) \quad \& \quad B_1 = \frac{E_0}{2J_1(\gamma R)}\left(1 - i\frac{1}{\sqrt{3}}\right)$$

$$E_z = (A_1 e^{i\phi} + B_1 e^{-i\phi}) J_1(\gamma\rho) = \left(\frac{E_0}{2J_1(\gamma R)}\left(1 + i\frac{1}{\sqrt{3}}\right)e^{i\phi} + \frac{E_0}{2J_1(\gamma R)}\left(1 - i\frac{1}{\sqrt{3}}\right)e^{-i\phi}\right) J_1(\gamma\rho)$$

$$= \frac{E_0}{J_1(\gamma R)} J_1(\gamma\rho)\left(\cos\phi - \frac{\sin\phi}{\sqrt{3}}\right)$$

$$\boxed{B.C. \Rightarrow \hat{n} \times \vec{H}\big|_{\rho=R} = 0 \Rightarrow -\hat{e}_\rho \times \vec{H}\big|_{\rho=R} = 0 \Rightarrow H_\phi\big|_{\rho=R} = 0}$$

$$H_\phi\big|_{\rho=R} = \frac{i\mu\gamma_{mn}}{\omega(\mu^2 - \kappa^2)}\left(\left(J'_m(\gamma_{mn}R) - \frac{m\kappa}{\mu\gamma_{mn}R}J_m(\gamma_{mn}R)\right)A_m e^{im\phi} - \left(J'_m(\gamma_{mn}R) + \frac{m\kappa}{\mu\gamma_{mn}R}J_m(\gamma_{mn}R)\right)B_m e^{-im\phi}\right) = 0$$

$$H_\phi(\rho = R, \phi) = \begin{cases} H_0 & \text{for } -\psi < \phi < \psi \\ H_0 & \text{for } 120^\circ - \psi < \phi < 120^\circ + \psi \\ 0 & \text{elsewhere} \end{cases}$$

$$H_\phi(\rho = R, \phi) = \sum_{m=-\infty}^{\infty} C_m e^{im\phi} = \frac{2H_0\psi}{\pi} + \frac{H_0}{\pi}\sum_{m=1}^{\infty}\frac{\sin(m\psi)}{m}\left(\left(1 + e^{-i\frac{2m\pi}{3}}\right)e^{im\phi} + \left(1 + e^{i\frac{2m\pi}{3}}\right)e^{-im\phi}\right)$$

$$m = 1 \ \Rightarrow \ H_\phi(\rho = R, \phi) = \frac{H_0 \sin\psi}{2\pi}\left(\left(1 - i\sqrt{3}\right)e^{i\phi} + \left(1 + i\sqrt{3}\right)e^{-i\phi}\right)$$

$$= -\frac{\gamma E_0}{2\sqrt{3}\omega\mu_e J_1(\gamma R)}\left(\left(J'_1(\gamma R) - \frac{\kappa}{\mu\gamma R}J_1(\gamma R)\right)\left(1 - i\sqrt{3}\right)e^{i\phi} + \left(J'_1(\gamma R) + \frac{\kappa}{\mu\gamma R}J_1(\gamma R)\right)\left(1 + i\sqrt{3}\right)e^{-i\phi}\right)$$

$$\begin{cases} \dfrac{H_0 \sin\psi}{2\pi} = -\dfrac{\gamma E_0}{2\sqrt{3}\omega\mu_e J_1(\gamma R)}\left(J'_1(\gamma R) - \dfrac{\kappa}{\mu\gamma R}J_1(\gamma R)\right) \\ \dfrac{H_0 \sin\psi}{2\pi} = \dfrac{\gamma E_0}{2\sqrt{3}\omega\mu_e J_1(\gamma R)}\left(J'_1(\gamma R) + \dfrac{\kappa}{\mu\gamma R}J_1(\gamma R)\right) \end{cases} \Rightarrow \begin{cases} J'_1(\gamma R) = 0 \\ \dfrac{H_0 \sin\psi}{\pi} = \dfrac{E_0}{\sqrt{3}\omega\mu_e}\dfrac{\kappa}{\mu R} \Rightarrow \dfrac{E_0}{H_0} = \dfrac{\sqrt{3}\omega\mu_e\mu R \sin\psi}{\pi\kappa} \end{cases}$$

$$H_\rho = -\frac{\kappa\gamma_{mn}}{\omega(\mu^2 - \kappa^2)}\left(\left(J'_m(\gamma_{mn}\rho) - \frac{m\mu}{\kappa\gamma_{mn}\rho}J_m(\gamma_{mn}\rho)\right)A_m e^{im\phi} + \left(J'_m(\gamma_{mn}\rho) + \frac{m\mu}{\kappa\gamma_{mn}\rho}J_m(\gamma_{mn}\rho)\right)B_m e^{-im\phi}\right)$$

$$A_1 = \frac{E_0}{2J_1(\gamma R)}\left(1 + i\frac{1}{\sqrt{3}}\right) \quad \& \quad B_1 = \frac{E_0}{2J_1(\gamma R)}\left(1 - i\frac{1}{\sqrt{3}}\right)$$

$m = 1, n = 1$

$$H_\rho = -\frac{\kappa\gamma_{11}}{\omega(\mu^2 - \kappa^2)}\left(\left(J'_1(\gamma_{11}\rho) - \frac{\mu}{\kappa\gamma_{11}\rho}J_1(\gamma_{11}\rho)\right)A_1 e^{i\phi} + \left(J'_1(\gamma_{11}\rho) + \frac{\mu}{\kappa\gamma_{11}\rho}J_1(\gamma_{11}\rho)\right)B_1 e^{-i\phi}\right)$$

$$= -\frac{\kappa\gamma_{11}E_0}{2\omega(\mu^2 - \kappa^2)J_1(\gamma_{11}R)}\left(\left(J'_1(\gamma_{11}\rho) - \frac{\mu}{\kappa\gamma_{11}\rho}J_1(\gamma_{11}\rho)\right)\left(1 + i\frac{1}{\sqrt{3}}\right)e^{i\phi} + \left(J'_1(\gamma_{11}\rho) + \frac{\mu}{\kappa\gamma_{11}\rho}J_1(\gamma_{11}\rho)\right)\left(1 - i\frac{1}{\sqrt{3}}\right)e^{-i\phi}\right)$$

$$= -\frac{\kappa\gamma_{11}E_0}{\omega(\mu^2 - \kappa^2)J_1(\gamma_{11}R)}\left(\left(J'_1(\gamma_{11}\rho) - i\frac{\mu}{\kappa\gamma_{11}\rho}\frac{J_1(\gamma_{11}\rho)}{\sqrt{3}}\right)\cos\phi - \left(\frac{J'_1(\gamma_{11}\rho)}{\sqrt{3}} + \frac{i\mu}{\kappa\gamma_{11}\rho}J_1(\gamma_{11}\rho)\right)\sin\phi\right)$$

$$H_\rho(\rho = R, \phi) = -\frac{\kappa\gamma_{mn}}{\omega(\mu^2 - \kappa^2)}\left(\left(-J_m(\gamma_{mn}R)\right)A_m e^{im\phi} + \left(\frac{m\mu}{\kappa\gamma_{mn}R}J_m(\gamma_{mn}R)\right)B_m e^{-im\phi}\right)$$

$$= \frac{m\mu}{\omega R(\mu^2 - \kappa^2)}\left(A_m e^{im\phi} - B_m e^{-im\phi}\right)J_m(\gamma_{mn}R) = \frac{\mu}{\omega R(\mu^2 - \kappa^2)}\left(A_1 e^{i\phi} - B_1 e^{-i\phi}\right)J_1(x'_{11})$$

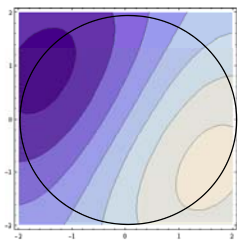
$$= \frac{\mu E_0}{2\omega R(\mu^2 - \kappa^2)}\left(\left(1 + i\frac{1}{\sqrt{3}}\right)e^{i\phi} - \left(1 - i\frac{1}{\sqrt{3}}\right)e^{-i\phi}\right) = \frac{iH_0}{x'_{11}}\left(\sin\phi + \frac{1}{\sqrt{3}}\cos\phi\right)$$

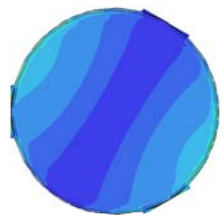
$$E_z = \frac{E_0}{J_1(\gamma R)}J_1(\gamma\rho)\left(\cos\phi - \frac{\sin\phi}{\sqrt{3}}\right) = \frac{E_0}{J_1(x'_{11})}J_1\left(\frac{x'_{11}\rho}{R}\right)\left(\cos\phi - \frac{\sin\phi}{\sqrt{3}}\right)$$

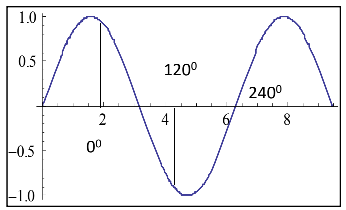
$$\begin{cases} J'_1(\gamma R) = 0 \quad \Rightarrow \boxed{\gamma_{11}R = x'_{11} = 1.841 = \omega_{110}\sqrt{\varepsilon\mu_e}R} \\ \dfrac{H_0 \sin\psi}{\pi} = \dfrac{E_0}{\sqrt{3}\omega\mu_e}\dfrac{\kappa}{\mu R} \Rightarrow \dfrac{E_0}{H_0} = \dfrac{\sqrt{3}\omega_{110}\mu_e\mu R \sin\psi}{\pi\kappa} = \dfrac{\sqrt{3}\mu_e\mu \sin\psi}{\pi\kappa}\dfrac{1.841}{\sqrt{\varepsilon\mu_e}} \end{cases}$$

$$\frac{E_0}{H_0} = \frac{\sqrt{3}\omega_{110}\mu_e\mu R \sin\psi}{\pi\kappa} = \sqrt{\frac{3\mu_e}{\varepsilon}}\frac{1.841\mu \sin\psi}{\pi\kappa} = \sqrt{\frac{\mu_e}{\varepsilon}} \Rightarrow \boxed{\sin\psi = \frac{\pi}{1.841\sqrt{3}}\left(\frac{\kappa}{\mu}\right)}$$

$$\sin\psi = \frac{\pi}{1.841\sqrt{3}}\left(\frac{\kappa}{\mu}\right) = \frac{\pi}{1.841\sqrt{3}}\left(\frac{\omega\omega_m}{\omega_0^2 - \omega^2 + \omega_0\omega_m}\right)$$







$$H_\phi(\rho,\phi)=-\frac{\gamma E_0}{2\sqrt{3}\omega\mu_e J_1(x'_{11})}\left(\left(J'_1(\gamma\rho)-\frac{\kappa}{\mu\gamma\rho}J_1(\gamma\rho)\right)\left(1-i\sqrt{3}\right)e^{i\phi}+\left(J'_1(\gamma\rho)+\frac{\kappa}{\mu\gamma\rho}J_1(\gamma\rho)\right)\left(1+i\sqrt{3}\right)e^{-i\phi}\right)$$

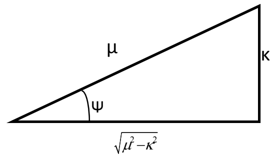
$$x=\gamma\rho\Rightarrow\begin{cases}J'_1(x)-\frac{\kappa}{\mu x}J_1(x)=0\\ J'_1(x)+\frac{\kappa}{\mu x}J_1(x)=0\end{cases}\Rightarrow\begin{cases}J'_1(x^-)-\frac{\kappa}{\mu x^-}J_1(x^-)=0\\ J'_1(x^+)+\frac{\kappa}{\mu x^+}J_1(x^+)=0\end{cases}$$

$$\Rightarrow x'_{11}=\gamma_{11}R=\sqrt{\omega_{110}^2\varepsilon\frac{\left(\mu^2-\kappa^2\right)}{\mu}}R=\omega_{110}\sqrt{\varepsilon\mu_e}R$$

$$R=\frac{x'_{11}}{\omega_{110}\sqrt{\varepsilon\mu_e}}$$

$$\omega_{110}^+=\frac{x'^+_{11}}{R\sqrt{\varepsilon\mu_e}}\quad\&\quad\omega_{110}^-=\frac{x'^-_{11}}{R\sqrt{\varepsilon\mu_e}}$$

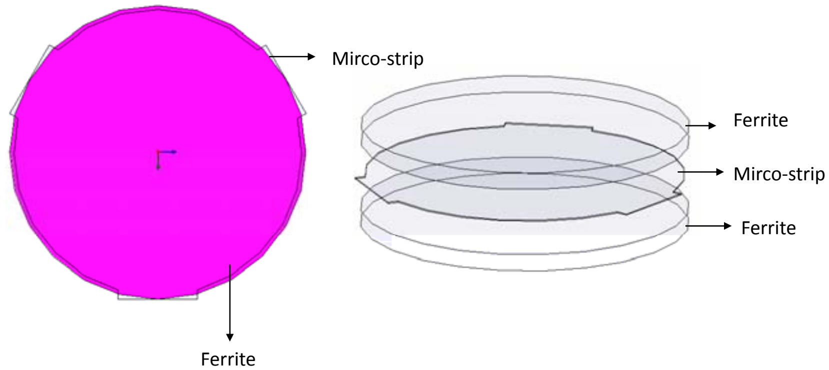
$$\sin\psi=\frac{\pi}{1.841\sqrt{3}}\left(\frac{\kappa}{\mu}\right)\approx\frac{\kappa}{\mu}$$



$$\cos\psi=\frac{\sqrt{\mu^2-\kappa^2}}{\mu}\Rightarrow\mu\cos^2\psi=\frac{\mu^2-\kappa^2}{\mu}=\mu_e$$

HSS Simulation

Simulation Model



Example 1 :

$$R=\frac{x'_{11}}{\omega_{110}\sqrt{\varepsilon\mu_e}}=19.843\text{mm}$$

$$f_0=\omega_0/2\pi=\mu_0\gamma H_0/2\pi=\left(2.8\ MHz/Oe\right)\times\left(H_0\ Oe\right)=10.82GHz$$

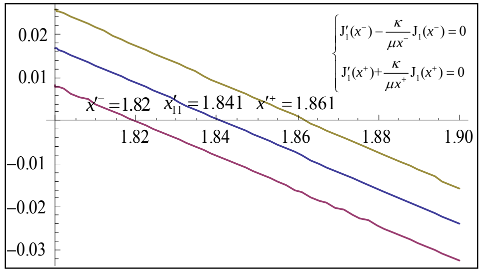
$$f_m=\omega_m/2\pi=\mu_0\gamma M_s/2\pi=\left(2.8\ MHz/Oe\right)\times\left(4\pi M_s\ Gauss\right)=4.9GHz$$

$$f=0.945GHz$$

$$\mu=\mu_0\left(1+\frac{\omega_0\omega_m}{\omega_0^2-\omega^2}\right)=1.4563\mu_0$$

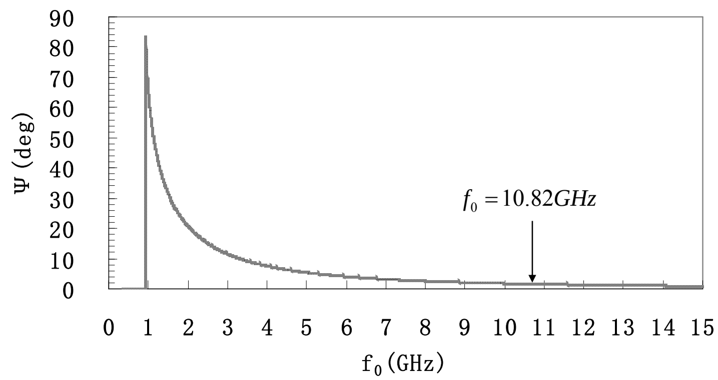
$$\kappa=\mu_0\frac{\omega\omega_m}{\omega_0^2-\omega^2}=0.03985\mu_0$$

$$\mu_e=\frac{\left(\mu^2-\kappa^2\right)}{\mu}=1.4552\mu_0$$

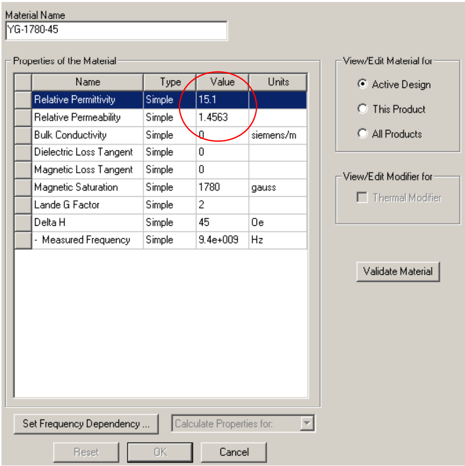


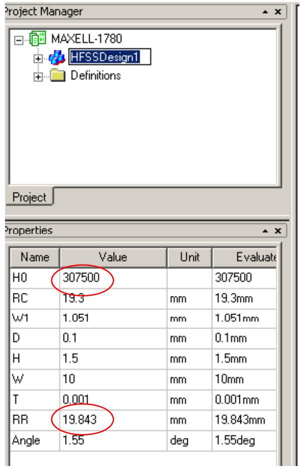
$$\omega_{110}^+=\frac{x'^+_{11}}{R\sqrt{\varepsilon\mu_e}}=\frac{x'^+_{11}}{x'_{11}}\omega_{110}\Rightarrow f_{110}^+=\frac{x'^+_{11}}{x'_{11}}f_{110}=\frac{1.863}{1.841}\times0.945\text{GHz}=0.956\text{GHz}$$

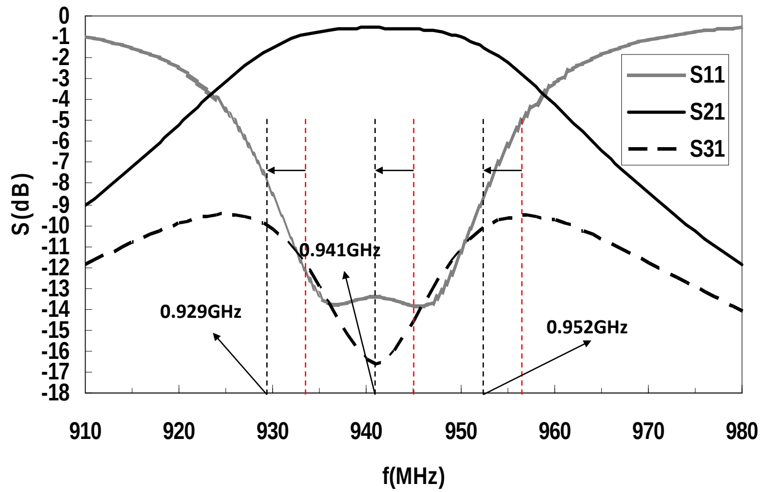
$$\omega_{110}^-=\frac{x'^-_{11}}{R\sqrt{\varepsilon\mu_e}}=\frac{x'^-_{11}}{x'_{11}}\omega_{110}\Rightarrow f_{110}^-=\frac{x'^-_{11}}{x'_{11}}f_{110}=\frac{1.820}{1.841}\times0.945\text{GHz}=0.934\text{GHz}$$

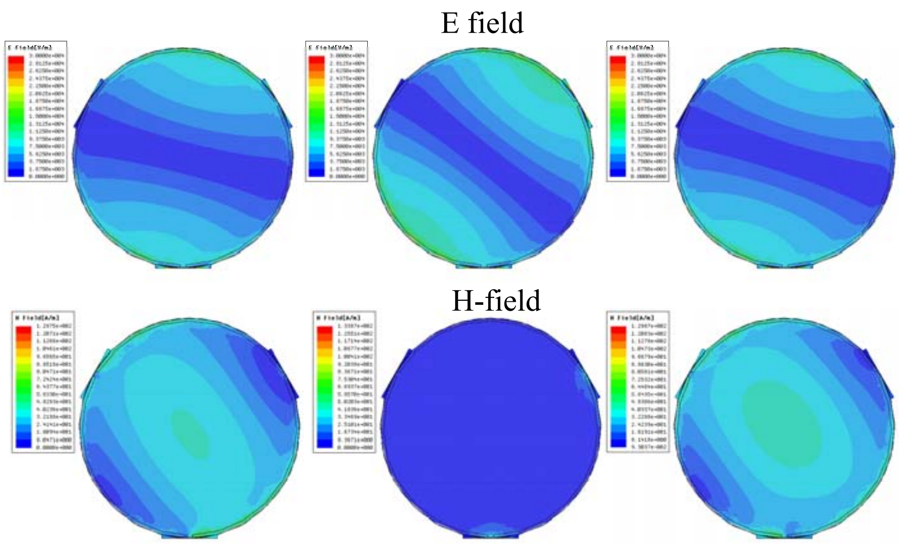


$$\sin\psi=\frac{\pi}{1.841\sqrt{3}}\left(\frac{f\times f_m}{f_0^2-f^2+f_0f_m}\right)\Rightarrow f=0.945GHz\quad\&\quad f_m=4.9GHz$$









0.929GHz 0.941GHz 0.952GHz

Example 2 :

$$R = \frac{x'_{11}}{\omega_{110}\sqrt{\varepsilon\mu_e}} = 22.024\text{mm}$$

$$f_0 = \omega_0 / 2\pi = \mu_0\gamma H_0 / 2\pi = (2.8\ MHz / Oe)\times(H_0\ Oe) = 5.65GHz$$

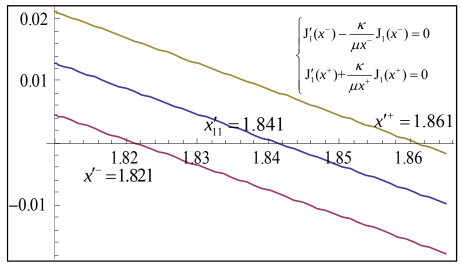
$$f_m = \omega_m / 2\pi = \mu_0\gamma M_s / 2\pi = (2.8\ MHz / Oe)\times(4\pi M_s\ Gauss) = 1GHz$$

$$f = 0.945GHz$$

$$\mu = \mu_0\left(1+\frac{\omega_0\omega_m}{\omega_0^2-\omega^2}\right) = 1.1821\mu_0$$

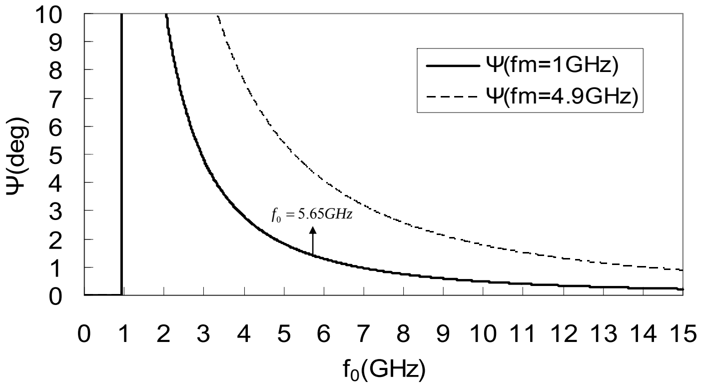
$$\kappa = \mu_0\frac{\omega\omega_m}{\omega_0^2-\omega^2} = 0.030455\mu_0$$

$$\mu_e = \frac{(\mu^2-\kappa^2)}{\mu} = 1.1813\mu_0$$

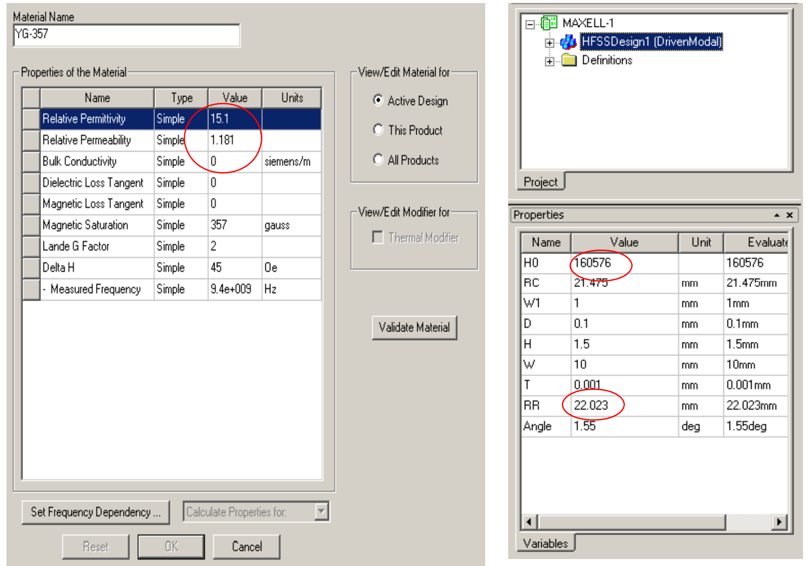


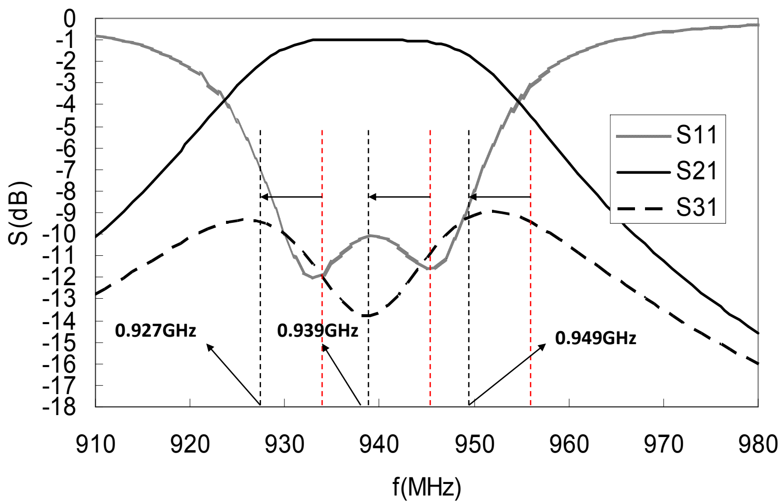
$$\omega_{110}^{+} = \frac{x_{11}'^{+}}{R\sqrt{\varepsilon\mu_e}} = \frac{x_{11}'^{+}}{x'_{11}}\omega_{110} \Rightarrow f_{110}^{+} = \frac{x_{11}'^{+}}{x'_{11}} f_{110} = \frac{1.861}{1.841}\times 0.945\text{GHz} = 0.955\text{GHz}$$

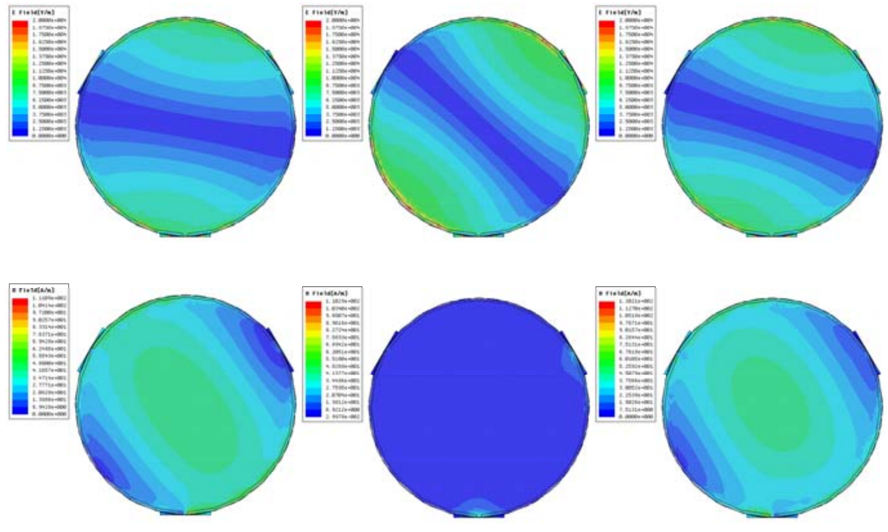
$$\omega_{110}^{-} = \frac{x_{11}'^{-}}{R\sqrt{\varepsilon\mu_e}} = \frac{x_{11}'^{-}}{x'_{11}}\omega_{110} \Rightarrow f_{110}^{-} = \frac{x_{11}'^{-}}{x'_{11}} f_{110} = \frac{1.821}{1.841}\times 0.945\text{GHz} = 0.935\text{GHz}$$



$$\sin\psi = \frac{\pi}{1.841\sqrt{3}}\left(\frac{f \times f_m}{f_0^2 - f^2 + f_0 f_m}\right) \Rightarrow \; f = 0.945GHz \;\; \& \;\; f_m = 1GHz$$







0.927GHz 0.939GHz 0.949GHz