2012 Fall PHYS535 **“**Introduction to Plasma Physics**”** (3 credits)
Instructor: Tsun-Hsu Chang

Textbook：
Dwight R. Nicholson, “Introduction to Plasma Theory” Chaps. 1, 2, 6, and 7.
It is supplemented by two Special Topics.

References：
Krall and Trivelpiece, "Principles of Plasma Physics".
Erwin Kreyszig, "Advanced Engineering Mathematics".

Conduct of Class：
Every Fridays 15:20 - 18:10, total 150 minutes, Physics building R501.
The course is offered in English, but physical concepts will be emphasized.
Students have to go through the algebra in the lecture notes before attending classes.
Questions are strongly encouraged.

Grading Policy：Midterm oral report 50%; Final oral report 50%; Attendance 10% (extra).
The overall score will be normalized to reflect an average consistent with other courses.

Schedule：See the table below.

Week	Date	Content
1	09/16	Introduction & Chap.1
2	09/23	Chap.1
3	09/30	Chap.1
4	10/07	Chap.2
5	10/14	Chap.2
6	10/21	Chap.6
7	10/28	Chap.6
8	11/04	Chap.6
9	11/11	**Oral presentation**
10	11/18	Chap.6
11	11/25	Chap.6
12	12/02	Chap.6
13	12/09	Chap.6
14	12/16	Special Topic I
15	12/23	Special Topic I
16	12/30	Special Topic II
17	01/06	Special Topic II
18	01/13	**Oral presentation**

* This table is for your reference only.
* The practical schedule will depend on the students' learning condition.

清大物理系"離子體物理導論" 任課老師:張存續

"Introduction to Plasma Physics" (PHYS 535)

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Fall Semester, 2012

1. **Textbook:** Dwight R. Nicholson,"Introduction to Plasma Theory" Chapters 1, 2, 6, and 7 (supplemented by two Special Topics).
2. **Principal Reference:** Krall and Trivelpiece, "Principles of Plasma Physics". Other books will be referenced in the lecture notes.
3. **Conduct of Class:** Physical concepts will be emphasized, while algebraic details in the lecture notes will often be skipped. *Questions are encouraged*. It is assumed that students have at least gone through the algebra in the lecture notes before attending classes (*important*!).
4. **Grading Policy:** Major: midterm and final; Minor: attendance. The overall score will be normalized to reflect an average consistent with other courses.

5. **Lecture Notes:**

The first three chapters of the lecture notes follow Nicholson and the rest on selected topics, all starting from basic equations.

As in Nicholson, we adopt the Gaussian unit system. Appendix A in Ch. 1 discusses the conversion between Gaussian and SI systems.

Equations numbered in the format of (1.1), (1.2)... refer to Nicholson. Supplementary equations derived in lecture notes, which will later be referenced, are numbered (1), (2)... [restarting from (1) in each chapter.] Equations in Appendices A, B…of each chapter are numbered (A.1), (A.2)…and (B.1), (B.2)…

Page numbers cited in the text (e.g. p. 120) refer to Nicholson.

Section numbers (e.g. Sec. 1.1) refer to Nicholson. Main topics within each section are highlighted by **boldfaced** characters. Some words are typed in *italicized* characters for attention. Technical terms which are introduced for the first time are <u>underlined</u>.

Chapter 1: Introduction

1.1 Introduction

Loosely defined, a plasma is a gas of charged particles (electrons and ions), whose behavior is dominated by electric and magnetic forces despite the (often unavoidable) presence of neutral particles. A more specific definition will be given later in this chapter, following the development of some basic concepts.

The Fourth State of Matter :

The plasma is often referred to as the fourth state of matter, as is clear from the following phase-transition process:

$$\text{Solid} \underset{\uparrow \atop \text{heat}}{\rightarrow} \text{Liquid} \underset{\uparrow \atop \text{heat}}{\rightarrow} \text{Gas} \underset{\uparrow \atop \text{heat}}{\rightarrow} \text{Plasma (an ionized gas)}$$

Before going into details, we outline below principal properties of the plasma and the existence of plasma in nature and laboratories.

Properties of the Plasma:

long-range interaction (through EM fields)
collective behavior (Debye shielding, waves, …)
self consistency ($\mathbf{E}, \mathbf{B} \leftrightarrow \mathbf{J}, \rho$)
non-Maxwellian distribution $[f \neq n_0 \exp(\frac{-mv^2}{2kT})]$
free energy
instability
classical diffusion (due to collisions)
anomalous diffusion (due to instabilities)

Existence of the Plasma:

gas discharges (spark gap, lightning, sprite, ...)
non-neutral plasmas (accelerators, microwave tubes,….)
controlled fusion (tokamak, laser fusion, ….)
space physics (ionosphere, solar plasma, ….)
astrophysics (stellar plasmas, radiative processes, …)

1.1 Introduction (*continued*)

Although plasma physics is generally considered to be a branch of classical physics, quantum effects will set in when inter particle distances ($\sim 1/n_0^{1/3}$, n_0: plasma density) are smaller or comparable to the de Broglie wavelength (h/p) of a thermal particle, i.e. when

$$\frac{1}{n_0^{1/3}} \leq \frac{h}{p}$$

For the plasmas of interest to most plasma physicists, quantum effects are negligible.

1.2 Debye Shielding

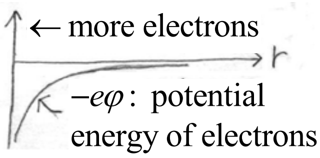
Consider an infinite, uniform plasma of equal electron and (singly-charged) ion density n_0. To test its property, we put a charge q_T at the origin. Then, at the eqilibrium state, the electrostatic potential φ obeys the Poisson equation: $\nabla^2\varphi = -4\pi\rho = 4\pi e(n_e - n_i) - 4\pi q_T \delta(\mathbf{r})$, (1.1)

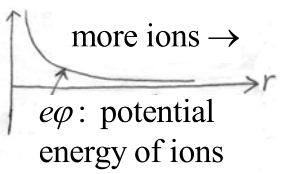
where n_e (n_i) is the electron (ion) density in the presence of q_T.

To solve (1.1), we make two assumptions:

Assumption (i): The electrons and ions are each in thermal equilibrium at temperatures T_e and T_i, respectively, with densities given by:

$$\begin{cases} n_e = n_0 e^{\frac{e\varphi}{kT_e}} & (1.2) \\ n_i = n_0 e^{\frac{-e\varphi}{kT_i}} & (1.3) \end{cases}$$





(1.2) and (1.3) are well-known statistical relations. They will later be derived in this chapter [see (17).]

Assumption (ii) : $\frac{e\varphi}{kT_e} \ll 1$ and $\frac{e\varphi}{kT_i} \ll 1$

Then, $\begin{cases} n_e = n_0 e^{\frac{e\varphi}{kT_e}} \approx n_0(1+\frac{e\varphi}{kT_e}) \\ n_i = n_0 e^{\frac{-e\varphi}{kT_i}} \approx n_0(1-\frac{e\varphi}{kT_i}) \end{cases}$ [Nicholson uses the notation T for both temperature and kinetic energy. Here, we denote the latter by kT.]

and $\nabla^2\varphi = 4\pi e(n_e - n_i) - 4\pi q_T\delta(\mathbf{r})$ [(1.1)] can be written

$$\nabla^2\varphi = 4\pi n_0 e^2(\frac{1}{kT_e}+\frac{1}{kT_i})\varphi - 4\pi q_T\delta(\mathbf{r}) \quad (1)$$

As $r \to 0$, (1) reduces to $\nabla^2\varphi \simeq -4\pi q_T\delta(\mathbf{r})$. Hence*,

$$\lim_{r\to 0}\varphi = \frac{q_T}{r}, \quad (2)$$

(2) is the Coulomb potential of q_T. Physically, this is because q_T makes the dominant contribution to φ as $r \to 0$.

*See Eq. (B.10) in Appendix B.

For $r > 0$, (1) reduces to

$$\nabla^2\varphi = \frac{1}{r^2}\frac{d}{dr}(r^2\frac{d\varphi}{dr}) = 4\pi n_0 e^2(\frac{1}{kT_e}+\frac{1}{kT_i})\varphi \quad (1.3)$$

Defining the electron Debye length λ_{De}, ion Debye length λ_{Di}, and plasma Debye length λ_D as

$$\lambda_{De,i} \equiv \sqrt{kT_{e,i}/(4\pi n_0 e^2)} \left[\approx 740\sqrt{kT_{e,i}(eV)/n_0(cm^{-3})} \text{ cm}\right] \quad (1.4)$$

$$\frac{1}{\lambda_D^2} \equiv \frac{1}{\lambda_{De}^2} + \frac{1}{\lambda_{Di}^2}, \quad (1.5)$$

we may write (1.3) as $\frac{1}{r^2}\frac{d}{dr}(r^2\frac{d\varphi}{dr}) = \frac{\varphi}{\lambda_D^2}$, (1.6)

which has the solution: $\varphi = A\frac{e^{-\frac{r}{\lambda_D}}}{r} + B\frac{e^{\frac{r}{\lambda_D}}}{r}$

Boundary conditions $\left\{\begin{array}{l} \lim_{r\to\infty}\varphi(r) = 0 \\ \lim_{r\to 0}\varphi(r) = \frac{q_T}{r} \;\; [(2)] \end{array}\right\}$ give $\begin{cases} B = 0 \\ A = q_T \end{cases}$

The solution of (1.1) is thus

$$\varphi = \frac{q_T}{r} e^{-\frac{r}{\lambda_D}} \qquad (1.8)$$

from which we obtain the only component of the electric field:

$$E_r = -\frac{d\varphi}{dr} = q_T\left(\frac{1}{r^2} + \frac{1}{r\lambda_D}\right) e^{-\frac{r}{\lambda_D}},$$

the charge density:

$$\rho = \frac{-1}{4\pi}\nabla^2\varphi = \overbrace{q_T\delta(\mathbf{r})}^{\text{test charge}} \overbrace{-\frac{q_T}{4\pi r\lambda_D^2} e^{-\frac{r}{\lambda_D}}}^{\text{shielding cloud}},$$

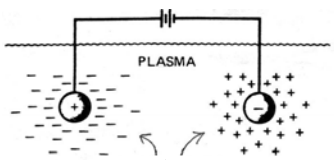
and the total charge in a sphere of radius r (including test charge):

$$Q_{Total}(r) = \frac{1}{4\pi}\oint \mathbf{E}\cdot d\mathbf{a} = r^2 E(r) = q_T\left(1 + \frac{r}{\lambda_D}\right) e^{-\frac{r}{\lambda_D}}$$

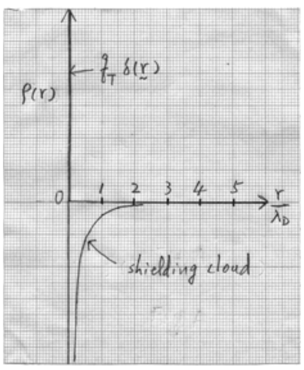
We find $Q_{Total}(r) \to 0$ as $r \to \infty$. This shows that the total charge in the shielding cloud is $-q_T$, which exactly cancels the test charge.

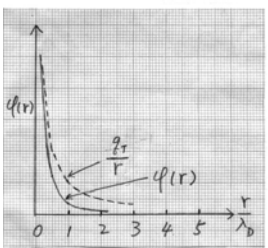
Discussion: Rewrite $\rho = q_T\delta(\mathbf{r}) - \frac{q_T}{4\pi r\lambda_D^2} e^{-\frac{r}{\lambda_D}}$

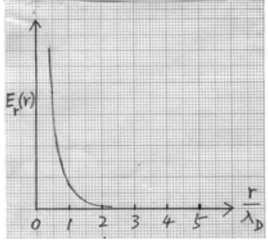
1. When a test charge q_T is placed in a plasma, the plasma particles form a charge cloud of the opposite sign (with total charge $-q_T$ and most of which in a radius of $\sim\lambda_D$) to shield q_T. As a result, φ due to q_T falls off as $\frac{1}{r}$ when $r \ll \lambda_D$ [(2)]. But as r increases, φ falls off much faster than $\frac{1}{r}$ due to the shielding cloud. This effect is called <u>Debye shielding</u>.



Debye shielding







2. Through the shielding charge cloud, the plasma effectively neutralizes the test charge over a distance of approximately λ_{D}. Hence, a plasme can be regarded as quasineutral ($n_e \approx n_i$) if $\lambda_D \ll L$, where L is the dimension of the plasma.

Approximate Electron Densities and Temperatures for Typical Plasmas

Plasma	Density (cm^{-3})	Temperature (°K)	λ_D (cm)
Interstellar gas	10^0	10^4	7×10^2
Gaseous nebula	10^2	10^4	7×10^1
Ionosphere (F layer)	10^6	10^3	2×10^{-1}
Solar corona	10^6	10^6	7
Tenuous laboratory plasma	10^{11}	10^4	2×10^{-3}
Solar atmosphere	10^{14}	10^4	7×10^{-5}
Dense laboratory plasma	10^{15}	10^5	7×10^{-5}
Thermonuclear plasma	10^{16}	10^8	7×10^{-4}
Metal	10^{23}	10^2	2×10^{-10}
Stellar interior	10^{27}	10^7	7×10^{-10}

3. Any plasma particle can be regarded as a test charge and is therefore shielded by the other particles just as the test charge.

4. $\lambda_{De,i} = \sqrt{\dfrac{kT_{e,i}}{4\pi n_0 e^2}} \quad \left[\dfrac{1}{\lambda_D^2} = \dfrac{1}{\lambda_{De}^2} + \dfrac{1}{\lambda_{Di}^2} \right]$

Physical reason for $\lambda_{De,i} \propto \sqrt{T_{e,i}}$: When $T_{e,i} \neq 0$, the neutralizing particles, after drawing to the shileding cloud, have the mobility (due to thermal velocities) to move out of the cloud, thus reducing their shielding ability.

Physical reason for $\lambda_{\mathrm{D}} \propto \dfrac{1}{\sqrt{n_0}}$: The shielding ability of the plasma increases with its density.

5. In a plasma, we can have $T_e \neq T_i$ for a limited time duration. As will be shown in Sec. 1.6 [Eqs. (36)-(39)], this is because of the difference in ralaxation time scales: $\tau_{ee} \ll \tau_{ii} \ll \tau_{ie}$. As a result, it takes a much longer time for T_e and T_i to equalize than for the electrons and ions to thermalize.

6. In the presence of a magnetic field **B**, forces acting on charges along **B** can be different from those acting perpendicular to **B**. Thus, even a single species (e.g. the electrons) can have two temperatures, $T_\perp$ and $T_\parallel$, before collisions equalize the two temperatures.

7. Temperatures of laboratory plasmas typically range from 1 eV to 10 keV (1 eV $\approx 11,600$ K), but with a density much lower than that of the air (see table two pages back). Hence, the thermal energy per unit volume (nkT) can be much lower than that of the ordinary matter. For example, T_e of a fluorescent light tube is $\sim 20,000$ K, but the energy content just barely warms up the glass tube.

8. In a neutral gas, particles interact with each other only when they come into close contact. In the plasma, a charged particle interact simultaneously with many particles through electromagnetic forces. The short-range binary interactions still take place in a plasma, but particle interactions are dominated by long range forces. These forces are important because of the large number of particles involved. A quantitative comparison between long- and short-range interactions will be given in Sec. 1.6.

9. Long-range interactions can result in a variety of collective behavior. Debye shielding is an example of such behavior, in which many particles respond cooperatively to shield the electrostatic field of the test charge. Collective behavior is responsible for the richness of plasma phenomena. For example, there are numerous types of plasma waves, but there is only one type of wave (the sound wave) in a neutral gas. However, long-range interactions can also be stochastic in nature, as in collisions discussed in Sec. 1.6.

1.3 Plasma Parameter

Derivation of λ_D (hence Debye shielding) has used the statistical relations (1.2) and (1.3), which are valid only when there are *many* particles in the shielding cloud of radius ~ λ_D, i.e.

$$\Lambda \equiv n_0 \lambda_D^3 \gg 1 \tag{1.14}$$

where Λ is called the plasma parameter.

Derivation of λ_D has also assumed $kT_{e,i} \gg e\varphi$. We take φ to be the average potential at a particle due to its nearest neighbor located a distance $1/n_0^{1/3}$ away:

$$\varphi \approx \frac{e}{r} \approx e n_0^{1/3} \left[\begin{array}{l}\text{Coulomb potential is appropriate to}\\ \text{use because particle distances} \ll \lambda_D.\end{array}\right] \tag{1.9}$$

Thus, $kT_{e,i} \gg e\varphi$ implies $kT_{e,i} \gg n_0^{1/3} e^2$ (1.12)

Using $\lambda_D \approx \sqrt{kT_{e,i}/(4\pi n_0 e^2)}$, (1.12) gives $n_0 \lambda_D^3 \gg 1$, which is again (1.14). Thus, (1.14) is the validity condition for the Debye theory.

1.4 Plasma Frequency

This section considers another familiar example of the collective behavior : plasma oscillations. We begin with a development of the basic equations for a detailed treatment of this important problem.

Basic Equations : First, we will derive the fluid equation (7.5) in Sec. 7.1 by following a small element of the α-th plasma species, which has a velocity $\mathbf{v}_\alpha(\mathbf{x}, t)$ at position $\mathbf{x}$ and time t. By Newton's law of motion, we write $n_\alpha(\mathbf{x}, t) m_\alpha \frac{d}{dt}\mathbf{v}_\alpha(\mathbf{x}, t) = \mathbf{f}_\alpha(\mathbf{x}, t)$, (3)

where n_α is the particle density of species α, m_α is the particle mass, and $\mathbf{f}_\alpha$ is the force per unit volume acting on the fluid element. Note that although $n_\alpha(\mathbf{x}, t)$ varies with t, it is not to be differentiated with repect to t because $\mathbf{f}_\alpha$ acts on the mass per unit volume $(n_\alpha m_\alpha)$.

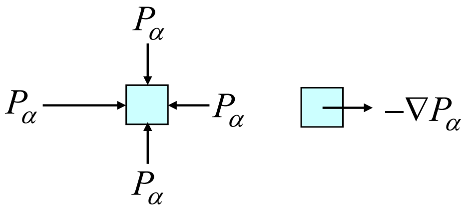
For the LHS, we have $\frac{d}{dt}\mathbf{v}_\alpha(\mathbf{x}, t) = \frac{\partial}{\partial t}\mathbf{v}_\alpha + (\frac{d}{dt}\mathbf{x} \cdot \nabla)\mathbf{v}_\alpha$

Since $\mathbf{x}$ is the position of the fluid element, $\frac{d}{dt}\mathbf{x}$ is the velocity of the fluid element. Thus, $\frac{d}{dt}\mathbf{v}_\alpha(\mathbf{x}, t) = \frac{\partial}{\partial t}\mathbf{v}_\alpha + (\mathbf{v}_\alpha \cdot \nabla)\mathbf{v}_\alpha$. (4)

Rewrite (3): $n_\alpha(\mathbf{x},t)m_\alpha \frac{d}{dt}\mathbf{v}_\alpha(\mathbf{x},t)=\mathbf{f}_\alpha(\mathbf{x},t),$ (3)

On the RHS, we consider only the Lorentz force density and the pressure force density: $\mathbf{f}_\alpha(\mathbf{x},t)=n_\alpha q_\alpha[\mathbf{E}+\frac{1}{c}\mathbf{v}_\alpha\times\mathbf{B}]-\nabla P_\alpha,$ (5)

where q_α is charge per particle. Note that the fluid pressure "P_α" has the unit of force/cm^2, while pressure force density "$-\nabla P_\alpha$" (force per unit volume due to P_α) has the unit of force/cm^3.



Sub. the expressions for $\frac{d}{dt}\mathbf{v}_\alpha(\mathbf{x},t)$ [(4)] and $\mathbf{f}_\alpha(\mathbf{x},t)$ [(5)] into (3), we obtain the fluid equation for species α [same as (7.5)]:

$$n_\alpha m_\alpha[\frac{\partial}{\partial t}\mathbf{v}_\alpha+(\mathbf{v}_\alpha\cdot\nabla)\mathbf{v}_\alpha]=n_\alpha q_\alpha(\mathbf{E}+\frac{1}{c}\mathbf{v}_\alpha\times\mathbf{B})-\nabla P_\alpha \quad (6)$$

Discussion: In writing the pressure force density as "$-\nabla P_\alpha$", we have assumed an isotropic velocity distribution for the particles. This has considerably simplified our derivation of the fluid equation. A rigorous derivation can be found in Nicholson Sec. 7.2, which also leads to (7.5), but with a specific expression for P_α:

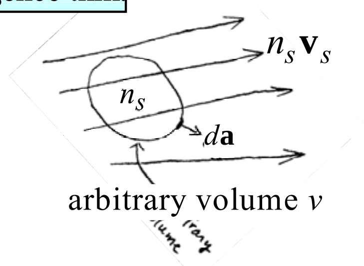
$$P_\alpha = n_\alpha kT_\alpha \quad (7)$$

Next, we note that conservation of particles requires (see figure)

$$\frac{d}{dt}\int_v n_\alpha d^3x+\int_v(n_\alpha\mathbf{v}_\alpha)\cdot d\mathbf{a}=0\Rightarrow\int_v\frac{\partial}{\partial t}n_\alpha d^3x+\int_v\nabla\cdot(n_\alpha\mathbf{v}_\alpha)d^3x$$

$$\Rightarrow \quad \frac{\partial}{\partial t}n_\alpha+\nabla\cdot(n_\alpha\mathbf{v}_\alpha)=0, \quad \boxed{\text{divergence thm.}} \quad (8)$$

which is the equation of continuity.



Rewrite the fluid equation:

$$n_\alpha m_\alpha[\frac{\partial}{\partial t}\mathbf{v}_\alpha + (\mathbf{v}_\alpha \cdot \nabla)\mathbf{v}_\alpha] = n_\alpha q_\alpha(\mathbf{E} + \frac{1}{c}\mathbf{v}_\alpha \times \mathbf{B}) - \nabla P_\alpha \qquad (6)$$

and the continuity equation:

$$\frac{\partial}{\partial t} n_\alpha + \nabla \cdot (n_\alpha \mathbf{v}_\alpha) = 0 \qquad (8)$$

Finally, the EM fields are governed by the Maxwell equations:

$$\begin{cases} \nabla \cdot \mathbf{E} = 4\pi \sum_\alpha n_\alpha q_\alpha \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial}{\partial t}\mathbf{B} \\ \nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial}{\partial t}\mathbf{E} + \frac{4\pi}{c}\sum_\alpha n_\alpha \mathbf{v}_\alpha \end{cases} \qquad (9)$$

(6), (8), and (9) form a set of coupled, self-consistent equations, i.e. the fluid motion is determined by dynamic equations (6) [$\alpha = 1,2,..$] through **E** and **B**, while **E** and **B** are determined by field equations (9) through the fluid motion (which produces ρ and **J**).

Plasma Oscillations :

Assume that the plasma is formed of two fluids (electrons and singly-charged ions), each obeying (6) and (8), and contributing to (9). Then, the complete set of equations are

$$\begin{cases} n_e m_e[\frac{\partial}{\partial t}\mathbf{v}_e + (\mathbf{v}_e \cdot \nabla)\mathbf{v}_e] = -n_e e(\mathbf{E} + \frac{1}{c}\mathbf{v}_e \times \mathbf{B}) - \nabla P_e & (10a) \\ n_i m_i[\frac{\partial}{\partial t}\mathbf{v}_i + (\mathbf{v}_i \cdot \nabla)\mathbf{v}_i] = n_i e(\mathbf{E} + \frac{1}{c}\mathbf{v}_i \times \mathbf{B}) - \nabla P_i & (10b) \\ \nabla \cdot \mathbf{E} = 4\pi e(n_i - n_e) & (10c) \\ \nabla \cdot \mathbf{B} = 0 & (10d) \\ \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial}{\partial t}\mathbf{B} & (10e) \\ \nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial}{\partial t}\mathbf{E} + \frac{4\pi e}{c}(-n_e \mathbf{v}_e + n_i \mathbf{v}_i) & (10f) \\ \frac{\partial}{\partial t} n_e + \nabla \cdot (n_e \mathbf{v}_e) = 0 & (10g) \\ \frac{\partial}{\partial t} n_i + \nabla \cdot (n_i \mathbf{v}_i) = 0 & (10h) \end{cases}$$

Although the oscillation behavior involves plasma currents, we start out with the assumption:

$$\mathbf{B} = 0 \tag{11a}$$

From (10e) and (10f), this is equivalent to assuming

$$\begin{cases} \nabla \times \mathbf{E} = 0 \quad \text{[The oscillation is an electrostatic phenomenon]} & (11b) \\ \frac{\partial}{\partial t}\mathbf{E} + 4\pi e(-n_e \mathbf{v}_e + n_i \mathbf{v}_i) = 0 \quad \begin{bmatrix} \text{displacement current} \\ +\text{particle current} = 0 \end{bmatrix} & (11c) \end{cases}$$

which need to be justified later on the basis of the results obtained.

We further make the simplifying assumption:

$$T_e = T_i = 0 \tag{11d}$$

which, by (7), implies that the plasma is "cold" and hence $P_e = P_i = 0$.

Under approximations (11a)-(11d), Eqs. (10a)-(10h) reduce to

$$\begin{cases} \frac{\partial}{\partial t}\mathbf{v}_e + (\mathbf{v}_e \cdot \nabla)\mathbf{v}_e = -\frac{e}{m_e}\mathbf{E} & (12a) \\ \frac{\partial}{\partial t}\mathbf{v}_i + (\mathbf{v}_i \cdot \nabla)\mathbf{v}_i = \frac{e}{m_i}\mathbf{E} & (12b) \\ \nabla \cdot \mathbf{E} = 4\pi e(n_i - n_e) & (12c) \\ \frac{\partial}{\partial t} n_e + \nabla \cdot (n_e \mathbf{v}_e) = 0 & (12d) \\ \frac{\partial}{\partial t} n_i + \nabla \cdot (n_i \mathbf{v}_i) = 0 & (12e) \end{cases}$$

As a simple but important first step, we find the solutions for the <u>equilibrium state</u> ($\frac{\partial}{\partial t} = 0$). The equilibrium solutions are obviously

$$\begin{cases} \mathbf{v}_{e0} = \mathbf{v}_{i0} = \mathbf{E}_0 = 0 \\ n_{e0} = n_{i0} = n_0 = const \ (\Rightarrow \text{ infinite and uniform plasma}) \end{cases}$$

where the equilibrium solutions are treated as <u>zero-order</u> quantities and denoted by subscript "0".

Next, assume that the equilibrium state is slightly perturbed, and denote the perturbations by subscript "1" as <u>first-order</u> quantities.

$$\begin{cases} \mathbf{v}_e = \cancel{\mathbf{v}_{e0}} + \mathbf{v}_{e1} \\ \mathbf{v}_i = \cancel{\mathbf{v}_{i0}} + \mathbf{v}_{i1} \\ n_e = n_0 + n_{e1} \\ n_i = n_0 + n_{i1} \\ \mathbf{E} = \cancel{\mathbf{E}_0} + \mathbf{E}_1 \end{cases} \tag{13}$$

Sub (13) into (12) and keep only first order terms,

$$\frac{\partial}{\partial t}\mathbf{v}_{e1} = -\frac{e}{m_e}\mathbf{E}_1 \tag{14a}$$

$$\frac{\partial}{\partial t}\mathbf{v}_{i1} = \frac{e}{m_i}\mathbf{E}_1 \tag{14b}$$

$$\nabla \cdot \mathbf{E}_1 = 4\pi e(n_{i1} - n_{e1}) \tag{14c}$$

$$\frac{\partial}{\partial t} n_{e1} + n_0 \nabla \cdot \mathbf{v}_{e1} = 0 \tag{14d}$$

$$\frac{\partial}{\partial t} n_{i1} + n_0 \nabla \cdot \mathbf{v}_{i1} = 0 \tag{14e}$$

$(\mathbf{v}_{e1} \cdot \nabla)\mathbf{v}_{e1}$ and $(\mathbf{v}_{i1} \cdot \nabla)\mathbf{v}_{i1}$ are 2nd order terms

The steps from (12) to (14) are called the <u>linearization</u> procedure. As a result, (12a)-(12e) reduce to a set of linear differential equations [(14)], which can be readily solved. However, the solutions are valid only when the perturbations are sufficiently small ($n_{e1} \ll n_0$, $n_{i1} \ll n_0$, $\cdots$) so that higher order perturbations can be neglected. Consider a <u>normal mode</u> by letting

$$\mathbf{v}_{e1}(\mathbf{x}, t) = v_{e1k} e^{-i\omega t + ik_z z} \mathbf{e}_z \tag{15a}$$

$$\mathbf{v}_{i1}(\mathbf{x}, t) = v_{i1k} e^{-i\omega t + ik_z z} \mathbf{e}_z \tag{15b}$$

$$n_{e1}(\mathbf{x}, t) = n_{e1k} e^{-i\omega t + ik_z z} \tag{15c}$$

$$n_{i1}(\mathbf{x}, t) = n_{i1k} e^{-i\omega t + ik_z z} \tag{15d}$$

$$\mathbf{E}_1(\mathbf{x}, t) = E_{1k} e^{-i\omega t + ik_z z} \mathbf{e}_z \tag{15e}$$

Subscripts "*e*" and "*i*" denote particle species. Subscript "0" denotes zero-order quantities. Subscript "1" denotes first-order quantities. Subscript "*k*" denotes a normal mode.

where the normal mode is independent of *x, y,* and v_{e1k}, v_{i1k}, n_{e1k}, n_{i1k}, E_{1k} are the (constant) amplitudes of the respective variables. It is understood that the LHS is given by the real part of the RHS.

Sub. (15) into (14), we obtain a set of linear algebraic equations:

$$\begin{cases} -i\omega v_{e1k} = -\frac{e}{m_e}E_{1k} & \text{(16a)} \\ -i\omega v_{i1k} = \frac{e}{m_i}E_{1k} & \text{(16b)} \\ ik_z E_{1k} = 4\pi e(n_{i1k} - n_{e1k}) & \text{(16c)} \\ -i\omega n_{e1k} + ik_z n_0 v_{e1k} = 0 & \text{(16d)} \\ -i\omega n_{i1k} + ik_z n_0 v_{i1k} = 0 & \text{(16e)} \end{cases}$$

From (16a,b) and (16d,e), we obtain v_{e1k}, v_{i1k}, n_{e1k}, and n_{i1k} in terms of E_{1k} :

$$\begin{cases} v_{e1k} = -i\frac{e}{\omega m_e}E_{1k} & \text{(17a)} \\ v_{i1k} = i\frac{e}{\omega m_i}E_{1k} & \text{(17b)} \\ n_{e1k} = -i\frac{k_z}{\omega^2}\frac{n_0 e}{m_e}E_{1k} & \text{(17c)} \\ n_{i1k} = i\frac{k_z}{\omega^2}\frac{n_0 e}{m_i}E_{1k} & \text{(17d)} \end{cases}$$

Sub. $n_{e1k} = -i\frac{k_z}{\omega^2}\frac{n_0 e}{m_e}E_{1k}$ [(17c)] and $n_{i1k} = i\frac{k_z}{\omega^2}\frac{n_0 e}{m_i}E_{1k}$ [(17d)] into $ik_z E_{1k} = 4\pi e(n_{i1k} - n_{e1k})$ [(16c)], we obtain

$$ik_z E_{1k} = 4\pi i\frac{k_z}{\omega^2}\left(\frac{n_0 e^2}{m_e} + \frac{n_0 e^2}{m_i}\right)E_{1k},$$

which can be written $(\omega^2 - \omega_{pe}^2 - \omega_{pi}^2)E_{1k} = 0$ (18)

where $$\begin{cases} \omega_{pe} \equiv \sqrt{\frac{4\pi n_0 e^2}{m_e}} \; [= 5.64\times 10^4\sqrt{n_0(\text{cm}^{-3})}\;\frac{\text{rad}}{\text{sec}}] \\ \omega_{pi} \equiv \sqrt{\frac{4\pi n_0 e^2}{m_i}} \; [= \sqrt{\frac{m_e}{m_i}}\omega_{pe} \ll \omega_{pe}] \end{cases}$$

For (18) to have a non-trivial solution ($E_{1k} \neq 0$), ω can only have a single frequency given by $\omega^2 = \omega_{pe}^2 + \omega_{pi}^2 = \omega_p^2$ (19)

where $$\omega_p^2 \equiv \omega_{pe}^2 + \omega_{pi}^2 \qquad (20)$$

ω_p is called the plasma frequency. It is a characteristic frequency of the plasma most frequently encountered in plasma studies.

To summarize, the solution of this problem is in (15) with the amplitude constants given by (17) and ω given by (19). As a final step, we need to justify the assumption $\mathbf{B}=0$ made in (11a), which

requires $\begin{cases} \nabla\times\mathbf{E}=0 & (11b) \\ \frac{\partial}{\partial t}\mathbf{E}+4\pi e(-n_e\mathbf{v}_e+n_i\mathbf{v}_i)=0 & (11c) \end{cases}$

or, upon linearization, $\begin{cases} \nabla\times\mathbf{E}_1=0 & (21a) \\ \frac{\partial}{\partial t}\mathbf{E}_1+4\pi n_0 e(-\mathbf{v}_{e1}+\mathbf{v}_{i1})=0 & (21b) \end{cases}$

From (15a,b,e) and (17a,b), we have $\begin{cases} \mathbf{E}_1=E_{1k}e^{-i\omega t+ik_z z}\mathbf{e}_z \\ \mathbf{v}_{e1}=-i\frac{e}{\omega m_e}\mathbf{E}_1 \\ \mathbf{v}_{i1}=i\frac{e}{\omega m_i}\mathbf{E}_1 \end{cases}$

(21a) is clearly satisfied. The LHS of (21b) gives

$$-i\omega\mathbf{E}_1+i\tfrac{1}{\omega}(\omega_{pe}^2+\omega_{pi}^2)\mathbf{E}_1=i\tfrac{1}{\omega}(-\omega^2+\omega_{pe}^2+\omega_{pi}^2)\mathbf{E}_1\overset{\text{by (17)}}{=}0$$

Hence, (21b) is also satisfied.

Discussion:

1. In (15), we have assumed a wave solution (i.e. all quantities are $\sim e^{-i\omega t+ik_z z}$). But (19) shows that the frequeny ω is fixed at a constant value ω_p independent of the wave number k_z. As a result, the wave does not propagate (group velocity $v_g=d\omega/dk_z=0$). Thus, what we have is an oscillation phenomenon rather than a wave phenomenon.

Rewrite (21b): $\frac{\partial}{\partial t}\mathbf{E}_1+4\pi n_0 e(-\mathbf{v}_{e1}+\mathbf{v}_{i1})=0$. This equation shows an exact cancellation of the plasma and displacement currents. Thus, no magnetic field is generated; the oscillation is electrostatic in nature.

However, if the plasma has a finite temperature (P_e and/or $P_i\neq 0$), thermal velocities will allow the charges to carry any disturbance away from the source. In this case, ω will be a function of k_z (hence $v_g\neq 0$) and plasma oscillations turns into a plasma wave. This will be treated in Chapter 6.

2. Since $\omega_{pe} \gg \omega_{pi}$, we have from (19) $\omega \approx \omega_{pe}$, which implies that plasma oscillations are basically an electron effect. Physically, this is because the wave field varies so rapidly that the much heavier ions can not respond fast enough to play a significant role. Hence, as a good approximation, we may let $m_i \to \infty$, Then, from (16b), $v_{i1} \to 0$, and from (16e), $n_{i1} \to 0$, i.e. the ions are essentially stationary. Thus, we may simply treat the ions as a uniform, neutralizing background with zero density and velocity perturbations and neglect the ion dynamic equation [(10b,h)] in the analysis of plasma oscillations, as is done in Nicholson Sec. 7.2. This approximation will lead to a plasma frequency of $\omega_p = \omega_{pe}$. Comparing it with the exact value in (20), $\omega_p = \omega_{pe}(1 + m_e / m_i)^{1/2}$, we find the error to be of the order of m_e / m_i.

3. In Section 1.2, we have shown that the *electrostatic* field of a test charge is localized within a distance ~ λ_D because of Debye shielding. Here, we find that a *time-varying* electric field can extend well beyond λ_D (i.e. we can have $k\lambda_D \ll 1$). The difference can be attributed to the "inertia effect" associated with the dynamic behavior of the plasma.

Consider Eq. (10a) without the nonlinear term $(\mathbf{v}_e \cdot \nabla)\mathbf{v}_e$ and the magnetic force $\mathbf{v} \times \mathbf{B} / c$,

$$n_e m_e \frac{\partial}{\partial t} \mathbf{v}_e = -n_e e \mathbf{E} - \nabla P_e \qquad (22)$$

The LHS is the inertia term due to the finite electron mass, the importance of which depends upon the rate of change of $\mathbf{v}_e : \frac{\partial}{\partial t} \mathbf{v}_e$.

For the static case $(\frac{\partial}{\partial t} = 0)$, the electron inertia ($m_e$) plays no role and we have

$$-n_e e\mathbf{E} - \nabla P_e = 0 \tag{23}$$

Physically, (23) states that the electric force are balanced by the pressure force, as is required for the static state. The solution of (23) is [write **E** = $-\nabla\varphi(\mathbf{x})$ and assume $P_e = n_e(\mathbf{x})kT_e$]

$$n_e(\mathbf{x}) = n_0 e^{\frac{e\varphi(\mathbf{x})}{kT_e}} \tag{24}$$

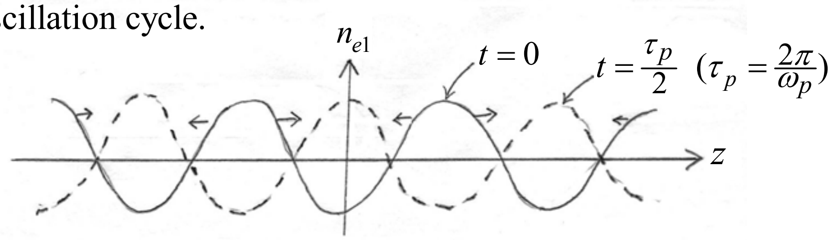
where n_0 and T_e are constants. (24) was used in our earlier derivation of λ_D [see (1.2)].

To consider the dynamic case ($\frac{\partial}{\partial t} \neq 0$), we rewrite (22):

$$n_e m_e \frac{\partial}{\partial t}\mathbf{v}_e = -n_e e\mathbf{E} - \nabla P_e,$$

which shows that, depending on its relative magnitude with respect to the other terms, the LHS may modify the static results (e.g. Debye shielding) only slightly (e.g. the case of a moving test charge, see Nicholson, p.3) or in a fundamental way (e.g. plasma oscillations).

4. The effect of electron inertia is best illustrated by the example of plasma oscillations. The solid curve in the figure below shows the perturbed electron density at $t = 0$. In order to restore neutrality, the excess electrons in regions with $n_{e1} > 0$ will be pushed by the electric field into regions with $n_{e1} < 0$. In doing so, the electrons will obtain maximum velocities at the state of complete neutrality (field energy becoming kinetic energy). The momentum will then carry the motion further (an inertia effect). As a result, half a plasma period later, the electron density turns nonuniform again in the opposite way (dashed curve). The reverse process then starts to complete the second half of the oscillation cycle.



1.4 Plasma Frequency (*continued*)

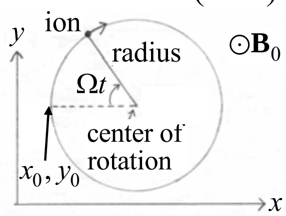
5. The analysis of plasma oscillations here is a typical example of fluid treatment of plasma modes. In Ch. 7 of Nicholson, a variety of plasma modes are derived with fluid equations. As we will show in Ch. 6, the same phenomena can be analyzed on the basis of a (more rigorous) kinetic equation called the Vlasov equation. So the plan of this course is to incorporate most of the materials in Ch. 7 of Nicholson into Ch. 6 of lecture notes for a kinetic treatment. Ch. 7 will be referenced in (but not part of) the lecture notes.

1.5 Other Parameters

The plasma is often immersed in an external magnetic field. Hence, cyclotron motion of charged particles can influence plasma behavior. Consider the equation of motion of a single particle of mass m and charge q in a uniform magnetic field $\mathbf{B}_0 = B_0\mathbf{e}_z$,

$$m\ddot{\mathbf{r}} = \frac{q}{c}\dot{\mathbf{r}} \times B_0\mathbf{e}_z \tag{1.24}$$

with initial conditions: $\begin{cases} \mathbf{r}(t=0) = (x_0,\ y_0,\ z_0) \\ \dot{\mathbf{r}}(t=0) = (0,\ v_\perp,\ v_z) \end{cases}$



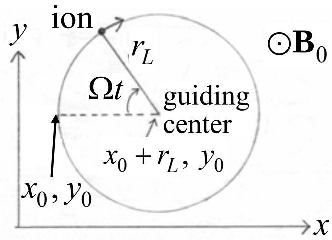
The solutions are:

$$\begin{cases} x(t) = x_0 + \dfrac{v_\perp}{\Omega}(1-\cos\Omega t) \\ y(t) = y_0 + \dfrac{v_\perp}{\Omega}\sin\Omega t \\ z(t) = z_0 + v_z t \end{cases} \quad \& \quad \begin{cases} v_x = v_\perp \sin\Omega t \\ v_y = v_\perp \cos\Omega t \\ v_z = const. \end{cases} \tag{1.25}$$

where $\Omega \equiv \dfrac{qB_0}{mc}$ [called gyrofrequency or cyclotron frequency] (1.26)

and q carries the sign of the charge.

The center of the gyrational motion is called the guiding center. The radius of gyration is called the Larmor radius or gyroradius given by



$$r_L = \frac{v_\perp}{|\Omega|} \tag{1.28}$$

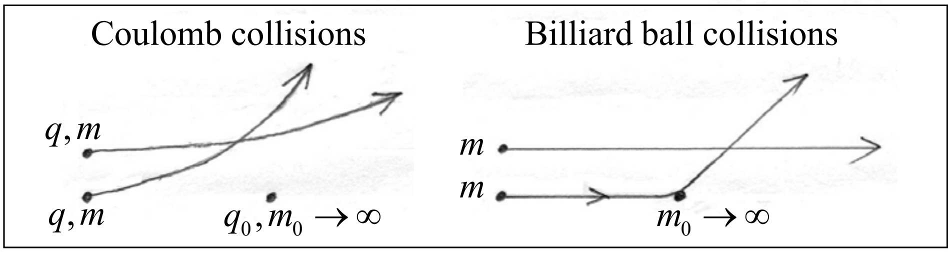
Relativistic correction: Since there is no electric field in (1.24), the particle energy does not change and its relativistic factor:

$$\gamma = \left(1 - \frac{v_\perp^2}{c^2} - \frac{v_z^2}{c^2}\right)^{-\frac{1}{2}}$$

is a constant. For the relativistic correction, we simply replace m with γm in all equations. Thus, $\Omega = \dfrac{qB_0}{\gamma mc}$ $\begin{bmatrix}\text{relativistic cyclotron} \\ \text{frequency}\end{bmatrix}$ (25)

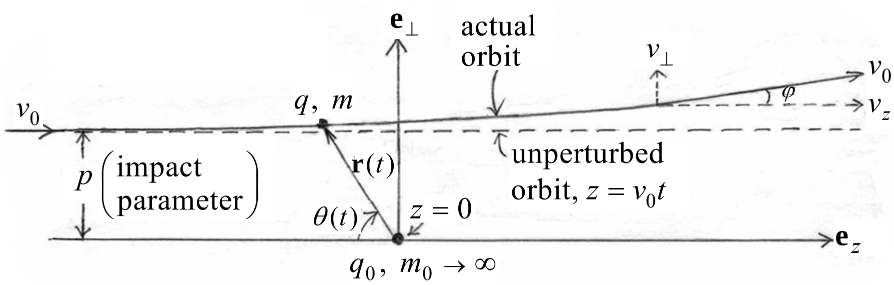
For an electron, $\Omega = \dfrac{eB_0}{\gamma m_e c} = 1.76 \times 10^7 \, \dfrac{B_0(\text{Gauss})}{\gamma} \; \dfrac{\text{rad}}{\text{sec}}$

1.6 Collisions



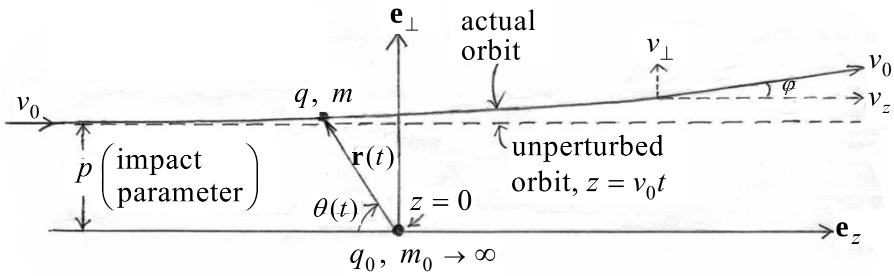
Properties of Coulomb Collisions in a Plasma:

1. Each particle sees the electric field of all the particles within the Debye sphere of this particle, i.e. it is undergoing approximately Λ simultaneous Coulomb collisions.
2. Most of the Coulomb collisions are small-angle collisions (i.e. collisions of small deflection angles). However, the cumulative effect of many small-angle collisions is greater than that of large-angle collisions, as will be shown in this section.



Small - Angle Collisions :

As shown in the figure above, an incident particle with charge q and mass m is colliding with (or scattered by) a particle of charge q_0 and mass m_0. Assume $m_0 \to \infty$, so the scatterer remains stationary. Define the impact parameter p to be the perpendicular distance from the scatterer to the unperturbed orbit ($\parallel$ to the z-axis) of the incident particle. The actual orbit of the incident particle is $\mathbf{r}(t)$, which makes an angle $\theta(t)$ with the negative z-axis. Although the figure shows a repulsive collision, the treantment applies also to attractive collisions.

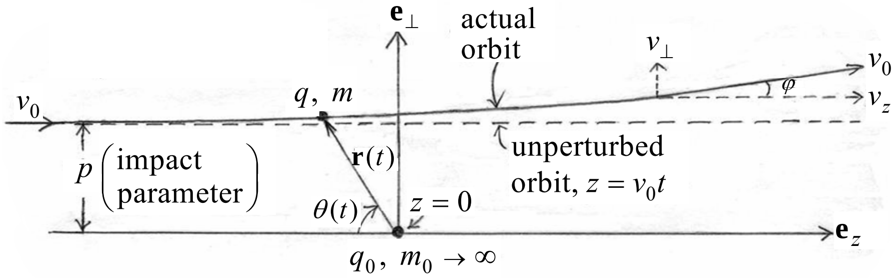


Our primary interest is the scattering angle φ, defined to be the angle betwen the initial ($t \to -\infty$) and final ($t \to \infty$) velicities of the incident particle, i.e. the angle between $v_0 \mathbf{e}_z$ and $v_z \mathbf{e}_z + v_\perp \mathbf{e}_\perp$, where $v_\perp$ is given by

$$m v_\perp = \int_{-\infty}^{\infty} F_\perp(t)dt = \int_{-\infty}^{\infty} \frac{qq_0}{r^2(t)} \sin\theta(t) dt \qquad (26)$$

The scattering angle φ can be evaluated exactly (see Appendix C). Here, we calculate it approximately for the case of small-angle collisions ($\varphi \ll 1$). For $\varphi \ll 1$, we have $v_\perp \ll v_z$ (or $v_z \approx v_0$), which gives the approximate relations:

$$\begin{cases} r\sin\theta \approx p & (27) \\ r\cos\theta \approx -v_0 t & (28) \end{cases}$$



Sub (27) into (26) $\Rightarrow\ v_\perp \approx \dfrac{qq_0}{mp^2}\displaystyle\int_{-\infty}^{\infty} \sin^3\theta dt$ (1.32)

Divide (28) by (27) $\Rightarrow \dfrac{p\cos\theta}{\sin\theta} \approx -v_0 t \Rightarrow dt \approx \dfrac{pd\theta}{v_0 \sin^2\theta}$ (1.34)

Sub (1.34) into (1.32) $\Rightarrow v_\perp \approx \dfrac{qq_0}{mv_0 p}\displaystyle\int_0^\pi \sin\theta d\theta = \dfrac{2qq_0}{mv_0 p}$ (1.35)

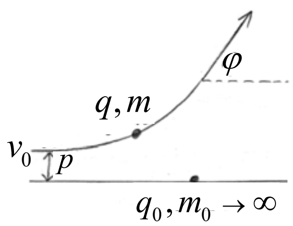
$\Rightarrow \dfrac{v_\perp}{v_0} \approx \dfrac{p_0}{p}$ or $\varphi \approx \dfrac{p_0}{p}$ [valid when $p \gg p_0$] (1.37)

where $p_0 \equiv \dfrac{2qq_0}{mv_0^2}$ $(\Rightarrow \frac{1}{2}mv_0^2 = \dfrac{qq_0}{p_0})$ (1.36)

p_0 is the distance of closest approach for a head-on collision ($p = 0$).

Large - Angle Collisions :

Refer to the right figure. The scattering angle is given by (derived in Appendix C)



$$\tan\frac{\varphi}{2} = \frac{p_0}{2p} \quad [p_0 \equiv \frac{2qq_0}{mv_0^2}] \qquad (29)$$

In the limit $\varphi \to 0$, we have $\varphi \approx \dfrac{p_0}{p}$, in agreement with (1.37).

If $p \le p_0$, (29) gives $\varphi \ge 53^o$, which we define to be large-angle scattering. Then, the cross section (σ_L) for large-angle scattering is

$$\sigma_L = \pi p_0^2$$

For an electron moving with velocity v_0 in a gas of ions ($m_0 \to \infty$), the large-angle collision frequency (γ_L) is

for singly-charged ions

$$\gamma_L = n_0 v_0 \sigma_L = n_0 v_0 \pi p_0^2 = \frac{4\pi n_0 e^2 q_0^2}{m^2 v_0^3} = \frac{4\pi n_0 e^4}{m_e^2 v_0^3},\ [\propto \frac{1}{v_0^3}] \qquad (1.38)$$

where n_0 is the ion density. Note that Nichoson uses the notation ν for collision freuencies. Here, we use γ to avoid confusion with velocity v.

Small - Angle Collision Frequency :

Random walk: Let's first consider the problem of one-dimensional random walk. A man starts out from $x = 0$ and and has walked N steps of equal length ℓ. The direction of each step ($\mathbf{e}_x$ or $-\mathbf{e}_x$) is completely random. The total distance away from $x = 0$ is: $\Delta x^{tot} = \sum_{i=1}^{N} \Delta x_i$,
where Δx_i ($= +\ell$ or $-\ell$) is the i-th step length, and Δx^{tot} can be any multiple of ℓ between $-N\ell$ and $N\ell$ for a single event. Statistically, if there are an arbitrarily large number of similar random-walk events, one can find the average of Δx_i, Δx^{tot}, $(\Delta x^{tot})^2$, etc. Such an average is called the ensemble average and we denote it by $\langle \; \rangle$. Evidently, $\langle \Delta x_i \rangle = 0$ (hence $\langle \Delta x^{tot} \rangle = 0$) because the i-th steps in all events are uncorrelated, and $\langle \Delta x_i \Delta x_j \rangle = 0$ if $i \neq j$ because the i-th and j-th steps of each event are uncorrelated.

Thus, $\left\langle (\Delta x^{tot})^2 \right\rangle = \left\langle (\sum_{i=1}^{N} \Delta x_i)^2 \right\rangle = \sum_{i=1}^{N} \overbrace{\left\langle (\Delta x_i)^2 \right\rangle}^{=\ell^2} + \sum_{i \neq j}^{N} \overbrace{\left\langle \Delta x_i \Delta x_j \right\rangle}^{0} = N\ell^2$ (30)

Taking the square root of $\left\langle (\Delta x^{tot})^2 \right\rangle = N\ell^2$ [(30)], we obtain

$$\sqrt{\left\langle (\Delta x^{tot})^2 \right\rangle} = \sqrt{N}\ell, \tag{31}$$

which shows that, in a great many random-walk events in which a man has randomly walked N steps of equal length ℓ, he will be at an average distance of $\sqrt{N}\ell$ from where he started. So the more steps he makes, the further away he will likely be from the starting point, while the average distance will be proportional to $\sqrt{N}$ rather than N. The key word here is "average". The actual distance traveled differs from event to event. It can be any multiple of ℓ between 0 and $N\ell$, with the highest probability for 0 and the lowest probability for $N\ell$.

The (3-dimensional) diffusion of a molecule in a gas is a typical random-walk phenomenon. If the molecule moves a mean distance ℓ between collisions, it will be at an average distance of $\sqrt{N}\ell$ from the starting point after N steps (ℓ^2 is the diffusion coefficient).

Electron-ion collision frequency: Return to the collision problem. Suppose a test electron with velocity $v_0\mathbf{e}_z$ is injected into an ion gas. After N *small-angle* collisions, all with the same p, the total perturbed velocities will be $\Delta v_x^{tot}=\sum_{i=1}^{N}\Delta v_{xi}$; $\Delta v_y^{tot}=\sum_{i=1}^{N}\Delta v_{yi}$ $\left[\Delta v_{xi}^2+\Delta v_{yi}^2=\Delta v_{\perp i}^2\right]$

Assume $m_i\to\infty$ and the electron velocity $\approx v_0\mathbf{e}_z$ in all collisions. Then, from (1.37), we have $\Delta v_{\perp i}^2=v_0^2p_0^2/p^2$ for all i (32)

This is a random-walk problem in the velocity space. If we repeat the injection for a large number of times, the ensemble averages are:

$$\begin{cases}\langle\Delta v_{xi}\rangle=\langle\Delta v_{yi}\rangle=0, \text{ for all } i\\ \langle\Delta v_{xi}\Delta v_{xj}\rangle=\langle\Delta v_{yi}\Delta v_{yj}\rangle=0,\ i\neq j\end{cases}$$

$\boxed{\langle(\Delta v_{xi})^2\rangle \text{ is the same for all } i.}$

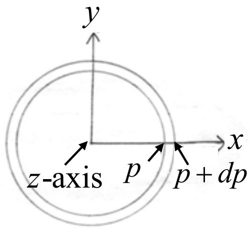
and $$\left\langle(\Delta v_x^{tot})^2\right\rangle=\left\langle\left(\sum_{i=1}^{N}\Delta v_{xi}\right)^2\right\rangle=\sum_{i=1}^{N}\left\langle(\Delta v_{xi})^2\right\rangle=N\left\langle(\Delta v_{xi})^2\right\rangle$$

Similarly, $\left\langle(\Delta v_y^{tot})^2\right\rangle=N\left\langle(\Delta v_{yi})^2\right\rangle.$ $\boxed{=v_0^2p_0^2/p^2 \text{ by (32)}}$

$$\Rightarrow\left\langle(\Delta v_\perp^{tot})^2\right\rangle=N\left\langle(\Delta v_{xi})^2\right\rangle+N\left\langle(\Delta v_{yi})^2\right\rangle=N\left\langle(\Delta v_{\perp i})^2\right\rangle=N\frac{v_0^2p_0^2}{p^2}\quad(33)$$

Rewrite (33): $$\left\langle(\Delta v_\perp)^2\right\rangle=N\frac{v_0^2p_0^2}{p^2}\quad\left[\begin{array}{l}\text{where we have dropped the}\\ \text{superscript "tot" for brevity.}\end{array}\right]\quad(33)$$

Assume that the electron (with initial velocity $v_0\mathbf{e}_z$) is injected along the z-axis (see figure) into the ion gas. (33) gives the average perpendicular velocity obtained by the electron after N small-angle collisions with surrounding ions which are located on a cylindrical shell of radius p (the impact parameter).



If the ion density is n_0, the rate of collisions made with ions located between p and $p+dp$ is

$$\frac{dN}{dt}=2\pi n_0v_0pdp\quad(34)$$

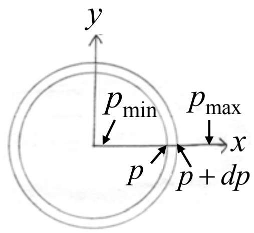
Differentiating (33) with respect to t, we obtain the rate of change of $\left\langle(\Delta v_\perp)^2\right\rangle$ due to collisions with ions located between p and $p+dp$:

$$\frac{d}{dt}\left\langle(\Delta v_\perp)^2\right\rangle=\frac{v_0^2p_0^2}{p^2}\frac{dN}{dt}=2\pi n_0v_0^3p_0^2\frac{dp}{p}\quad(35)$$

Rewrite (35): $\frac{d}{dt}\left\langle (\Delta v_\perp)^2 \right\rangle = \frac{v_0^2 p_0^2}{p^2}\frac{dN}{dt} = 2\pi n_0 v_0^3 p_0^2 \frac{dp}{p}$ (35)

Integrating (35) over p from $p_{\min}$ to $p_{\max}$, we obtain the rate of change of $\left\langle (\Delta v_\perp)^2 \right\rangle$ due to collisions with ions located between $p_{\min}$ and $p_{\max}$:

$$\frac{d}{dt}\left\langle (\Delta v_\perp)^2 \right\rangle = 2\pi n_0 v_0^3 p_0^2 \int_{p_{\min}}^{p_{\max}} \frac{dp}{p} \quad (1.45)$$



Small-angle collisions are, by our definition, for impact parameters of $p > |p_0|$, so we set $p_{\min} = |p_0|$. Because of the Debye shielding effect, the Coulomb field is negligible at $p > \lambda_D$. So we set $p_{\max} = \lambda_D$. Then,

$$\frac{d}{dt}\left\langle (\Delta v_\perp)^2 \right\rangle = 2\pi n_0 v_0^3 p_0^2 \ln\frac{\lambda_D}{|p_0|} \quad (1.46)$$

The log factor in (1.46), $\ln(\lambda_D/|p_0|)$, is insensitive to the argument $\lambda_D/|p_0|$. This justifies our rough choices of $p_{\min}$ and $p_{\max}$.

Rewrite (1.46): $\frac{d}{dt}\left\langle (\Delta v_\perp)^2 \right\rangle = 2\pi n_0 v_0^3 p_0^2 \ln\frac{\lambda_D}{|p_0|}$ (1.46)

For an estimate of $\ln(\lambda_D/|p_0|)$, which is insensitive to $\lambda_D/|p_0|$, we let $v_0 \approx v_{Te} = \sqrt{kT_e/m_e}$ = electron thermal velocity (Nicholson denotes v_{Te} by v_e). This gives $|p_0| = \frac{2e^2}{m_e v_0^2} \approx \frac{2e^2}{kT_e}$ (36)

Ignoring the difference between λ_D and λ_{De}, we have from (1.4)

$$\lambda_D \approx \sqrt{\frac{kT_e}{4\pi n_0 e^2}}$$

Hence, $\frac{\lambda_D}{|p_0|} \approx \lambda_D \frac{kT_e}{2e^2} = \lambda_D 2\pi n_0 \frac{kT_e}{4\pi n_0 e^2} \approx 2\pi n_0 \lambda_D^3 \overset{\boxed{(1.14)}}{=} 2\pi\Lambda,$ (1.47)

In (1.46), setting $|p_0| \approx \frac{2e^2}{m_e v_0^2}$ and $\frac{\lambda_D}{|p_0|} \approx \Lambda$ (neglect the factor 2π),

we obtain $\frac{d}{dt}\left\langle (\Delta v_\perp)^2 \right\rangle \approx \frac{8\pi n_0 e^4}{m_e^2 v_0}\ln\Lambda$ (1.48)

The collision time (τ_c) is roughly the time it takes for the particle to be deflected by 90^0. We define it to be the time for $\langle(\Delta v_\perp)^2\rangle$ to increase to v_0^2 according to the *initial* rate of change in (1.48):

$$\frac{d}{dt}\langle(\Delta v_\perp)^2\rangle \approx \frac{8\pi n_0 e^4}{m_e^2 v_0}\ln\Lambda \qquad (1.48)$$

although v_0 will have changed considerably during this time. Thus, $\frac{d}{dt}\langle(\Delta v_\perp)^2\rangle\cdot\tau_c = v_0^2$ or $\tau_c = \frac{m_e^2 v_0^3}{8\pi n_0 e^4 \ln\Lambda}$, which gives the small-angle collision frequency:

$$\gamma_c = \frac{1}{\tau_c} = \frac{8\pi n_0 e^4}{m_e^2 v_0^3}\ln\Lambda \qquad (1.49)$$

In (1.49), the factor $\ln\Lambda$ is insensitive to Λ. For most plasmas, $\ln\Lambda \sim 10$. Comparing γ_c with the large-angle collision frequency in (1.38): $\gamma_L = \frac{4\pi n_0 e^4}{m_e^2 v_0^3}$ (note both γ_c and γ_L are $\propto \frac{1}{v_0^3}$), we find

$$\frac{\gamma_c}{\gamma_L} = 2\ln\Lambda \gg 1 \qquad (37)$$

i.e. small-angle collisions dominate over large-angle collisions.

Rewrite (1.49): $$\gamma_c = \frac{8\pi n_0 e^4}{m_e^2 v_0^3}\ln\Lambda \qquad (1.49)$$

γ_c in (1.49) applies to *a test electron* of velocity v_0 colliding with an ion gas. In a plasma, we have *an electron gas* colliding with an ion gas. We may estimate the electron-ion collision frequency (γ_{ei}) in the gas by letting $v_0 \approx v_{Te} = \sqrt{kT_e/m_e}$. Thus,

$$\gamma_{ei} = \gamma_c(v_0 = v_{Te}) = \frac{8\pi n_0 e^4}{m_e^{1/2}(kT_e)^{3/2}}\ln\Lambda \quad \begin{bmatrix}\text{electron-ion}\\ \text{collision frequency}\end{bmatrix} \qquad (38)$$

To compare the relative magnitude of γ_{ei} and ω_{pe}, we write

$$\begin{cases} \gamma_{ei} = \dfrac{v_{Te}}{2\pi n_0}\dfrac{(4\pi n_0 e^2)^2}{(kT_e)^2}\ln\Lambda \approx \dfrac{v_{Te}}{2\pi n_0 \lambda_D^4}\ln\Lambda \\ \omega_{pe} = \sqrt{\dfrac{4\pi n_0 e^4}{m_e}} = \sqrt{\dfrac{4\pi n_0 e^4}{kT_e}}\sqrt{\dfrac{kT_e}{m_e}} = \dfrac{v_{Te}}{\lambda_D} \end{cases} \quad \begin{bmatrix}\text{Ignore the difference}\\ \text{between } \lambda_D \text{ and } \lambda_{De}\end{bmatrix}$$

Thus, $$\frac{\gamma_{ei}}{\omega_{pe}} = \frac{\ln\Lambda}{2\pi n_0 \lambda_D^3} = \frac{\ln\Lambda}{2\pi\Lambda} \approx \frac{1}{\Lambda} \ll 1 \qquad (1.51)$$

Rewrite (1.51): $\dfrac{\gamma_{ei}}{\omega_{pe}} = \dfrac{\ln\Lambda}{2\pi n_0 \lambda_D^3} = \dfrac{\ln\Lambda}{2\pi\Lambda} \sim \dfrac{1}{\Lambda} \ll 1$ (1.51)

This shows that Coulomb collisions are not effective in damping plasma oscillations or waves. For example, if $\Lambda = 10^6$, the damping time will be $\sim 10^6$ oscillation periods. As will be considered in Ch.6, a damping mechanism called Landau damping is much more effective.

Electron - electron collision frequency:

When an electron collides with another electron, the scatterer is no longer stationary. The collision frequency can be calculated in the same way by moving to the center-of-mass frame*. The result, within a factor or ~ 2, is the same as (38). So, we have

$$\gamma_{ee} \approx \gamma_{ei} = \frac{8\pi n_0 e^4}{m_e^{1/2}(kT_e)^{3/2}} \ln\Lambda \quad \begin{bmatrix}\text{electron-electron} \\ \text{collision frequency}\end{bmatrix} \qquad (39)$$

*See L. Spitzer, Jr., "Physics of Ionized Gases," 2nd ed., Ch. 5.

1.6 Collisions (*continued*)

Ion - ion collision frequency:

Rewrite (39) : $\gamma_{ee} = \dfrac{8\pi n_0 e^4}{m_e^{1/2}(kT_e)^{3/2}} \ln\Lambda$ (39)

The ion-ion collision frequency can be obtained from (39) by replacing m_e with m_i and T_e with T_i. Thus,

$$\gamma_{ii} = \frac{8\pi n_0 e^4}{m_i^{1/2}(kT_i)^{3/2}} \ln\Lambda \quad \begin{bmatrix}\text{ion-ion collision} \\ \text{frequency}\end{bmatrix} \qquad (40)$$

Comparing (38)-(40), we find, for $T_e = T_i$, γ_{ii} is smaller by the factor $\sqrt{\dfrac{m_e}{m_i}}$ i.e $\quad \gamma_{ii} = \sqrt{\dfrac{m_e}{m_i}}\gamma_{ei}$ (or $\sqrt{\dfrac{m_e}{m_i}}\gamma_{ee}$) (41)

Ion - electron collision frequency:

Scattering of ions by electrons (γ_{ie}) is like scattering of billiard balls by ping-pong balls. A similar calculation in the center-of-mass frame shows the γ_{ie} is another factor of $\sqrt{m_e / m_i}$ smaller, so that

$$\gamma_{ie} = \frac{m_e}{m_i}\gamma_{ei} \ \left(\text{or } \frac{m_e}{m_i}\gamma_{ee}\right) \qquad (42)$$

Relaxation Times : When a plasma is first formed (e.g. by an electrical discharge), a charged - particle species may be in directed motion. The directed velocity will then randomize on the time scale of one collision. Thus, from (38) and (39), we obtain

$$\begin{cases} \text{Electrons } \underline{\text{thermalize}} \text{ on the time scale of } \tau_{ee} = \gamma_{ee}^{-1}. & (43) \\ \text{Ions } \underline{\text{thermalize}} \text{ on the time scale of } \tau_{ii} = \gamma_{ii}^{-1} \ (\approx \sqrt{\frac{m_i}{m_e}}\tau_{ee}). & (44) \end{cases}$$

In addition, electron and ion energies may not be equal when a plasma is first formed. The energies will equalize via ion-electron collisions on the time scale of

$$\tau_{ie} = \gamma_{ie}^{-1} \tag{45}$$

to reach the final state of equipartition of energy ($kT_e = kT_i$).

Comparing (45) with (43) and (44), we find τ_{ie} is greater than τ_{ee} and τ_{ie} by the factors:

$$\tau_{ie} \approx \frac{m_i}{m_e}\tau_{ee}; \ \ \tau_{ie} \approx \sqrt{\frac{m_i}{m_e}}\tau_{ii}, \tag{46}$$

which explains why we can have $T_e \neq T_i$ for a limited time.

Discussion:

1. All the Coulomb collision frequencies have the $\gamma \propto 1/T^{3/2}$ dependence, i.e. less collisions at higher temperatures. Physically, this is because thermal velocities, but not Coulomb forces, are greater at higher temperatures. Hence, it takes a longer time to deflect the velocity of a particle by 90^0.

2. Heavier particles collide less because they have greater momenta at the same T and hence need more collisions to deflect.

3. Coulomb collisions, though due to long-range forces, are not a collective effect because the scatterers do not act cooperatively. This is in contrast to Debye shielding and plasma oscillations as summarized below.

$$\text{Long-range forces} \Rightarrow \begin{cases} \text{Debye shielding} & \text{(collective effect)} \\ \text{plasma oscillations} & \text{(collective effect)} \\ \text{Coulomb collisions} & \text{(single-particle effect)} \end{cases}$$

Electrical Conductivity of a Plasma:

Coulomb forces tend to slow down the directed motion, hence affect the plasma conductivity. The current density **J** is given by

$$\mathbf{J} = -n_0 e\mathbf{v}, \quad \text{[Ion current is negligible.]}$$

where n_0 is the electron density, and the electron velocity **v** obeys

$$m_e \frac{d\mathbf{v}}{dt} = -e\mathbf{E} - \underbrace{m_e \gamma_e \mathbf{v}} \quad [\gamma_e: \text{ electron collision frequency}]$$

rate of change of electron momentum

$$\text{Let } \begin{Bmatrix} \mathbf{E} \\ \mathbf{v} \\ \mathbf{J} \end{Bmatrix} = \begin{Bmatrix} \mathbf{E}_0 \\ \mathbf{v}_0 \\ \mathbf{J}_0 \end{Bmatrix} e^{-i\omega t} \Rightarrow -im_e\omega\mathbf{v}_0 = -e\mathbf{E}_0 - m_e\gamma_e\mathbf{v}_0$$

$$\Rightarrow \mathbf{v}_0 = \frac{-e}{m_e(\gamma_e - i\omega)}\mathbf{E}_0 \Rightarrow \mathbf{J}_0 = -n_0 e\mathbf{v}_0 = \frac{n_0 e^2}{m_e(\gamma_e - i\omega)}\mathbf{E}_0$$

$$\Rightarrow \mathbf{J}_0 = \sigma\mathbf{E}_0, \text{ where } \sigma \equiv \frac{n_0 e^2}{m_e(\gamma_e - i\omega)} \approx \frac{n_0 e^2}{m_e \gamma_e} \propto T_e^{3/2}$$

plasma conductivity

if $\omega \ll \gamma_e$

Appendix A: Unit Systems and Dimensions

(Ref. J. D. Jackson, "Classical Electrodynamics," pp. 775-784)

Unit Systems:

Two systems of electromagnetic units are in common use today: the SI and Gaussian systems. Regardless of one's personal preference, it is important to be familiar with both systems and, in particular, the conversion from one system to the other. Conversion formulae can be divided into two categories: "symbol/equation conversion (such as E and $E = q/r^2$)" and "unit conversion (such as coulomb)".

Conversion formulae for symbols and equations are listed in Table 3 on p. 782 of Jackson and conversion formulae for units in Table 4 on p. 783 (both tables attached on next page). These two tables are all we need to convert between SI and Gaussian systems. Correct use of the tables requires practices.

Appendix A: Unit Systems and Dimensions (*continued*)

Table 3 Conversion Table for Symbols and Formulas

The symbols for mass, length, time, force, and other not specifically electromagnetic quantities are unchanged. To convert any equation in SI variables to the corresponding equation in Gaussian quantities, on both sides of the equation replace the relevant symbols listed below under "SI" by the corresponding "Gaussian" symbols listed on the left. The reverse transformation is also allowed. Residual powers of $\mu_0\epsilon_0$ should be eliminated in favor of the speed of light ($c^2\mu_0\epsilon_0 = 1$). Since the length and time symbols are unchanged, quantities that differ dimensionally from one another only by powers of length and/or time are grouped together where possible.

Quantity	Gaussian	SI
Velocity of light	c	$(\mu_0\epsilon_0)^{-1/2}$
Electric field (potential, voltage)	$\mathbf{E}(\Phi, V)/\sqrt{4\pi\epsilon_0}$	$\mathbf{E}(\Phi, V)$
Displacement	$\sqrt{\epsilon_0/4\pi}\,\mathbf{D}$	$\mathbf{D}$
Charge density (charge, current density, current, polarization)	$\sqrt{4\pi\epsilon_0}\,\rho(q, \mathbf{J}, I, \mathbf{P})$	$\rho(q, \mathbf{J}, I, \mathbf{P})$
Magnetic induction	$\sqrt{\mu_0/4\pi}\,\mathbf{B}$	$\mathbf{B}$
Magnetic field	$\mathbf{H}/\sqrt{4\pi\mu_0}$	$\mathbf{H}$
Magnetization	$\sqrt{4\pi/\mu_0}\,\mathbf{M}$	$\mathbf{M}$
Conductivity	$4\pi\epsilon_0\sigma$	σ
Dielectric constant	$\epsilon_0\epsilon$	ϵ
Magnetic permeability	$\mu_0\mu$	μ
Resistance (impedance)	$R(Z)/4\pi\epsilon_0$	$R(Z)$
Inductance	$L/4\pi\epsilon_0$	L
Capacitance	$4\pi\epsilon_0 C$	C

$$c = 2.997\,924\,58 \times 10^8 \text{ m/s}$$
$$\epsilon_0 = 8.854\,187\,8 \ldots \times 10^{-12} \text{ F/m}$$
$$\mu_0 = 1.256\,637\,0 \ldots \times 10^{-6} \text{ H/m}$$
$$\sqrt{\frac{\mu_0}{\epsilon_0}} = 376.730\,3 \ldots \,\Omega$$

Jackson, p.782, Table 3

Table 4 Conversion Table for Given Amounts of a Physical Quantity

The table is arranged so that a given amount of some physical quantity, expressed as so many SI or Gaussian units of that quantity, can be expressed as an equivalent number of units in the other system. Thus the entries in each row stand for the same amount, expressed in different units. All factors of 3 (apart from exponents) should, for accurate work, be replaced by (2.997 924 58), arising from the numerical value of the velocity of light. For example, in the row for displacement (D), the entry ($12\pi \times 10^5$) is actually ($2.997\,924\,58 \times 4\pi \times 10^5$) and "9" is actually $10^{-16} c^2 = 8.987\,55 \ldots$. Where a name for a unit has been agreed on or is in common usage, that name is given. Otherwise, one merely reads so many Gaussian units, or SI units.

Physical Quantity	Symbol	SI		Gaussian
Length	l	1 meter (m)	10^2	centimeters (cm)
Mass	m	1 kilogram (kg)	10^3	grams (g)
Time	t	1 second (s)	1	second (s)
Frequency	ν	1 hertz (Hz)	1	hertz (Hz)
Force	F	1 newton (N)	10^5	dynes
Work Energy	$\left.\begin{matrix} W \\ U \end{matrix}\right\}$	1 joule (J)	10^7	ergs
Power	P	1 watt (W)	10^7	ergs s^{-1}
Charge	q	1 coulomb (C)	3×10^9	statcoulombs
Charge density	ρ	1 C m^{-3}	3×10^3	statcoul cm^{-3}
Current	I	1 ampere (A)	3×10^9	statamperes
Current density	J	1 A m^{-2}	3×10^5	statamp cm^{-2}
Electric field	E	1 volt m^{-1} (Vm^{-1})	$\frac{1}{3} \times 10^{-4}$	statvolt cm^{-1}
Potential	Φ, V	1 volt (V)	$\frac{1}{300}$	statvolt
Polarization	P	1 C m^{-2}	3×10^5	dipole moment cm^{-3}
Displacement	D	1 C m^{-2}	$12\pi \times 10^5$	statvolt cm^{-1} (statcoul cm^{-2})
Conductivity	σ	1 mho m^{-1}	9×10^9	s^{-1}
Resistance	R	1 ohm (Ω)	$\frac{1}{9} \times 10^{-11}$	s cm^{-1}
Capacitance	C	1 farad (F)	9×10^{11}	cm
Magnetic flux	ϕ, F	1 weber (Wb)	10^8	gauss cm^2 or maxwells
Magnetic induction	B	1 tesla (T)	10^4	gauss (G)
Magnetic field	H	1 A m^{-1}	$4\pi \times 10^{-3}$	oersted (Oe)
Magnetization	M	1 A m^{-1}	10^{-3}	magnetic moment cm^{-3}
Inductance*	L	1 henry (H)	$\frac{1}{9} \times 10^{-11}$	

Jackson, p. 783, Table 4

Appendix A: Unit Systems and Dimensions (*continued*)

Conversion of symbols and equations:

Consider, for example, the conversion of the SI equation

$$E = \frac{q}{4\pi\varepsilon_0 r^2} \tag{A.1}$$

into the Gaussian system.

This involves the conversion of symbols and equations. So we use Table 3. First, we note from Table 3 (top) that mechanical symbols (e.g. time, length, mass, force, energy, and frequency) are unchanged in the conversion. Thus, we only need to deal with electromagnetic symbols on *both* sides of (A.1).

From Table 3, we find $E^{SI} \to \dfrac{E^G}{\sqrt{4\pi\varepsilon_0}}$ and $q^{SI} \to \sqrt{4\pi\varepsilon_0}\,q^G$ (A.2)

Sub. $E^G/\sqrt{4\pi\varepsilon_0}$ and $\sqrt{4\pi\varepsilon_0}\,q^{SI}$, respectively, for E and q in (A.1), we obtain the corresponding equation in the Gaussian system:

$$\frac{E^G}{\sqrt{4\pi\varepsilon_0}} = \frac{\sqrt{4\pi\varepsilon_0}\,q^G}{4\pi\varepsilon_0 r^2} \Rightarrow E^G = \frac{q^G}{r^2} \tag{A.3}$$

Conversion of units and evaluation of physical quantities:

Consider again the SI equation: $E = \dfrac{q}{4\pi\varepsilon_0 r^2}$ (A.1)

Given $r = 0.01$ m, $q = 1$ statcoulomb, we may evaluate E in 3 steps:

Step 1: Express r, q, and ε_0 in SI units. From Table 3 (bottom) and Table 4, we find

$$\begin{cases} \varepsilon_0 = 8.854\times10^{-12} \text{ Farad/m} = \dfrac{1}{36\pi\times10^9} \text{ Farad/m} \\ r = 0.01 \text{ m (same as given)} \\ q(=1 \text{ statcoulomb}) = \dfrac{1}{3\times10^9} \text{ coulomb} \end{cases} \quad (A.4)$$

Step 2: Sub. the numbers (*but not the units*) from (A.4) into (A.1).

This gives $E = \dfrac{q}{4\pi\varepsilon_0 r^2} = \dfrac{\frac{1}{3\times10^9}}{4\pi\times\frac{1}{36\pi\times10^9}\times(0.01)^2} = 3\times10^4$

Step 3: Look up Table 4 for the SI unit of E. As shown in Table 4, the SI unit of E is V/m. Thus, $E = 3\times10^4$ V/m (A.5)

As another exercise, we write (A.1) in the Gaussian system:

$$E = \frac{q}{r^2} \quad (A.3)$$

and evaluate E for the same r ($= 0.01$ m) and q ($= 1$ statcoulomb).

Step 1: Express r and q in Gaussian units. From Table 4, we find

$$\begin{cases} r(=0.01 \text{ m}) = 1 \text{ cm} \\ q = 1 \text{ statcoulomb (same as given)} \end{cases} \quad (A.6)$$

Step 2: Sub. the numbers (*but not the units*) from (A.6) into (A.3).

This gives $E = \dfrac{q}{r^2} = \dfrac{1}{1} = 1$

Step 3: Look up Table 4 for the Gaussian unit of E. We find the unit to be statvolt/cm. Thus, $E = 1$ statvolt/cm (A.7)

Table 4 shows 1 statvolt/cm $= 3\times10^4$ V/m. Hence, the 2 results

in (A.5) and (A.7): $\left\{\begin{array}{l} E = 3\times10^4 \text{ V/m} \\ E = 1 \text{ statvolt/cm} \end{array}\right\}$ are identical as expected.

Units and Dimensions:

In the Gaussian system, the basic units are length (ℓ), mass (m), and time (t). In the SI system, they are the above plus the current (I). [See Table 1 (top) on p. 779 of Jackson.] All other units are derived units.

If a physical quantity is expressed in term of the basic units, we have the dimension of this quantity.

A mechanical quantity has the same dimension in both systems. For example, the acceleration a $(= d^2x/dt^2)$ has the dimension of ℓt^{-2}. From $f = ma$, we obtain the dimension of force: $m\ell t^{-2}$, which in turn gives the dimension of work ($f \cdot \ell$) or energy: $m\ell^2 t^{-2}$.

An electromagnetic quantity has different dimensions in different systems. For example, the charge q has the SI dimension of It. In the Gaussian system, from the equation $f = q^2/r^2$ and the dimension of force, we find the Gaussian dimension of q to be $m^{1/2}\ell^{3/2}t^{-1}$. Since $q\phi$ has the dimension of energy ($m\ell^2t^{-2}$), the potential ϕ has the SI dimension of $m\ell^2 t^{-3} I^{-1}$ and the Gaussian dimension of $m^{1/2}\ell^{1/2}t^{-1}$.

All physical quantities in an equation must be expressed in the same unit system and all terms must have the same dimension. For example, by Stokes's theorem, we have

$$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = \int_S (\nabla \times \mathbf{E}) \cdot \mathbf{n}\, da \qquad \text{(A.8)}$$

where both terms have the dimension of $\ell \times$(the dimension of E).

In the definition of the delta function:

$$\int_{a_1}^{a_2} \delta(x-a)dx = 1, \qquad \text{(A.9)}$$

the RHS is dimensionless. Thus, if x has the dimension of ℓ, $\delta(x-a)$ must have the dimension of ℓ^{-1}. However, "0" is not to be regarded as a dimensionless quantity. This is clear if we write (A.8) as

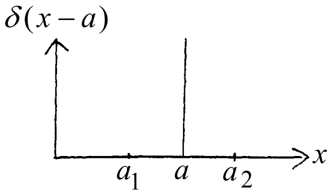
$$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} - \int_S (\nabla \times \mathbf{E}) \cdot \mathbf{n}\, da = 0.$$

Well known equations need not be checked for dimensional consistency. However, for newly derived equations, a dimensional check can be a convenient way to find mistakes.

Appendix B: Delta Functions

Definition of Delta Function :

$$\begin{cases} \delta(x-a)=0\ , & \text{if } x \neq a \\ \int_{a_1}^{a_2} \delta(x-a)dx = 1\ , & \text{if } a_1 < a < a_2 \end{cases} \quad \text{(B.1)}$$



Note: Since the delta function is defined in terms of an integral, it takes an integration to bring out its full meaning.

Properties of Delta Function :

(i) $\int_{a_1}^{a_2} f(x)\delta(x-a)dx = f(a)$ (B.2)

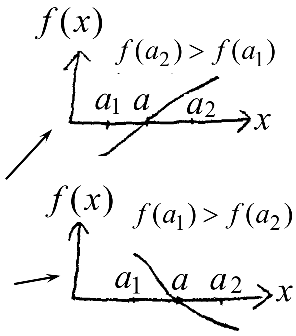
(ii) $\int_{a_1}^{a_2} f(x)\delta'(x-a)dx = \overbrace{f(x)\delta(x-a)\,|_{a_1}^{a_2}}^{0} - \int_{a_1}^{a_2} f'(x)\delta(x-a)dx$

$= -f'(a)$ (B.3)

(iii) Let $x = a$ be the root of $f(x) = 0$, then

$$\int_{a_1}^{a_2} \delta[f(x)]dx = \int_{f(a_1)}^{f(a_2)} \delta[f(x)] \frac{1}{\frac{d}{dx}f(x)} df(x)$$

$$= \begin{cases} \int_{f(a_1)}^{f(a_2)} \frac{1}{f'} \delta(f) df = \frac{1}{f'(a)} = \frac{1}{|f'(a)|}, & f'(a) > 0 \\ -\int_{f(a_2)}^{f(a_1)} \frac{1}{f'} \delta(f) df = -\frac{1}{f'(a)} = \frac{1}{|f'(a)|}, & f'(a) < 0 \end{cases}$$



Note: In both expressions above, the integration is from a samller value to a larger value, as in the definition of the delta function.

Compare with (2) $\Rightarrow \delta[f(x)] = \frac{1}{|f'(a)|}\delta(x-a)\ \left[= \frac{1}{|f'(x)|}\delta(x-a)\right]$ (B.4)

If $f(x)$ has multiple roots x_i $[f(x_i) = 0,\ i = 1, 2, \cdots]$, then

$$\delta[f(x)] = \sum_i \frac{1}{|f'(x_i)|}\delta(x - x_i)\ \left[= \sum_i \frac{1}{|f'(x)|}\delta(x - x_i)\right] \quad \text{(B.5)}$$

Exercise: Show $\delta(a-x) = \delta(x-a)$ and $\delta(cx) = \delta(x)/|c|$

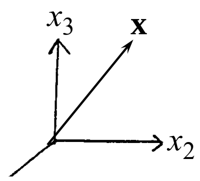
Extension to 3 Dimensions :

1. Cartesian coordinates: $\mathbf{x} = (x_1, x_2, x_3)$

$$\delta(\mathbf{x}-\mathbf{x}') \equiv \delta(x_1 - x_1')\delta(x_2 - x_2')\delta(x_3 - x_3') \tag{B.6}$$

$$\Rightarrow \int_V \delta(\mathbf{x}-\mathbf{x}')d^3x = \int \delta(x_1 - x_1')dx_1 \int \delta(x_2 - x_2')dx_2 \int \delta(x_3 - x_3')dx_3$$

$$= \begin{cases} 0, & \text{if } \mathbf{x}' \text{ lies outside } V \\ 1, & \text{if } \mathbf{x}' \text{ lies inside } V \end{cases}$$



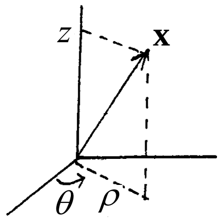
2. Cylindrical coordinates: $\mathbf{x} = (\rho, \theta, z)$

$$\delta(\mathbf{x}-\mathbf{x}') \equiv \tfrac{1}{\rho}\delta(\rho-\rho')\delta(\theta-\theta')\delta(z-z') \tag{B.7}$$

$$\Rightarrow \int_V \delta(\mathbf{x}-\mathbf{x}')d^3x = \int_V \delta(\mathbf{x}-\mathbf{x}')\rho d\rho d\theta dz$$

$$= \int \delta(\rho-\rho')d\rho \int \delta(\theta-\theta')d\theta \int \delta(z-z')dz$$

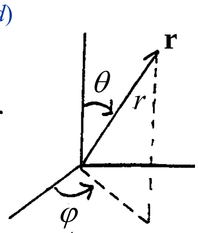
$$= \begin{cases} 0, & \text{if } \mathbf{x}' \text{ lies outside } V \\ 1, & \text{if } \mathbf{x}' \text{ lies inside } V \end{cases}$$



Question: If x and $\mathbf{x}$ both have the dimension of cm, what are the dimensions of $\delta(x)$ and $\delta(\mathbf{x})$? [See Appendix (A), Eq. (A.9).]

3. Spherical coordinates: $\mathbf{r} = (r, \theta, \varphi)$

$$\delta(\mathbf{r}-\mathbf{r}') \equiv \begin{cases} \dfrac{1}{r^2 \sin\theta}\delta(r-r')\delta(\theta-\theta')\delta(\varphi-\varphi'), \text{ or} \\ \dfrac{1}{r^2}\delta(r-r')\delta(\cos\theta-\cos\theta')\delta(\varphi-\varphi') \end{cases} \tag{B.8}$$



$\boxed{\text{By (B.4), } \delta(\cos\theta-\cos\theta') = \dfrac{1}{|\sin\theta|}\delta(\theta-\theta') = \dfrac{1}{\sin\theta}\delta(\theta-\theta'),\ 0 \le \theta \le \pi}$

$$\int_V \delta(\mathbf{r}-\mathbf{r}')d^3x = \int_V \frac{\delta(r-r')}{r^2}\delta(\cos\theta-\cos\theta')\delta(\varphi-\varphi')\underbrace{r^2 dr d(\cos\theta)d\varphi}_{\boxed{\substack{d^3x \\ \text{[see (B.9) below]}}}}$$

$$= \begin{cases} 0, & \text{if } \mathbf{r}' \text{ lies outside } V \\ 1, & \text{if } \mathbf{r}' \text{ lies inside } V \end{cases}$$

Note: Volume integration in spherical coordinates

$$\int_0^\infty dr \int_0^\pi r d\theta \int_0^{2\pi} r\sin\theta d\varphi = \int_0^\infty r^2 dr \underbrace{\int_0^\pi \sin\theta d\theta}_{\boxed{-\int_1^{-1} d(\cos\theta)}} \int_0^{2\pi} d\varphi$$

$$= \int_0^\infty r^2 dr \int_{-1}^1 d(\cos\theta) \int_0^{2\pi} d\varphi$$

Variables are to be integrated from smaller to larger values.

$$\Rightarrow d^3x = r^2 \sin\theta dr d\theta d\varphi \text{ or } r^2 dr d(\cos\theta)d\varphi \tag{B.9}$$

Approximate Representations of the Delta Function :

The delta function, $\delta(x)$, can be represented analytically by the following functions because they satisfy the definition of the delta function in the limit $\gamma \to 0$ $(\gamma > 0)$.

$$\delta(x) = \lim_{\gamma\to 0} \frac{1}{\pi} \frac{\gamma}{x^2+\gamma^2}$$

$$\delta(x) = \lim_{\gamma\to 0} \frac{1}{\sqrt{2\pi}\gamma} \mathrm{e}^{-\frac{x^2}{2\gamma^2}}$$

$$\delta(x) = \lim_{\gamma\to 0} \begin{cases} \frac{1}{\gamma}, & \text{for } -\frac{\gamma}{2} < x < \frac{\gamma}{2} \\ 0, & \text{otherwise} \end{cases}$$

Problem 1: A total charge Q is uniformly distributed around a circular ring of radius a and infinitesimal thickness. Write the charge density $\rho(\mathbf{x})$ in cylindrical coordinates.

Solution:

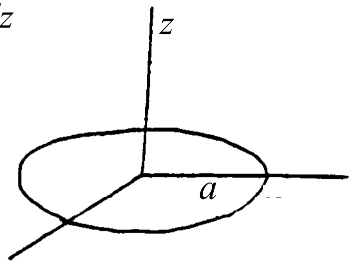
Let $\rho(\mathbf{x}) = K\delta(r-a)\delta(z)$ and find K as follows.

$$\int \rho(\mathbf{x})d^3x = K \int \delta(r-a)\delta(z) r dr d\theta dz$$

$$= 2\pi K a = Q$$

$$\Rightarrow K = \frac{Q}{2\pi a}$$

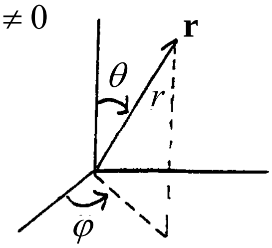
$$\Rightarrow \rho(\mathbf{x}) = \frac{Q}{2\pi a}\delta(r-a)\delta(z)$$



Note: ρ has the dimension of "charge/volume" as expected.

Problem 2: Prove $\nabla^2 \frac{1}{r} = -4\pi\delta(\mathbf{r})$ (B.10)

Solution: Definition of $\delta(\mathbf{r}): \begin{cases} \delta(\mathbf{r}) = 0, \text{ if } r \neq 0 \\ \int \delta(\mathbf{r}) d^3x = 1 \end{cases}$



Hence, we need to prove

(i) $\nabla^2 \frac{1}{r} = 0, \text{ if } r \neq 0$

(ii) $\int \nabla^2 \frac{1}{r} d^3x = -4\pi \int \delta(\mathbf{r}) d^3x = -4\pi$

It is convenient to use the spherical coordinates. To prove (i), we we write ∇^2 as (see back cover of Jackson)

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

$$\Rightarrow \nabla^2 \frac{1}{r} = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d}{dr}\frac{1}{r}\right) = -\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{r^2}\right) = 0 \quad \text{if} \quad r \neq 0$$

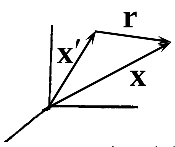
Note: $\frac{r^2}{r^2}$ is undetermined at $r = 0$. However, here we are only concerned with the region $r > 0$.

To prove (ii), we integrate $\nabla^2 \frac{1}{r}$ over a spherical volume V

$$\int_V \nabla^2 \frac{1}{r} d^3x = \int_V \nabla \cdot \nabla \frac{1}{r} d^3x \underset{\boxed{\text{divergence thm.}}}{=} \oint_S \mathbf{e}_r \cdot \underbrace{\nabla \frac{1}{r}}_{-\frac{1}{r^2}\mathbf{e}_r} \underset{\boxed{r^2 d\Omega}}{da} = -\oint_S r^2 \frac{1}{r^2} d\Omega = -4\pi$$

Note: Since $r > 0$ on the spherical surface, again we do not have the problem of evaluating r^2 / r^2 at $r = 0$.

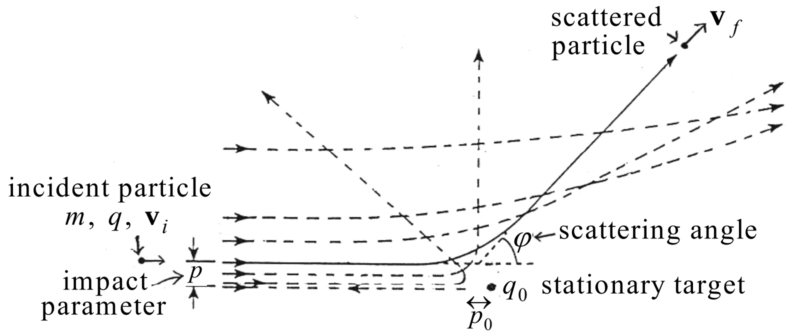
Change to a coordinate system in which $\mathbf{r} = \mathbf{x} - \mathbf{x}'$ and $r = |\mathbf{x} - \mathbf{x}'|$. We obatin from $\nabla^2 \frac{1}{r} = -4\pi\delta(\mathbf{r})$



$$\nabla^2 \frac{1}{|\mathbf{x} - \mathbf{x}'|} = -4\pi\delta(\mathbf{x} - \mathbf{x}') \quad \text{(B.11)}$$

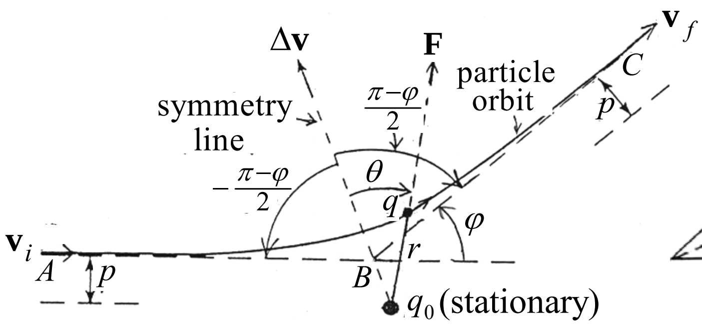
Appendix C: Rutherford Scattering

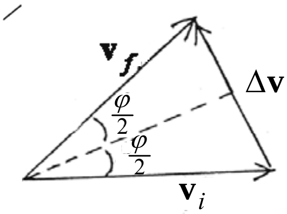
Reference: A. Beiser, "Concepts of Modern Physics"



The figure above shows the scattering of an incident particle ($m,\ q$) by a stationary target (q_0) for different impact parameters (p). Note that for $p = 0,$ the incident particle will be reflected (i.e. 180° deflection) upon reaching the distance of closest approach p_0.

Since the particle moves in a static electric field, its final velocity $\mathbf{v}_f$ (at $t = \infty$) will be equal in magnitude to the initial velocity $\mathbf{v}_i$ (at $t = -\infty$), i.e. $|\mathbf{v}_f| = |\mathbf{v}_i| = v_0$, but oriented at an angle φ with respect to $\mathbf{v}_i$.

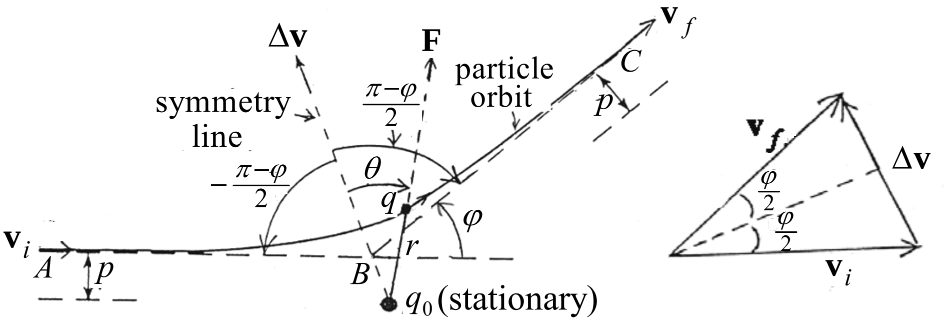




Our main interest here is to express the scattering angle φ as a function of p. From the triangle (righe figure) formed of $\mathbf{v}_f$, $\mathbf{v}_i$, and $\Delta\mathbf{v}\ (= \mathbf{v}_f - \mathbf{v}_i)$ and the equality $|\mathbf{v}_f| = |\mathbf{v}_i| = v_0$, we obtain

$$\Delta v = |\mathbf{v}_f - \mathbf{v}_i| = 2v_0 \sin\frac{\varphi}{2} \qquad \text{(C.1)}$$

Refer now to the left figure. Lines AB and BC are tangential to the the orbits at $t = -\infty$ and $t = \infty$, respectively. Since the angular momentum ($L = mpv_0$) is conserved in the central-force field, the orbit must be symmetric about the line, which equally divides the angle $\angle$ABC.



$\Delta\mathbf{v}$ is due to the impulse of the Coulomb force $\mathbf{F}$: $\Delta\mathbf{v} = \frac{1}{m}\int_{-\infty}^{\infty}\mathbf{F}(t)dt.$

By symmetry of the orbit, $\Delta\mathbf{v}$ must be along the symmetry line.

Thus, we need only to evaluate the *magnitude* of $\Delta\mathbf{v}$ along this line:

$\Delta v = \frac{1}{m}\int_{-\infty}^{\infty}F\cos\theta dt.$ [θ: angle between $\mathbf{F}$ and the symmetry line]

Writing $dt = \frac{dt}{d\theta}d\theta$ and noting $\theta(t=\pm\infty) = \pm\frac{1}{2}(\pi-\varphi),$ we obtain

$$\Delta v = \frac{1}{m}\int_{-\infty}^{\infty}F\cos\theta dt = \frac{1}{m}\int_{-\frac{1}{2}(\pi-\varphi)}^{\frac{1}{2}(\pi-\varphi)}F\cos\theta\frac{dt}{d\theta}d\theta \qquad \text{(C.2)}$$

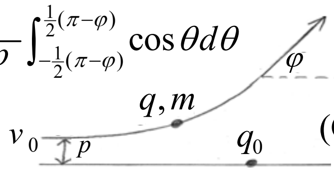
Conservation of angular momentum requires $mr^2\frac{d\theta}{dt} = mpv_0,$

which gives $\frac{dt}{d\theta} = \frac{r^2}{v_0 p}$ (C.3)

Sub. (C.3) into (C.2) [$\Delta v = \frac{1}{m}\int_{-\frac{1}{2}(\pi-\varphi)}^{\frac{1}{2}(\pi-\varphi)}F\cos\theta\frac{dt}{d\theta}d\theta$], we obtain

$$\Delta v = \frac{1}{m}\int_{-\frac{1}{2}(\pi-\varphi)}^{\frac{1}{2}(\pi-\varphi)}F\frac{r^2}{v_0 p}\cos\theta d\theta = \frac{qq_0}{mv_0 p}\int_{-\frac{1}{2}(\pi-\varphi)}^{\frac{1}{2}(\pi-\varphi)}\cos\theta d\theta$$

$$= \frac{qq_0}{mv_0 p}2\cos\frac{\varphi}{2} \qquad \boxed{F = \frac{qq_0}{r^2}} \qquad \text{(C.4)}$$



From (C.4) and (C.1) [$\Delta v = 2v_0\sin\frac{\varphi}{2}$], we obtain

$$v_0\sin\frac{\varphi}{2} = \frac{qq_0}{mv_0 p}\cos\frac{\varphi}{2} \Rightarrow \tan\frac{\varphi}{2} = \frac{p_0}{2p} \quad \left[p_0 \equiv \frac{2qq_0}{mv_0^2}\right] \qquad \text{(C.5)}$$

(C.5) was derived by Rutherford in 1911 to show that the nuclear model of the atom (the positive charge in an atom is concentrated in a small nucleus) was the only model that could explain the measured results of α particle scattering in passing through a thin foil.