Chapter 2: Single Particle Motion

2.1 Introduction

A plasma moves in *self-consistent* electric and magnetic fields, i.e. a superposition of the *self fields* produced by the plasma under study and the *prescribed fields* from external sources (if any). The plasma motion and the fields are governed by a set of coupled dynamic and field equations [e.g. Eq. (3) of Ch. 1].

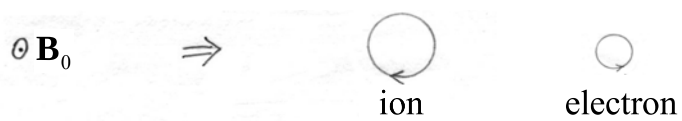
Here in Ch. 2, we consider single-particle (mass m, charge q) motion in external fields ($\mathbf{E}_0$, $\mathbf{B}_0$) with self-fields neglected. The particle obeys the equation of motion: $m\dot{\mathbf{v}} = q(\mathbf{E}_0 + \frac{1}{c}\mathbf{v}\times\mathbf{B}_0)$ (2.3) with $\mathbf{E}_0$ and $\mathbf{B}_0$ prescribed in a way to satisfy the Maxwell equaions.

$$\text{Example: } \begin{cases} \mathbf{B}_0 = B(0)(1+\frac{z}{L})\mathbf{e}_z \Rightarrow \nabla\cdot\mathbf{B}_0 \neq 0 \text{ (incorrect)} \\ \mathbf{B}_0 = -\dfrac{B(0)}{2L}r\mathbf{e}_r + B(0)(1+\frac{z}{L})\mathbf{e}_z \Rightarrow \nabla\cdot\mathbf{B}_0 = 0 \text{ (correct)} \end{cases}$$

Understanding of single-particle behavior in prescribed fields is an important first step toward the understanding of plasma behavior.

2.2 E×B Drifts

A Physical Picture :

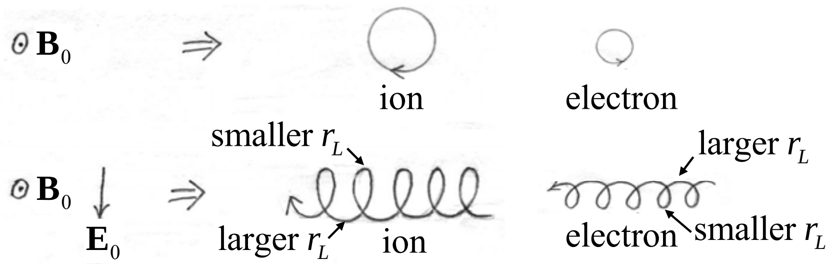


The $\mathbf{E}\times\mathbf{B}$ drift motion is the most common form of single-particle behavior in a plasma. Refer to the figure above. In a static and uniform magnetic field $\mathbf{B}_0$, an ion or electron performs (circular) cyclotron motion with a Larmor radius given by (see Sec. 1.5)

$$r_L = \frac{v_\perp}{|\Omega|}, \qquad (1.28)$$

where $$\Omega \equiv \frac{qB_0}{mc} \quad [\text{cyclotron frequency}] \qquad (1.26)$$

Note that here and in subsequent equations, q (hence Ω) carries the sign of the charge.



If now a static electric field $\mathbf{E}_0$ is added, and $\mathbf{E}_0$ is perpendicular to $\mathbf{B}_0$ and points downward (see lower figure), the ion will then be decelerated when it moves upward whereas the electron will be accelerated in its upward motion. This will result in a smaller (larger) instantaneous Larmor radius for the ion (electron). The opposite takes place in downward motions. As shown in the lower figure, the overall effect will be a drift motion in the $\mathbf{E}\times\mathbf{B}$ direction for both the ion and the electron. With a qualitative knowledge of the $\mathbf{E}\times\mathbf{B}$ drift motion, we peoceed with quantitative analyses by different methods.

A Simple Derivation of the $\mathbf{E}\times\mathbf{B}$ Drift Velocity :

Rewrite the equation of motion: $m\dot{\mathbf{v}} = q(\mathbf{E}_0 + \frac{1}{c}\mathbf{v}\times\mathbf{B}_0)$ (2.3)

If $\mathbf{E}_0, \mathbf{B}_0$ are static and uniform, and $\mathbf{E}_0 \perp \mathbf{B}_0$, then the particle will be continuously turned around by $\mathbf{B}_0$ while being accelerated by $\mathbf{E}_0$. The time-averaged acceleration $\langle\dot{\mathbf{v}}\rangle_t$ will thus be 0. Hence,

$$m\langle\dot{\mathbf{v}}\rangle_t = 0 = q[\mathbf{E}_0 + \tfrac{1}{c}\langle\mathbf{v}\rangle_t \times \mathbf{B}_0] \quad \begin{bmatrix}\text{balance of electric force} \\ \text{and average magnetic force}\end{bmatrix} \quad (2.1)$$

$$\mathbf{B}_0 \times (2.1) \Rightarrow \mathbf{E}_0 \times \mathbf{B}_0 = \tfrac{1}{c}\mathbf{B}_0 \times (\langle\mathbf{v}\rangle_t \times \mathbf{B}_0) = \tfrac{1}{c}[\langle\mathbf{v}\rangle_t B_0^2 - \mathbf{B}_0(\langle\mathbf{v}\rangle_t \cdot \mathbf{B}_0)]$$

Assume 2-D motion on the plane $\perp \mathbf{B}_0 \Rightarrow \langle\mathbf{v}\rangle_t \cdot \mathbf{B}_0 = 0$. Then,

$$\langle\mathbf{v}\rangle_t = \mathbf{v}_d \equiv \frac{c\mathbf{E}_0\times\mathbf{B}_0}{B_0^2}, \quad \begin{bmatrix}\text{Question: If } E_0 > B_0, \text{ we have} \\ v_d = cE_0/B_0 > c. \text{ What is wrong?}\end{bmatrix} \quad (2.2)$$

where $\mathbf{v}_d$ is called the <u>$\mathbf{E}\times\mathbf{B}$ drift</u> velocity. $\mathbf{v}_d$ is independent of q and m, i.e. electrons and ions drift in the same direction with the same speed. Hence, $\mathbf{E}\times\mathbf{B}$ drifts do not result in a net current in the plasma.

2.2 E×B Drifts (*continued*)

A More Detailed Analysis :

Rewrite the equation of motion: $m\dot{\mathbf{v}} = q(\mathbf{E}_0 + \frac{1}{c}\mathbf{v}\times\mathbf{B}_0)$ (2.3)

Let $\mathbf{B}_0$ be along $\mathbf{e}_z$ and again assume 2-D motion on the x-y plane.

We let $$\mathbf{v} = \mathbf{v}_d + \mathbf{v}_\perp^{osc} \qquad (2.4)$$

$\boxed{\mathbf{v}_d = c\mathbf{E}_0\times\mathbf{B}_0 / B_0^2 \;[(2.2)]}$ $\boxed{\text{oscillatory velocity} \perp \mathbf{B}_0}$

Sub. (2.4) into (2.3), we obtain

$$m\dot{\mathbf{v}}_\perp^{osc} = q(\mathbf{E}_0 + \tfrac{1}{c}\mathbf{v}_\perp^{osc}\times\mathbf{B}_0 + \underbrace{\tfrac{1}{c}\mathbf{v}_d\times\mathbf{B}_0}) = \tfrac{q}{c}\mathbf{v}_\perp^{osc}\times\mathbf{B}_0 \qquad (2.5)$$

$$\boxed{= \frac{1}{B_0^2}(\mathbf{E}_0\times\mathbf{B}_0)\times\mathbf{B}_0 = \frac{1}{B_0^2}\mathbf{B}_0(\cancel{\mathbf{E}_0\cdot\mathbf{B}_0}) - \mathbf{E}_0 = -\mathbf{E}_0}$$

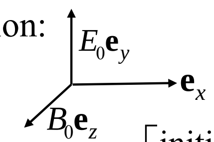
(2.5) is in the form of (1.24) in Ch. 1, where we have obtained the solution: $$\mathbf{v}_\perp^{osc} = v_\perp(\sin\Omega t\mathbf{e}_x + \cos\Omega t\mathbf{e}_y) \qquad (2.6)$$

Thus, the total solution is $\mathbf{v} = \dfrac{c\mathbf{E}_0\times\mathbf{B}_0}{B_0^2} + v_\perp(\sin\Omega t\mathbf{e}_x + \cos\Omega t\mathbf{e}_y),$ (2.7)

where $v_\perp$ is a constant. For $v_\perp = 0$, the particle moves in a straight line.

2.2 E×B Drifts (*continued*)

A Thorough Analysis : In obtaining (2.7), we have ignored the possibility of a contant velocity along $\mathbf{B}_0$, because it is completely decoupled from the drift motion. Also, to get the general form of the solution, initial conditions for determining the amplitude and phase angle of the oscillatory motion have been ignored. For a complete solution, we start from the equation of motion:



$$m\frac{d}{dt}\mathbf{v} = q(\mathbf{E}_0 + \tfrac{1}{c}\mathbf{v}\times\mathbf{B}_0) \qquad (2.3)$$

with $\begin{cases}\mathbf{E}_0 = E_0\mathbf{e}_y \\ \mathbf{B}_0 = B_0\mathbf{e}_z\end{cases}$ and $\mathbf{v}(t=0) = v_{x0}\mathbf{e}_x + v_{y0}\mathbf{e}_y + v_{z0}\mathbf{e}_z$ $\begin{bmatrix}\text{initial} \\ \text{condition}\end{bmatrix}$ (1)

(2.3) is a linear differential equation in $\mathbf{v}$. Let $\mathbf{v} = v_x\mathbf{e}_x + v_y\mathbf{e}_y + v_z\mathbf{e}_z$, we obtain from (2.3) a set of coupled linear differential equations:

$$\begin{cases} m\frac{d}{dt}v_x = \frac{q}{c}B_0 v_y \\ m\frac{d}{dt}v_y = qE_0 - \frac{q}{c}B_0 v_x \\ m\frac{d}{dt}v_z = 0 \end{cases} \Rightarrow \begin{cases} \frac{d}{dt}v_x = \Omega v_y & (2) \\ \frac{d}{dt}v_y = \frac{q}{m}E_0 - \Omega v_x & (3) \\ \frac{d}{dt}v_z = 0 & (4) \end{cases}$$

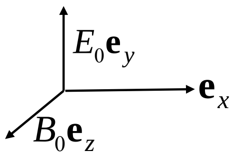
Rewrite: $\begin{cases} \dfrac{d}{dt}v_x = \Omega v_y & (2) \\ \dfrac{d}{dt}v_y = \dfrac{q}{m}E_0 - \Omega v_x & (3) \\ \dfrac{d}{dt}v_z = 0 & (4) \end{cases}$

by (3)

$$\frac{d}{dt}(2) \Rightarrow \frac{d^2}{dt^2}v_x = \Omega\frac{d}{dt}v_y \overset{\text{by (3)}}{=} \Omega\left(\frac{q}{m}E_0 - \Omega v_x\right) = \Omega^2\left(\frac{cE_0}{B_0} - v_x\right) \qquad (5)$$

from (5) — $\mathbf{E}\times\mathbf{B}$ drift speed

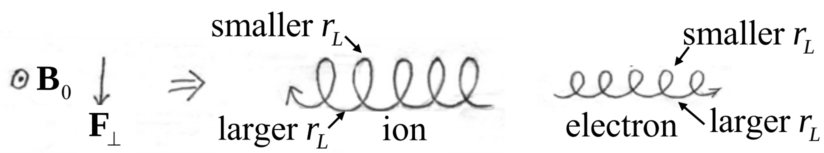
$$\Rightarrow \begin{cases} v_x \overset{\text{from (5)}}{=} v_\perp \sin(\Omega t + \varphi) + \dfrac{cE_0}{B_0} \\ v_y = \dfrac{1}{\Omega}\dfrac{d}{dt}v_x = v_\perp \cos(\Omega t + \varphi) \\ v_z = v_{z0} \leftarrow \text{constant speed along } \mathbf{B}_0 \end{cases} \qquad (6)$$



where the constants $v_\perp$ and φ are to be determined from the initial

conditions: $\begin{cases} v_x(t=0) = v_{x0} = v_\perp \sin\varphi + \dfrac{cE_0}{B_0} \\ v_y(t=0) = v_{y0} = v_\perp \cos\varphi \end{cases}$

Extension to F × B Drift :



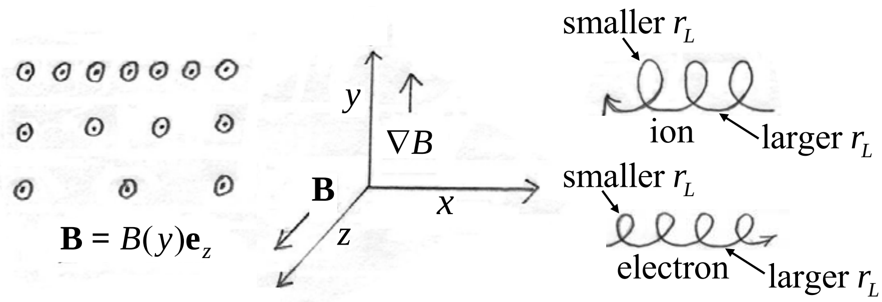
If the electric force $q\mathbf{E}_0$ in the previous model is replaced by a constant (in time and space) force $\mathbf{F}_\perp$ perpendicular to $\mathbf{B}_0$ (e.g. the gravitational force), there will still be a drift motion. The new drift velocity can be obtained from $\mathbf{v}_d = \dfrac{c\mathbf{E}_0\times\mathbf{B}_0}{B_0^2}$ [(2.2)] by replacing $\mathbf{E}_0$ with $\mathbf{F}_\perp / q$. Thus,

$$\mathbf{v}_d = \frac{c\mathbf{F}_\perp\times\mathbf{B}_0}{qB_0^2} \quad [\underline{\mathbf{F}\times\mathbf{B}\text{ drift}}\text{ velocity}] \qquad (2.8)$$

Note that, under the action of $\mathbf{F}_\perp$, the electron and ion will drift in opposite directions (see figure above).

2.3 Grad-B Drift

A Physical Picture :



The grad-**B** drift ("grad" stands for "gradient") takes place in a static but nonuniform magnetic field. If the **B**-field increases in the the positive y-direction (see figure above), the instantaneous Larmor radius of the charged particle will get smaller (larger) as it moves upward (downward). As shown the figure, the net effect will be a drift in the direction of $q\mathbf{B}\times\nabla B,$ the sign of which depends upon the sign of q and ∇B.

A Quantitative Analysis:

In a non-uniform **B**-field, the particle only "sees" the variation of the field over a distance of the particle's Larmor radius, which is usually much smaller than the scale length of the field. Thus we may expand the **B**-field about the guiding center of the particle using the formula for Taylor expansion [see (A.4) in Appendix A]:

$$\mathbf{A}(\mathbf{x}) = \mathbf{A}(\mathbf{a}) + \left[(\mathbf{x}-\mathbf{a})\cdot\nabla\right]\mathbf{A}(\mathbf{a}) + \cdots \tag{7}$$

where, in Cartesian coordinates,

$$\begin{aligned}(\mathbf{x}-\mathbf{a})\cdot\nabla &= (x_1-a_1)\tfrac{\partial}{\partial x_1} + (x_2-a_2)\tfrac{\partial}{\partial x_2} + (x_3-a_3)\tfrac{\partial}{\partial x_3} \\ &= \sum_i (x_i-a_i)\tfrac{\partial}{\partial x_i}\end{aligned} \tag{8}$$

$$\begin{aligned}\left[(\mathbf{x}-\mathbf{a})\cdot\nabla\right]\mathbf{A}(\mathbf{a}) &= \sum_i (x_i-a_i)[\tfrac{\partial}{\partial x_i}\sum_j A_j(\mathbf{x})\mathbf{e}_j]_{\mathbf{x}=\mathbf{a}} \\ &= \sum_j \{\sum_i (x_i-a_i)[\tfrac{\partial}{\partial x_i}A_j(\mathbf{x})]_{\mathbf{x}=\mathbf{a}}\}\mathbf{e}_j\end{aligned} \tag{9}$$

Rewrite (7): $\mathbf{A}(\mathbf{x}) = \mathbf{A}(\mathbf{a}) + \left[(\mathbf{x}-\mathbf{a})\cdot\nabla\right]\mathbf{A}(\mathbf{a}) + \cdots$ (7)

Let the vector field $\mathbf{A}$ be the magnetic field $\mathbf{B}$ and let $\mathbf{r}$ be the position vector of the particle: $\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$ (10)

Then, we have from (7)

$$\mathbf{B}(\mathbf{r}) = \mathbf{B}(\mathbf{a}) + \left[(\mathbf{r}-\mathbf{a})\cdot\nabla\right]\mathbf{B}(\mathbf{a}) + \cdots \tag{11}$$

Let the particle's guiding center be the origin of coordinetes. We expand $\mathbf{B}(\mathbf{x})$ about the guiding center by setting $\mathbf{a} = 0$ in (11) and neglecting higher order terms, then

$$\mathbf{B}(\mathbf{r}) = \mathbf{B}(0) + (\mathbf{r}\cdot\nabla)\mathbf{B}(0) \tag{12}$$

where $\mathbf{r}$ is now the vector distance from the guiding center and

$$(\mathbf{r}\cdot\nabla)\mathbf{B}(0) = \sum_j \{\sum_i r_i [\tfrac{\partial}{\partial x_i} B_j(\mathbf{r})]_{\mathbf{r}=0}\}\mathbf{e}_j.$$

For simplicity, we write (12) as $\quad \mathbf{B} = \mathbf{B}_0 + (\mathbf{r}\cdot\nabla)\mathbf{B}_0$ (2.16)

The equation of motion is thus

$$m\dot{\mathbf{v}} = \tfrac{q}{c}\mathbf{v}\times\mathbf{B} = \tfrac{q}{c}\mathbf{v}\times\mathbf{B}_0 + \tfrac{q}{c}[\mathbf{v}\times(\mathbf{r}\cdot\nabla)\mathbf{B}_0] \tag{2.17}$$

Rewrite (2.17): $m\dot{\mathbf{v}} = \tfrac{q}{c}\mathbf{v}\times\mathbf{B}_0 + \tfrac{q}{c}[\mathbf{v}\times(\mathbf{r}\cdot\nabla)\mathbf{B}_0]$ (2.17)

Unlike (2.3), (2.17) is a nonlinear differential equation of variables $\mathbf{r}$ and $\mathbf{v}\ (=\dot{\mathbf{r}})$. If the Lamor radius r_L is much samller than the scale length L of the magnetic field, then $(\mathbf{r}\cdot\nabla)\mathbf{B}_0 \sim \frac{r_L}{L}B_0 \ll B_0$ and the 2nd term on the RHS of (2.17) is much smaller than the 1st term. We may then solve (2.17) by the perturbation method.

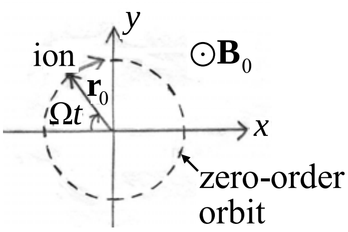
Let $\begin{cases} \mathbf{r}(t) = \mathbf{r}_0(t) + \mathbf{r}_1(t) \\ \mathbf{v}(t) = \dot{\mathbf{r}}(t) = \mathbf{v}_0(t) + \mathbf{v}_1(t) \end{cases}$ $\begin{bmatrix} \text{no free motion along } \mathbf{B}_0 \\ \Rightarrow \mathbf{r} \text{ and } \mathbf{v} \text{ on } x\text{-}y \text{ plane.} \end{bmatrix}$ (2.18)

and assume $|\mathbf{r}_1| \ll |\mathbf{r}_0|, |\mathbf{v}_1| \ll |\mathbf{v}_0|$. Sub. (2.18) into (2.17) and equate the zero-order terms, we obtain [$\mathbf{v}_0\times(\mathbf{r}_0\cdot\nabla)\mathbf{B}_0$ is a first-order term]

$$m\dot{\mathbf{v}}_0 = \tfrac{q}{c}\mathbf{v}_0\times\mathbf{B}_0, \tag{2.19}$$

which yields gyromotion given by

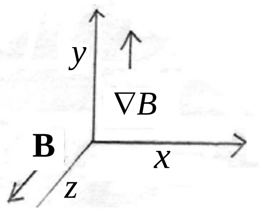
$$\begin{cases} \mathbf{r}_0(t) = \frac{v_0}{\Omega}(-\cos\Omega t,\ \sin\Omega t,\ 0) & (2.25) \\ \mathbf{v}_0(t) = v_0(\sin\Omega t,\ \cos\Omega t,\ 0) & (2.26) \end{cases}$$



2.3 Grad-B Drift (*continued*)

Sub. (2.18) into (2.17) and equate the first-order terms, we obtain

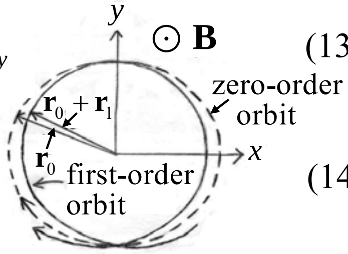
$$m\dot{\mathbf{v}}_1 = \tfrac{q}{c}\mathbf{v}_1 \times \mathbf{B}_0 + \tfrac{q}{c}\mathbf{v}_0 \times (\mathbf{r}_0 \cdot \nabla)\mathbf{B}_0 \qquad (2.20)$$



Let $\mathbf{B} = B(y)\mathbf{e}_z$, then $(\mathbf{r}_0 \cdot \nabla)\mathbf{B}_0 = y_0 \dfrac{\partial B_0}{\partial y}\mathbf{e}_z \qquad (2.23)$

$$\Rightarrow \mathbf{v}_0 \times (\mathbf{r}_0 \cdot \nabla)\mathbf{B}_0 = v_{0y} y_0 \frac{\partial B_0}{\partial y}\mathbf{e}_x - v_{0x} y_0 \frac{\partial B_0}{\partial y}\mathbf{e}_y$$

$$= \frac{v_0^2}{\Omega}\sin\Omega t\cos\Omega t \frac{\partial B_0}{\partial y}\mathbf{e}_x - \frac{v_0^2}{\Omega}\sin^2\Omega t \frac{\partial B_0}{\partial y}\mathbf{e}_y \qquad (13)$$



Sub. (13) into (2.20) and average over t.

$$\Rightarrow m\underbrace{\langle \dot{\mathbf{v}}_1 \rangle_t}_{0} = \frac{q}{c}\langle \mathbf{v}_1 \rangle_t \times \mathbf{B}_0 - \frac{q}{c}\frac{v_0^2}{2\Omega}\frac{\partial B_0}{\partial y}\mathbf{e}_y \qquad (14)$$

$$\mathbf{B}_0 \times (14) \Rightarrow \langle \mathbf{v}_1 \rangle_t B_0^2 - \overbrace{(\langle \mathbf{v}_1 \rangle_t \cdot \mathbf{B}_0)}^{0}\mathbf{B}_0 + \frac{v_0^2}{2\Omega}\frac{\partial B_0}{\partial y}B_0\mathbf{e}_x = 0$$

$$\Rightarrow \langle \mathbf{v}_1 \rangle_t = \frac{-1}{2B_0}\frac{v_0^2}{\Omega}\frac{\partial B_0}{\partial y}\mathbf{e}_x = \frac{1}{2B_0^2}\frac{v_0^2}{\Omega}\mathbf{B}_0 \times \nabla B_0 \quad \text{[\underline{grad-}\mathbf{B}\text{ \underline{drift}} velocity]} \qquad (2.28)$$

Question: Does $\mathbf{B} = B(y)\mathbf{e}_z$ satisfy $\nabla \cdot \mathbf{B} = 0$?

2.4 Curvature Drifts

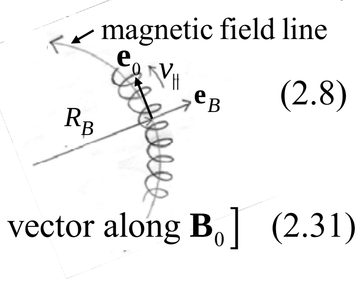
By turning a charged particle around to perform gyrational motion, the magnetic field tends to resist a particle's motion *across* its field lines, while the particle is completely free to move *along* the field line. This results in a helical orbit for the particle, with its guiding center following a field line. However, when the field line curves, the partcile will feel an outward centrifugal force given by

$$\mathbf{F}_c = \frac{mv_\parallel^2}{R_B}\mathbf{e}_B, \quad [\mathbf{e}_B \text{ is } \hat{\mathbf{R}}_B \text{ in Nicholson}] \qquad (2.29)$$

where R_B is the radius of curvature of the magnetic field line, and $\mathbf{e}_B$ is an outward unit vector along the radius of curvature and is $\perp$ to $\mathbf{B}_0$.

Sub. $\mathbf{F}_c$ for $\mathbf{F}_\perp$ in the $\mathbf{F}\times\mathbf{B}$ drift formula:

$$\mathbf{v}_d = \frac{c\mathbf{F}_\perp \times \mathbf{B}_0}{qB_0^2} \quad [\mathbf{F}\times\mathbf{B} \text{ drift velocity}] \qquad (2.8)$$



we obtain the <u>curvature drift</u> velocity:

$$\mathbf{v}_d = \frac{cmv_\parallel^2}{qR_B B_0^2}\mathbf{e}_B \times \mathbf{B}_0 = \frac{v_\parallel^2}{\Omega R_B}\mathbf{e}_B \times \mathbf{e}_0 \quad [\mathbf{e}_0\text{: unit vector along } \mathbf{B}_0] \qquad (2.31)$$

A field gradient is usually associated with curved field lines. Consider, for example, the magnetic field due to a straight, thin wire carrying current I : $\mathbf{B}_0 = B_0(r)\mathbf{e}_\theta$ with $B_0(r) = \frac{2I}{cr}$ $\boxed{\mathbf{e}_B}$ (15)

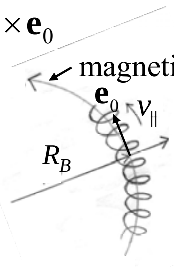
We have from (15): $\nabla B_0 = -\frac{2I}{cr^2}\mathbf{e}_r = -\frac{B_0}{r}\mathbf{e}_r = -\frac{B_0}{R_B}\mathbf{e}_B$, (16)

which results in a drift velocity given by the $\boxed{R_B}$

formula in (2.28): $\mathbf{v}_d = \frac{1}{2B_0^2}\frac{v_0^2}{\Omega}\mathbf{B}_0 \times \nabla B_0$, [grad-**B** drift velocity] (2.28)

where v_0 is the particle velocity $\perp \mathbf{B}_0$, which we denote below by $v_\perp$ because the particle also has a $v_\parallel$. Sub. ∇B_0 from (16) into (2.28),

we obtain $\mathbf{v}_d = \frac{v_\perp^2}{2B_0^2\Omega}\mathbf{B}_0 \times \nabla B_0 = \frac{v_\perp^2}{2\Omega R_B}\mathbf{e}_B \times \mathbf{e}_0$ (17)



Combining (2.31) and (17), we get the total drift velocity in the curved magnetic field of (16): $\mathbf{v}_d^{tot} = \frac{v_\parallel^2 + \frac{1}{2}v_\perp^2}{\Omega R_B}\mathbf{e}_B \times \mathbf{e}_0$ (2.32)

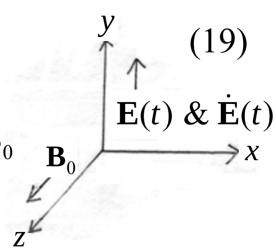
2.5 Polarization Drift

Rewrite the $\mathbf{E} \times \mathbf{B}$ drift velocity: $\mathbf{v}_d = \frac{c\mathbf{E}\times\mathbf{B}_0}{B_0^2}$ (2.2)

Assume that **E** varies with t, then $\dot{\mathbf{v}}_d = \frac{c\dot{\mathbf{E}}(t)\times\mathbf{B}_0}{B_0^2}$ (18)

For simplicity, we assume $\dot{\mathbf{E}}(t) \parallel \mathbf{E}(t) \perp \mathbf{B}_0$ (see figure below). The acceleration of $\dot{\mathbf{v}}_d$ requires a force $m\dot{\mathbf{v}}_d$, which cannot come from **E** because $\mathbf{E} \perp \dot{\mathbf{v}}_d$. Hence, a polarization drift velocity $\mathbf{v}_p$ in the direction of $\dot{\mathbf{E}}(t)$ is developed to provide a magnetic force equal to $m\dot{\mathbf{v}}_d$:

$$\frac{q}{c}\mathbf{v}_p \times \mathbf{B}_0 = m\dot{\mathbf{v}}_d = \frac{mc\dot{\mathbf{E}}(t)\times\mathbf{B}_0}{B_0^2} \qquad (19)$$



$$\mathbf{B}_0 \times (19) \Rightarrow \frac{q}{c}\overbrace{\mathbf{B}_0 \times (\mathbf{v}_p \times \mathbf{B}_0)}^{B_0^2\mathbf{v}_p - (\overbrace{\mathbf{v}_p\cdot\mathbf{B}_0}^{0})\mathbf{B}_0} = \frac{mc\overbrace{\mathbf{B}_0\times[\dot{\mathbf{E}}(t)\times\mathbf{B}_0]}^{B_0^2\dot{\mathbf{E}}(t) - [\overbrace{\dot{\mathbf{E}}(t)\cdot\mathbf{B}_0}^{0}]\mathbf{B}_0}}{B_0^2}$$

$$\Rightarrow \mathbf{v}_p = \frac{mc^2\dot{\mathbf{E}}(t)}{qB_0^2} = \frac{c\dot{\mathbf{E}}(t)}{\Omega B_0} \quad \text{[\underline{polarization drift} velocity]} \qquad (2.41)$$

Polarization Current :

Rewrite: $$\mathbf{v}_p = \frac{mc^2\dot{\mathbf{E}}(t)}{qB_0^2} = \frac{c\dot{\mathbf{E}}(t)}{\Omega B_0} \qquad (2.41)$$

Note: Since $\mathbf{v}_p$ is proportional to m, the polarization drift velocity is much faster for the ion than for the electron.

Question : Why is $\mathbf{v}_p$ proportional to m ?

In a plasma with $n_e = n_i = n_0$, the polarization drifts lead to a polarization current density $\mathbf{J}_p$ given by

$$\mathbf{J}_p = n_0 e(\mathbf{v}_{pi} - \mathbf{v}_{pe}) = \frac{n_0 c^2}{B_0^2}(m_i + m_e)\dot{\mathbf{E}}$$
$$= \frac{\rho_m c^2}{B_0^2}\dot{\mathbf{E}} \qquad (2.43)$$

where $\rho_m = n_0(m_i + m_e) \approx n_0 m_i$ is the mass density.

Applications of the $\mathbf{E}\times\mathbf{B}$ drift and polarization drift can be found in Sec. 6.12 on the interpretation of the Alfv′en wave and the magnetosonic wave.

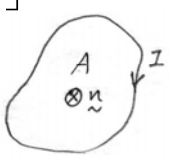
2.6 Magnetic Moment

We have thus far considered drift motion across the **B**-field. In this section, we consider particle motion along the **B**-field.

Magnetic Moment of a Gyrating Particle : A current loop of surface area A and current I produces a mgnetic (dipole) moment :

$$\boldsymbol{\mu} = \frac{IA}{c}\mathbf{n} \qquad \left[\text{See, for example, Jackson, "Classical Electrodynamics," Eq. (5.57).}\right] \qquad (2.44)$$

where **n** is a unit vector ⊥ surface area A and pointing in the direction determined by the right-hand rule.

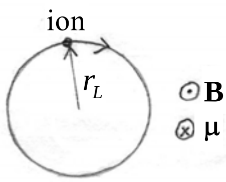


For a gyrating particle in a **B**-field, we have

$$\begin{cases} I = \dfrac{q}{\tau_c} = \dfrac{q\Omega}{2\pi} \\ A = \pi r_L^2 = \dfrac{\pi v_\perp^2}{\Omega^2} \end{cases} \Rightarrow \mu = \frac{q\Omega}{c2\pi}\frac{\pi v_\perp^2}{\Omega^2} = \frac{q v_\perp^2}{2c\Omega} = \frac{\frac{1}{2}m v_\perp^2}{B} = \frac{W_\perp}{B} \qquad (2.46)$$

where $W_\perp$ is the perpendicular energy of the particle

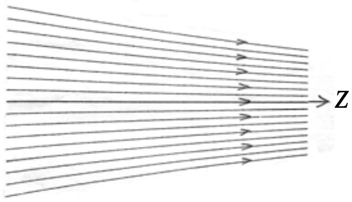
$\boldsymbol{\mu}$ has a direction opposite to **B** independent of the sign of q. ⇒ the plasma is diamagnetic.



Conservation Laws in a Slowly-Varying B-Field :

Expression of near-axis magnetic field :

Before considering the motion of a charged particle in a static, axisymmetric, nonuniform **B**-field, we first express the field in cylindrical coordinates as



$$\mathbf{B}(r,z) = B_r(r,z)\mathbf{e}_r + B_z(r,z)\mathbf{e}_z \tag{20}$$

$$\nabla\cdot\mathbf{B} = 0 \Rightarrow \frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{\partial B_z}{\partial z} = 0$$

$$\Rightarrow \frac{\partial}{\partial r}(rB_r) = -r\frac{\partial B_z}{\partial z} \tag{21}$$

On or near the z-axis, we have $B_r \ll B_z$, i.e. the **B**-field lines have a negligible angular divergence. Hence, the r-dependence of B_z can be neglected and we may approximate B_z by B. Integrating (21) over r,

we obtain $$B_r \approx -\frac{r}{2}\frac{\partial B_z}{\partial z} \approx -\frac{r}{2}\frac{\partial B}{\partial z} \tag{22}$$

Note: The z-axis here is the x-axis in Nicholson Sec. 2.6

Forces on a gyating particle in a slowly-varying static **B**-*field*:

Consider the motion of a gyrating particle with its guiding center on the z-axis (see figure). The Larmor radius will vary as the particle moves axially into a stronger or weaker field, which implies a radial motion (v_r) for the particle. We assume $\frac{\Delta B}{B} \ll 1$, (23)

where ΔB is the change in B seen by the particle in one gyro-period.

Under (23), we may neglect v_r

and write $$\mathbf{v} = v_\theta \mathbf{e}_\theta + v_z \mathbf{e}_z \tag{24}$$

Then, with $\mathbf{B} = B_r\mathbf{e}_r + B_z\mathbf{e}_z$ [(20)],

we can write down the 3 components

of the magnetic force on the particle and identify the function of each.

($Note$: $qv_\theta B_z < 0$)

$$\mathbf{F} = \frac{q}{c}\mathbf{v}\times\mathbf{B} = \frac{q}{c}(\underbrace{v_\theta B_z \mathbf{e}_r}_{\text{centripetal force for gyrations}} + \overbrace{v_z B_r \mathbf{e}_\theta}^{\text{increase (decrease) } v_\theta} \underbrace{-v_\theta B_r \mathbf{e}_z}_{\text{decrease (increase) } v_z}) \tag{25}$$

Conservation of the magnetic moment - adiabatic invariant:

Rewrite (25): $\mathbf{F} = \frac{q}{c}\mathbf{v}\times\mathbf{B} = \frac{q}{c}(v_\theta B_z \mathbf{e}_r + v_z B_r \mathbf{e}_\theta - v_\theta B_r \mathbf{e}_z)$ (25)

$$\Rightarrow F_z = -\frac{q}{c}v_\theta B_r \approx \frac{q}{c}v_\perp B_r \approx -\frac{q}{c}\frac{v_\perp}{2} r\frac{\partial B}{\partial z} = -\frac{qv_\perp^2}{2c\Omega}\frac{\partial B}{\partial z} = -\underbrace{\frac{mv_\perp^2}{2B}}_{\mu}\frac{\partial B}{\partial z}$$

$$= -\mu\frac{\partial B}{\partial z}, \quad \boxed{v_r \approx 0} \quad \boxed{B_r \approx -\frac{r}{2}\frac{\partial B}{\partial z} \ (22)} \quad \boxed{r \approx r_L = \frac{v_\perp}{\Omega}} \qquad (26)$$

where F_z is independent of the sign of q. In a static **B**-field, the total kinetic energy (W) of the particle is a constant. Hence,

$$\frac{d}{dt}W = \frac{d}{dt}(W_\perp + \tfrac{1}{2}mv_z^2) = 0$$

$$\Rightarrow \frac{d}{dt}W_\perp = -\frac{m}{2}\frac{d}{dt}v_z^2 = -mv_z\frac{d}{dt}v_z = -v_z F_z \underset{\boxed{(26)}}{=} v_z\mu\frac{\partial B}{\partial z} = v_z\frac{W_\perp}{B}\frac{\partial B}{\partial z} \quad (27)$$

$$\Rightarrow \quad \frac{d\mu}{dt} = \frac{d}{dt}\left(\frac{W_\perp}{B}\right) = \frac{1}{B}\frac{dW_\perp}{dt} - W_\perp\frac{1}{B^2}\frac{dB}{dt}$$

$$\boxed{(27)} \rightarrow = W_\perp v_z \frac{1}{B^2}\frac{\partial B}{\partial z} - W_\perp\frac{1}{B^2}v_z\frac{\partial B}{\partial z} = 0 \qquad (2.55)$$

$$\Rightarrow \quad \mu = \frac{W_\perp}{B} = const \quad \left[\begin{array}{l}\text{valid under assumption (23);}\\ \mu \text{ is called an } \underline{\text{adiabatic invariant}}\end{array}\right] \qquad (2.56)$$

Conservation of the magnetic flux:

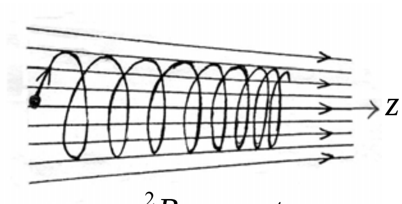
Rewrite (2.56): $\mu = \frac{W_\perp}{B} = const$ (2.56)

$$(2.56) \Rightarrow \mu = \frac{\frac{1}{2}mv_\perp^2}{B} = const$$

$$\Rightarrow B\frac{v_\perp^2}{B^2} = const$$

$$\Rightarrow B\frac{v_\perp^2}{\Omega^2} = const$$

$$\Rightarrow Br_L^2 = const$$



$\pi r_L^2 B = const$

$\Rightarrow$ The magnetic flux enclosed by the particle orbit is conserved.

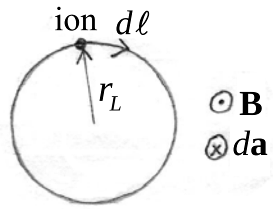
Problem: Show that μ is conserved in a **B**-field which varies slowly with time.

Solution: For simplicity, we assume that $\mathbf{B}$ is constant in space, while varying with time. Assume also that the guiding center is stationary. If the variation of the field (hence the Larmor radius) is sufficiently slow so that the orbit almost closes on itself in one revolution, then the change of $W_\perp$ in one revolution is

$$\delta W_\perp = \frac{q}{4\pi}\oint \mathbf{E}\cdot d\ell = q\int_S (\nabla\times\mathbf{E})\cdot d\mathbf{a} \quad [S\text{: surface spanning the orbit}]$$

$$= -\frac{q}{c}\int_S \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{a} = \frac{q}{c}\pi r_L^2 \frac{\partial B}{\partial t} = \frac{q}{c}\pi \frac{v_\perp^2}{\Omega^2}\frac{\partial B}{\partial t}$$

$$= \frac{q}{c}\pi \frac{mc}{qB}\frac{v_\perp^2}{\Omega}\frac{\partial B}{\partial t} = \frac{1}{2}mv_\perp^2 \frac{1}{B}\frac{2\pi}{\Omega}\frac{\partial B}{\partial t}$$



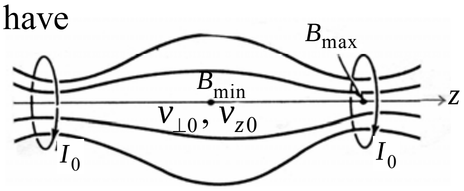
$$\Rightarrow \frac{\delta W_\perp}{W_\perp} = \frac{\delta B}{B} \quad \left[\delta B = \frac{2\pi}{\Omega}\frac{\partial B}{\partial t} = \text{change of } B \text{ in on revolution}\right]$$

$$\Rightarrow \delta(W_\perp / B) = \delta\mu = 0 \;\Rightarrow\; \mu \text{ is conserved.} \qquad (28)$$

Magnetic Mirror - An Example of Adiabatic Motion :

The simplest magnetic mirror field is produced by 2 coils with equal currents (I_0) flowing in the same direction (see figure). If condition (23) is satisfied, we have

$$\mu = \frac{mv_\perp^2}{2B} = const. \qquad (2.56)$$



At $B = B_{min}$, let $v_\perp = v_{\perp 0}$ and $v_z = v_{z0}$. Then, $\mu = \dfrac{mv_{\perp 0}^2}{2B_{min}}$ (29)

Since $F_z = -\mu \dfrac{\partial B}{\partial z}$ [(26)], as the particle moves away from B_{min}, v_z decreases (hence $v_\perp$ increases). Define a reflection field B_{refl} by the relation:

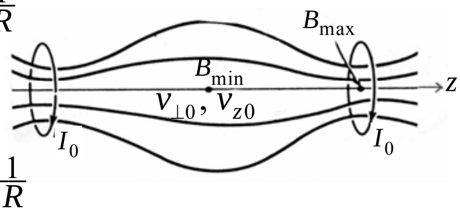
$$\mu = \frac{mv_{\perp 0}^2}{2B_{min}} = \frac{m(v_{\perp 0}^2 + v_{z0}^2)}{2B_{refl}} \quad \left[\begin{array}{l}\text{i.e. at } B = B_{refl}\text{, we have} \\ v_z = 0 \text{ and } v_\perp^2 = v^2 = v_{\perp 0}^2 + v_{z0}^2\end{array}\right]$$

or

$$B_{refl} \equiv \frac{v_{\perp 0}^2 + v_{z0}^2}{v_{\perp 0}^2} B_{min} \qquad (30)$$

Rewrite (30): $B_{refl} \equiv \dfrac{v_{\perp 0}^2 + v_{z0}^2}{v_{\perp 0}^2} B_{\min}$ and define $R \equiv \dfrac{B_{\max}}{B_{\min}}$ $\begin{bmatrix}\text{mirror}\\ \text{ratio}\end{bmatrix}$ (31)

Case (i): $B_{\max} < B_{refl}$ or $\dfrac{v_{\perp 0}^2}{v_{\perp 0}^2 + v_{z0}^2} < \dfrac{1}{R}$



$\Rightarrow$ The particle will pass through the $B_{\max}$ point.

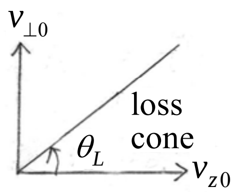
Case (ii): $B_{\max} = B_{refl}$ or $\dfrac{v_{\perp 0}^2}{v_{\perp 0}^2 + v_{z0}^2} = \dfrac{1}{R}$

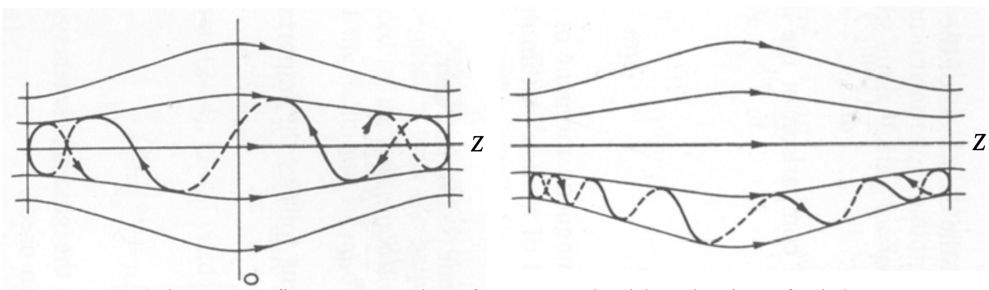
$\Rightarrow$ The particle will stop at the $B_{\max}$ point.

Case (iii): $B_{\max} > B_{refl}$ or $\dfrac{v_{\perp 0}^2}{v_{\perp 0}^2 + v_{z0}^2} > \dfrac{1}{R}$

$\Rightarrow$ The particle will be reflected before reaching the $B_{\max}$ point.

Thus, if we define a "loss cone" in the $(v_\perp^0, v_z^0)$ space by $\theta < \theta_L \equiv \sin^{-1}\dfrac{1}{\sqrt{R}}$, the particle will be lost if its $v_\perp^0$ and v_z^0 lie in the loss cone.





(These two figures are taken from G. Schmidt,"Physics of High Temperature Plasmas".)

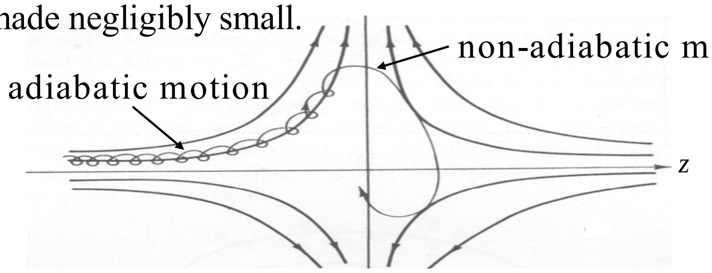
Discussion:

For simplicity, we have considered particle motion with the guiding center on the z-axis (left figure). The off-axis motion (right figure) is complicated by the grad-**B** and curvature drifts, which cause the guiding center to rotate slowly around the z-axis. However, the same results (conservation of μ and condition for reflection, etc.) still hold true.

Magnetic Cusp - An Example of Non-adiabatic Motion :

If the two coils for the mirror field have opposite currents, a cusp field will be generated. In the figure below, we show without derivation a particle orbit in the cusp field. At the far left, the field varies slowly and the motion is adiabatic. But as the partcile moves into the region of rapidly varying field, the motion becomes non-adiabatic and the orbit eventually encircles the z-axis.

The cusp field is often used to generate an axis-encircling electron beam, in which case the Larmor radius in the adiabatic portion of the orbit is made negligibly small.

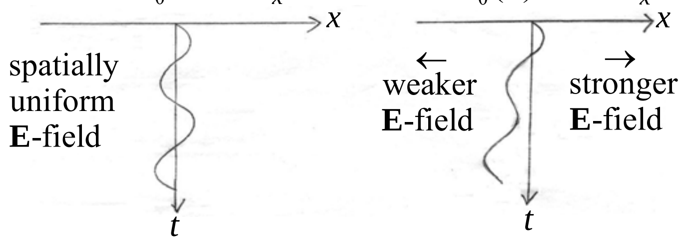


(figure taken from G. Schmidt,"Physics of High Temperature Plasmas")

2.8 Ponderomotive Force

The ponderomotive force is a single-particle effect occurring in spatially-varying, high-frequency electric fields, with or without a **B**-field. We assume no **B**-field in the following analysis.

$$\mathbf{E} = E_0 \cos \omega t \mathbf{e}_x \qquad \mathbf{E} = E_0(x) \cos \omega t \mathbf{e}_x$$



A Physical Picture : In a uniform **E**-field (left figure), a particle will oscillate with a constant amplitude. If the **E**-field is stronger to the right (right figure), the particle will be given a stronger stopping force, followed by a stronger pushing force, as it turns around on the right side. The reverse is true as it turns around on the left side. Thus, the net result is a gradual acceleration to the left.

A Quantitative Analysis :

This is a one-dimensional problem with the following equation of motion for the particle:

$$m\ddot{x} = qE_0(x)\cos\omega t \quad (2.71)$$

$$\text{Let} \begin{cases} x(t) = x_0(t) + x_1(t) \\ E_0(x) \approx E_0 + x_1 \dfrac{dE_0}{dx}, \end{cases} \quad (32)$$

where
- $x_0(t)$ is a slowly-varing component of $x(t)$, called the oscillation center.
- $x_1(t)$ is a rapidly-oscillating component of $x(t)$.
- E_0 (treated as a constant) is $E_0(x)$ evaluated at x_0.
- $\dfrac{dE_0}{dx}$ (treated as a constant) is $\dfrac{dE_0(x)}{dx}$ evaluated at x_0.

Sub. (32) into (2.71), we obtain

$$m(\ddot{x}_0 + \ddot{x}_1) = q(E_0 + x_1 \frac{dE_0}{dx})\cos\omega t \quad (2.72)$$

Rewrite (2.72): $m(\ddot{x}_0 + \ddot{x}_1) = q(E_0 + x_1 \dfrac{dE_0}{dx})\cos\omega t,$ (2.72)

where $\ddot{x}_0$ varies slowly while $\ddot{x}_1$ oscillates rapidly. Averaging (2.72) over one oscillation period gives $m\ddot{x}_0 = q\dfrac{dE_0}{dx}\langle x_1 \cos\omega t\rangle_t$ (2.73)

Since $\ddot{x}_1 \gg \ddot{x}_0$, $E_0 \gg x_1 \dfrac{dE_0}{dx}$, (2.72) gives $m\ddot{x}_1 \approx qE_0 \cos\omega t$ (2.74)

with the solution: $x_1 \approx -\dfrac{qE_0}{m\omega^2}\cos\omega t$ (33)

Sub. (33) into (2.73), we obtain

$$m\ddot{x}_0 = -\frac{q^2 E_0}{m\omega^2}\frac{dE_0}{dx}\langle\cos^2\omega t\rangle_t = -\frac{q^2 E_0}{2m\omega^2}\frac{dE_0}{dx} = F_p,$$

where $F_p \equiv -\dfrac{q^2}{4m\omega^2}\dfrac{dE_0^2}{dx} = -\dfrac{m}{4}\dfrac{d\tilde{v}^2}{dx}$ [ponderomotive force] (2.77)

and we have defined $\tilde{v} \equiv (\dot{x}_1)_{max} = \dfrac{qE_0}{m\omega}$ [quiver velocity].

The pondermotive force F_p is related to (wave field amplitude)^2 and hence is important for the study of nonlinear plasma behavior.

2.8 Ponderomotive Force (*continued*)

A computational exercise:

So far in this chapter, we have considered the following types of drift motion: $\mathbf{E}\times\mathbf{B}$ drift, grad-$\mathbf{B}$ drift, curvature drift, and polarization drift. In addition, we have shown that the magnetic moment of a single particle is an "adiabatic invariant"; namely, it is a constant in a slowly varying magnetic field. We have aslo derived a nonlinear force, called the pondermotive force, in a spatially nonuniform electric field which varies rapidly in time.

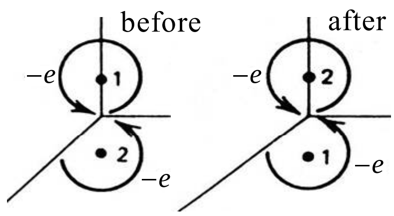
All these effects can be readily tested by computations. The student is encouraged to write a simple computer program to solve the single particle orbits numerically and compare the results with predictions by the formulae developed in this chapter.

It is also worthwhile to use the relativistic equation of motion [see Eq. (2) in Special Topic I] in this exercise. This will allow us to verify that the nonrelativistic $\mathbf{E}\times\mathbf{B}$ drift theory breaks down when $E > B$.

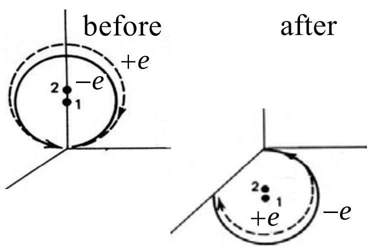
2.9 Diffusion (across the magnetic field)

The guiding center motion gives a simple picture of collisional effects on particle diffusion across the **B**-field. For simplicity, we consider head-on (180°) collisions, but the conclusion also applies to small-angle collisions.

(1) head-on collisions between like particles with the same energy (upper figure)
$\Rightarrow$ no diffusion



(2) head-on collisions between unlike particles with slightly different energies (lower figure)
$\Rightarrow$ significant diffusion



Appendix A. Taylor Expansion

Define $e^{\mathbf{a}\cdot\nabla} \equiv \sum\limits_{n=0}^{\infty} \frac{1}{n!}\left(\mathbf{a}\cdot\nabla\right)^n$ [a translational operator, which translates the argument of the function it operates on to a distance **a** away from the argument.]

Taylor expansion of $f(\mathbf{x}+\mathbf{a})$ and $\mathbf{A}(\mathbf{x}+\mathbf{a})$ about point $\mathbf{x}$:

$$\begin{cases} f(\mathbf{x}+\mathbf{a}) = e^{\mathbf{a}\cdot\nabla} f(\mathbf{x}) = \sum\limits_{n=0}^{\infty} \frac{1}{n!}\left(\mathbf{a}\cdot\nabla\right)^n f(\mathbf{x}) \\ \qquad = f(\mathbf{x}) + \left(\mathbf{a}\cdot\nabla\right) f(\mathbf{x}) + \frac{1}{2}\left(\mathbf{a}\cdot\nabla\right)\left(\mathbf{a}\cdot\nabla\right) f(\mathbf{x}) + \cdots \qquad \text{(A.1)} \\ \mathbf{A}(\mathbf{x}+\mathbf{a}) = e^{\mathbf{a}\cdot\nabla} \mathbf{A}(\mathbf{x}) = \sum\limits_{n=0}^{\infty} \frac{1}{n!}\left(\mathbf{a}\cdot\nabla\right)^n \mathbf{A}(\mathbf{x}) \\ \qquad = \mathbf{A}(\mathbf{x}) + \left(\mathbf{a}\cdot\nabla\right) \mathbf{A}(\mathbf{x}) + \frac{1}{2}\left(\mathbf{a}\cdot\nabla\right)\left(\mathbf{a}\cdot\nabla\right) \mathbf{A}(\mathbf{x}) + \cdots \qquad \text{(A.2)} \end{cases}$$

Similarly, operating $f(\mathbf{x})\big|_{\text{at } \mathbf{x}=\mathbf{a}}$ and $\mathbf{A}(\mathbf{x})\big|_{\text{at } \mathbf{x}=\mathbf{a}}$ with $e^{(\mathbf{x}-\mathbf{a})\cdot\nabla}$, we obtain the Taylor expansion of $f(\mathbf{x})$ and $\mathbf{A}(\mathbf{x})$ about point $\mathbf{a}$:

$$\begin{cases} f(\mathbf{x}) = f(\mathbf{a}) + \left[(\mathbf{x}-\mathbf{a})\cdot\nabla\right] f(\mathbf{a}) + \frac{1}{2}[(\mathbf{x}-\mathbf{a})\cdot\nabla][(\mathbf{x}-\mathbf{a})\cdot\nabla] f(\mathbf{a}) + \cdots \quad \text{(A.3)} \\ \mathbf{A}(\mathbf{x}) = \mathbf{A}(\mathbf{a}) + \left[(\mathbf{x}-\mathbf{a})\cdot\nabla\right] \mathbf{A}(\mathbf{a}) + \frac{1}{2}[(\mathbf{x}-\mathbf{a})\cdot\nabla][(\mathbf{x}-\mathbf{a})\cdot\nabla] \mathbf{A}(\mathbf{a}) + \cdots \quad \text{(A.4)} \end{cases}$$

In (A.1) and (A.2), we have [in Cartesian coordinates]

$$\mathbf{a}\cdot\nabla = a_1 \tfrac{\partial}{\partial x_1} + a_2 \tfrac{\partial}{\partial x_2} + a_3 \tfrac{\partial}{\partial x_3} = \sum_{i=1}^{3} a_i \tfrac{\partial}{\partial x_i} \qquad \text{(A.5)}$$

$$\left(\mathbf{a}\cdot\nabla\right)\left(\mathbf{a}\cdot\nabla\right) = \sum_i a_i \tfrac{\partial}{\partial x_i} \sum_j a_j \tfrac{\partial}{\partial x_j} = \sum_{ij} a_i a_j \tfrac{\partial^2}{\partial x_i \partial x_j} \qquad \text{(A.6)}$$

$$\left(\mathbf{a}\cdot\nabla\right) f(\mathbf{x}) = \sum_i a_i \tfrac{\partial}{\partial x_i} f(\mathbf{x}) = \mathbf{a}\cdot\nabla f(\mathbf{x}) \qquad \text{(A.7)}$$

$$\left(\mathbf{a}\cdot\nabla\right) \mathbf{A}(\mathbf{x}) = \sum_i a_i \tfrac{\partial}{\partial x_i} \Big(\sum_j A_j \mathbf{e}_j\Big) = \sum_j \Big(\sum_i a_i \tfrac{\partial}{\partial x_i} A_j\Big) \mathbf{e}_j \qquad \text{(A.8)}$$

Example: $\left(\mathbf{a}\cdot\nabla\right)\left(\mathbf{x}-\mathbf{x}'\right) = \sum_j \Big[\sum_i a_i \underbrace{\tfrac{\partial}{\partial x_i}(x_j - x'_j)}_{\delta_{ij}}\Big]\mathbf{e}_j = \sum_j a_j \mathbf{e}_j = \mathbf{a}$

For scalar functions with a scalar argument, (A.1) & (A.3) reduce to

$$f(x+a) = f(x) + a f'(x) + \tfrac{1}{2} a^2 f''(x) + \cdots \qquad \text{(A.9)}$$

$$f(x) = f(a) + (x-a) f'(a) + \tfrac{1}{2}(x-a)^2 f''(a) + \cdots \qquad \text{(A.10)}$$