CHAPTER 6: Vlasov Equation

Part 1

6.1 Introduction

Distribution Function, Kinetic Equation, and Kinetic Theory :

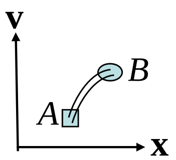
The distribution function $f(\mathbf{x},\mathbf{v},t)$ gives the particle density of a certain species in the 6-dimensional phase space of $\mathbf{x}$ and $\mathbf{v}$ at time t. Thus, $f(\mathbf{x},\mathbf{v},t)d^3xd^3v$ is the total number of particles in the differential volume d^3xd^3v at point $(\mathbf{x},\mathbf{v})$ and time t.

A kinetic equation describes the time evolution of $f(\mathbf{x},\mathbf{v},t)$. The kinetic theories in Chs. 3-5 derive various forms of kinetic equations. In most cases, however, the plasma behavior can be described by an approximate kinetic equation, called the Vlasov equation, which simply neglects the complications caused by collisions.

By ignoring collisions, we may start out without the knowledge of Chs. 3-5 and proceed directly to the derivation of the Vlasov equation (also called the collisionless Boltzmann equation).

The Vlasov Equation : As shown in Sec. 2.9, a collision can result in an *abrupt* change of two colliding particles' velocities and their *instant* escape from a small element in the **x-v** space, which contains the colliding particles before the collision.

However, if collisions are neglected (valid on a time scale < the collision time, see Sec. 1.6), particles in an element at position A in the **x-v** space will wander in continuous curves to position B (see figure). Thus, the total number in the element is conserved, and $f(\mathbf{x},\mathbf{v},t)$ obeys an equation of continuity, which takes the form (see next page):

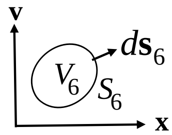


$$\frac{\partial}{\partial t} f(\mathbf{x},\mathbf{v},t) + \nabla_{\mathbf{x},\mathbf{v}} \cdot [f(\mathbf{x},\mathbf{v},t)(\dot{\mathbf{x}}, \dot{\mathbf{v}})] = 0 \qquad (1)$$

In (1), $\nabla_{\mathbf{x},\mathbf{v}}\ [= (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial v_x}, \frac{\partial}{\partial v_y}, \frac{\partial}{\partial v_z})]$ is a 6-dimensional divergence operator and $(\dot{\mathbf{x}}, \dot{\mathbf{v}})\,[= (\dot{x}, \dot{y}, \dot{z}, \dot{v}_x, \dot{v}_y, \dot{v}_z)]$ can be regarded as a 6-dimensional "velocity" vector in the **x-v** space.

6.1 Introduction (*continued*)

To show that $\frac{\partial}{\partial t} f(\mathbf{x},\mathbf{v},t) + \nabla_{\mathbf{x},\mathbf{v}} \cdot [\, f(\mathbf{x},\mathbf{v},t)(\dot{\mathbf{x}}, \dot{\mathbf{v}})] = 0$ [(1)] implies conservation of particles, we integrate it over an arbitrary volume V_6 enclosed by surface S_6 in the **x-v** space:



$$\underbrace{\frac{d}{dt}\int_{V_6} f(\mathbf{x},\mathbf{v},t)d^3x d^3v}_{N(t)} + \underbrace{\int_{V_6} \nabla_{\mathbf{x},\mathbf{v}} \cdot [\, f(\mathbf{x},\mathbf{v},t)(\dot{\mathbf{x}}, \dot{\mathbf{v}})] d^3x d^3v}_{\oint_{S_6} f(\mathbf{x},\mathbf{v},t)(\dot{\mathbf{x}}, \dot{\mathbf{v}})\cdot d\mathbf{s}_6} = 0$$

(by 6-dimensional divergence theorem)

$$\Rightarrow \qquad \frac{d}{dt} N(t) + \oint_{S_6} f(\mathbf{x},\mathbf{v},t)(\dot{\mathbf{x}}, \dot{\mathbf{v}}) \cdot d\mathbf{s}_6 = 0, \qquad (2)$$

where $N(t)$ is the total number of particles in V_6, $f(\mathbf{x},\mathbf{v},t)(\dot{\mathbf{x}}, \dot{\mathbf{v}})$ is the 6-dimensional "particle flux" in the **x-v** space, $d\mathbf{s}_6$ is a 6-dimensional differential surface area of S_6, with a direction normal to S_6 pointing outward. Thus, (2) states that the rate of increase (decrease) of the total number of particles in V_6 equals the partcle flux into (out of) V_6.

6.1 Introduction (*continued*)

Rewrite: $$\frac{\partial}{\partial t} f(\mathbf{x},\mathbf{v},t) + \nabla_{\mathbf{x},\mathbf{v}} \cdot [\, f(\mathbf{x},\mathbf{v},t)(\dot{\mathbf{x}}, \dot{\mathbf{v}})] = 0 \qquad (1)$$

In (1), $\nabla_{\mathbf{x},\mathbf{v}} \cdot [\, f(\mathbf{x},\mathbf{v},t)(\dot{\mathbf{x}}, \dot{\mathbf{v}})]$

$$= \frac{\partial}{\partial x}(f\dot{x}) + \frac{\partial}{\partial y}(f\dot{y}) + \frac{\partial}{\partial z}(f\dot{z}) + \frac{\partial}{\partial v_x}(f\dot{v}_x) + \frac{\partial}{\partial v_y}(f\dot{v}_y) + \frac{\partial}{\partial v_z}(f\dot{v}_z)$$

$$= \nabla_{\mathbf{x}} \cdot (f\dot{\mathbf{x}}) + \nabla_{\mathbf{v}} \cdot (f\dot{\mathbf{v}}) \qquad (6.1)$$

$$\boxed{\dot{\mathbf{x}} = \mathbf{v}} \rightarrow \; = f\nabla_{\mathbf{x}} \cdot \mathbf{v} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + f\nabla_{\mathbf{v}} \cdot \dot{\mathbf{v}} + \dot{\mathbf{v}} \cdot \nabla_{\mathbf{v}} f$$

where, because **v** is an independent variable, we have $\nabla_{\mathbf{x}} \cdot \mathbf{v} = 0$ (3)

In general, $\nabla_{\mathbf{v}} \cdot \dot{\mathbf{v}}$ does not vanish. But for $\dot{\mathbf{v}} = \frac{q}{m}(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B})$,

we have $$\begin{cases} \nabla_{\mathbf{v}} \cdot \mathbf{E} = 0 \quad [\mathbf{E} \text{ does not depend on } \mathbf{v}] \\ \nabla_{\mathbf{v}} \cdot (\mathbf{v} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla_{\mathbf{v}} \times \mathbf{v}) - \mathbf{v}(\nabla_{\mathbf{v}} \times \mathbf{B}) = 0 - 0 = 0 \end{cases}$$

which gives $$\nabla_{\mathbf{v}} \cdot \dot{\mathbf{v}} = \frac{q}{m}\nabla_{\mathbf{v}} \cdot (\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}) = 0 \qquad (4)$$

Thus, $\nabla_{\mathbf{x},\mathbf{v}} \cdot [\, f(\mathbf{x}, \mathbf{v}, t)(\dot{\mathbf{x}}, \dot{\mathbf{v}})] = \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \frac{q}{m}(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f$

and (1) becomes $$\frac{\partial}{\partial t} f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \frac{q}{m}(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f = 0, \qquad (6.5)$$

which is the Vlasov equation.

Physical Interpretation of the Vlasov Equation :

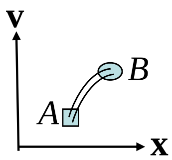
The Valsov equation $\frac{\partial}{\partial t} f + \mathbf{v}\cdot\nabla_{\mathbf{x}} f + \frac{q}{m}(\mathbf{E}+\frac{1}{c}\mathbf{v}\times\mathbf{B})\cdot\nabla_{\mathbf{v}} f = 0$ (6.5)
can be written

$$\frac{d}{dt} f(\mathbf{x},\mathbf{v},t) = \frac{\partial}{\partial t} f + \overbrace{\frac{d\mathbf{x}}{dt}}^{\mathbf{v}}\cdot\nabla_{\mathbf{x}} f + \overbrace{\frac{d\mathbf{v}}{dt}}^{\frac{q}{m}(\mathbf{E}+\frac{1}{c}\mathbf{v}\times\mathbf{B})}\cdot\nabla_{\mathbf{v}} f = 0, \quad (5)$$

where $\frac{df}{dt}$ is the total time derivative of f. The total time derivative (also called a convective derivative) follows the orbit* of a particle in the **x-v** space. It evaluates the variation of f due to the change of the particle position in the **x-v** space as well as the explicit time variation of f. Thus, (6.5) or (5) can be interpreted as: Along a particle's orbit in the **x-v** space, the particle density $f(\mathbf{x},\mathbf{v},t)$ remains unchanged.

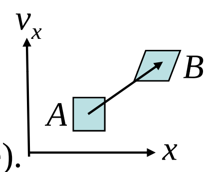
*In the 6-dimensional **x-v** space, particles at a point ($\mathbf{x},\mathbf{v}$) have the same velociy **v**. Hence, the orbit of a particle is also the orbit of an infinistesimal element containing the particle. In the 3-dimensional **x**-space, by contrast, particles at a point **x** have a range of velocities.

This interpretation is consistent with the fact that the sum of (3) and (4) gives $\nabla_{\mathbf{x},\mathbf{v}}\cdot(\dot{\mathbf{x}},\dot{\mathbf{v}}) = \nabla_{\mathbf{x}}\cdot\dot{\mathbf{x}} + \nabla_{\mathbf{v}}\cdot\dot{\mathbf{v}} = 0$, (6)
i.e. the 6-dimensional divergence of the 6-dimensional "velocity" $(\dot{\mathbf{x}},\dot{\mathbf{v}})$ vanishes. This implies that a collisionless plasma is incompressible. Hence, the volume of an elements (thus the particle density f) will be unchanged as it moves from A to B.



A specific example:

Consider the simple case of a group of particles initially located in a square in the x-v_x space (area A, lower figure). If the particles are force free, those on the upper edge will move at the fastest (equal and constant) speed, while those on the lower edge move at the slowest (equal and constant) speed. Then, some time later, the square will become a parallelogram of the same area (area B, lower figure).



Effect of collisions:

If there are collisions, they will cause a variation of f at the symbolic rate of $(\frac{\partial}{\partial t} f)_{coll}$, which should be added to (6.5) to give

$$\frac{\partial}{\partial t} f + \mathbf{v}\cdot\nabla_{\mathbf{x}} f + \frac{q}{m}(\mathbf{E}+\frac{1}{c}\mathbf{v}\times\mathbf{B})\cdot\nabla_{\mathbf{v}} f = (\frac{\partial}{\partial t} f)_{coll},$$

while the specfic form of $(\frac{\partial}{\partial t} f)_{coll}$ depends on interparticle forces. Throughout this course, the $(\frac{\partial}{\partial t} f)_{coll}$ term will be neglected.

Complete Set of Equations: We now have the following set of self-consistent, coupled particle and field equations:

$$\begin{cases} \frac{\partial}{\partial t} f_\alpha + \mathbf{v}\cdot\nabla f_\alpha + \frac{q}{m}(\mathbf{E}+\frac{1}{c}\mathbf{v}\times\mathbf{B})\cdot\nabla_{\mathbf{v}} f_\alpha = 0 & (6.5) \\ \nabla\cdot\mathbf{B} = 0 & (7) \\ \nabla\cdot\mathbf{E} = 4\pi\rho & (8) \\ \nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial}{\partial t}\mathbf{B} & (9) \\ \nabla\times\mathbf{B} = \frac{1}{c}\frac{\partial}{\partial t}\mathbf{E} + \frac{4\pi}{c}\mathbf{J} & (10) \end{cases}$$

For simplicity, we shall henthforth denote $\nabla_{\mathbf{x}}$ by ∇.

where
$$\begin{cases} \rho(\mathbf{x},t) = \sum_\alpha q_\alpha \int f_\alpha(\mathbf{x},\mathbf{v},t)d^3v & (11) \\ \mathbf{J}(\mathbf{x},t) = \sum_\alpha q_\alpha \int f_\alpha(\mathbf{x},\mathbf{v},t)\mathbf{v}d^3v & (12) \end{cases}$$

Each particle species, denoted by the subscript "α", is governed by a separate Valsov equation, and q_α carries the sign of the charge.

6.2 Equilibrium Solutions

General Form of Equilibrium Solutions : As shown in (5), the Vlasov equation can be written as a total time derivative:

$$\frac{d}{dt} f(\mathbf{x},\mathbf{v},t) = \frac{\partial}{\partial t} f + \frac{d\mathbf{x}}{dt}\cdot\nabla f + \frac{d\mathbf{v}}{dt}\cdot\nabla_{\mathbf{v}} f = 0, \quad (5)$$

where $\frac{d}{dt}$ follows the orbit of a particle whose position and velocity at time t is $\mathbf{x}$ and $\mathbf{v}$, respectively. Thus, any fuction of constants of the motion along the orbit of the particle, $C_i = C_i(\mathbf{x},\mathbf{v},t)$, is a solution of the Vlasov equation, i.e. $\frac{d}{dt} f(C_1, C_2 ...) = \sum_i \frac{\partial f}{\partial C_i}\frac{dC_i}{dt} = 0,$

because, by the definition of constant of the motion,

$$\frac{dC_i}{dt} = \frac{\partial}{\partial t} C_i + \frac{d\mathbf{x}}{dt}\cdot\nabla C_i + \frac{d\mathbf{v}}{dt}\cdot\nabla_{\mathbf{v}} C_i = 0.$$

The equilibrium solution (denoted by subscript "0") of interest to us is a steady-state solution formed of constants of the motion that do not depend explicitly on t, i.e. $f_0 = f_0(C_1, C_2 ...)$ with $C_i = C_i(\mathbf{x},\mathbf{v})$.

Examples of Constants of the Motion :

1. If $\mathbf{B}_0 = \mathbf{E}_0 = 0$, v_x, v_y, and v_z are constants of the motion.

2. If $\mathbf{E}_0 = 0$, $\mathbf{B}_0 = B_0\mathbf{e}_z = const.$, $v_\perp$, v_z are constants of the motion.

3. The motion of a charged particle (mass m and charge q) in EM fields (represented by potentials $\mathbf{A}$ and ϕ) is governed by Lagrange's equation: [Goldstein, Poole, & Safko, "Classical Mechanics," 3rd ed., p. 21]

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}, \ i = 1, 2, 3 \quad (13)$$

$(13) \Rightarrow m\frac{d\mathbf{v}}{dt} = q\mathbf{E} + \frac{q}{c}\mathbf{v}\times\mathbf{B}$
See Goldstein, Poole, & Safko, Sec. 1.5.

where $\begin{cases} q_i \text{ is a position coordinate.} \\ L = \frac{1}{2}mv^2 + \frac{q}{c}\mathbf{v}\cdot\mathbf{A} - q\phi \end{cases}$ [L: Lagrangian; $\mathbf{v}$: particle velocity]

In cylindrical coordinates, we have $q_i = (r, \theta, z)$, $\dot{q}_i = (\dot{r}, \dot{\theta}, \dot{z})$, $v^2 = \dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2$, and $\mathbf{v}\cdot\mathbf{A} = \dot{r}A_r + r\dot{\theta}A_\theta + \dot{z}A_z$. If the fields $\mathbf{A}$ and ϕ are independent of θ, (13) gives $\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = \frac{\partial L}{\partial\theta} = 0$. Hence,

$$\frac{\partial L}{\partial\dot{\theta}} = mrv_\theta + \frac{q}{c}rA_\theta = const \quad \text{[canonical angular momentum]} \quad (14)$$

Examples of Equilibrium Solutions:

In contrast to the Boltzmann equation, which has only one equilibrium solution (the Maxwellian distribution), the Vlasov equation has an infinite number of possible equilibrium solutions. But they exist on a time scale short compared with the collision time.

The choice of the equilibrium solution depends on how the plasma is formed. For example, if we inject two counter streaming electrons of velocity $v_0 \mathbf{e}_z$ and $-v_0 \mathbf{e}_z$ into a neutralizing background of cold ions, we have the following equilibrium solutions for the electrons and ions (of equal density n_0):

$$\begin{cases} f_{e0} = \frac{1}{2} n_0 \delta(v_x)\delta(v_y)[\delta(v_z - v_0) + \delta(v_z - v_0)] \\ f_{i0} = n_0 \delta(v_x)\delta(v_y)\delta(v_z) \end{cases} \quad (15)$$

which correctly represent the electron/ion distributions on a time scale short compared with the collision time.

Given sufficient time, collisions will first randomize electron velocities and eventually equalize electron and ion temperatures. The fianl state will be an equilibrium solution in the form of the Maxwellian distribution for both the electrons and ions:

$$f_0(v) = n_0 \left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\frac{mv^2}{2kT}\right) = \frac{n_0}{(2\pi)^{3/2} v_T^3} \exp\left(-\frac{v^2}{2v_T^2}\right) \quad (16)$$

where $v_T = \sqrt{kT/m}$ is the thermal speed, T is the same for both species, $m = m_e$ for the electrons and $m = m_i$ for the ions. In (16), f_0 has been normalized to give a uniform particle density of n_0 in $\mathbf{x}$-space $[\int f_0(v) d^3v = n_0]$.

$$\textit{Useful formulae:} \begin{cases} \int_0^\infty x^{2n} e^{-ax^2} dx = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^{n+1} a^n} \sqrt{\frac{\pi}{a}} \\ \int_0^\infty x^{2n+1} e^{-ax^2} dx = \frac{n!}{2a^{n+1}} \end{cases} \quad (a > 0) \quad (17)$$

$$\Rightarrow \int_0^\infty e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}; \ \int_0^\infty x^2 e^{-ax^2} dx = \frac{1}{4a}\sqrt{\frac{\pi}{a}}; \ \int_0^\infty x^4 e^{-ax^2} dx = \frac{3}{8a^2}\sqrt{\frac{\pi}{a}} \quad (18)$$

Discussion: In constructing the equilibrium solution f_0, we must also consider the self-consistency of the solution. For example, using (11) and (12), we find that both (15) and (16) give $\rho_0 = \rho_{e0} + \rho_{i0} = 0$ and $\mathbf{J}_0 = \mathbf{J}_{e0} + \mathbf{J}_{i0} = 0$. Hence, the plasma produces no self fields and the assumption of $\mathbf{E}_0 = \mathbf{B}_0 = 0$ (which makes v_x, v_y, and v_z constants of the motion) is valid. However, if $f_{e0} = \frac{1}{2} n_0 \delta(v_x)\delta(v_y)[\delta(v_z - v_0) + \delta(v_z - v_0)]$ in (15) is replace by $f_{e0} = n_0 \delta(v_x)\delta(v_y)\delta(v_z - v_0)$, then $J_{ez} \neq 0$ and there will be a self magnetic field, in which v_x, v_y, v_z are no longer constants of the motion. As a result, $f_{e0}(v_x, v_y, v_z)$ does not satisfy the Vlasov equation. In other words, the complete equilibrium solution includes not just f_0, but also the self-consistent fields.

When the air in equilibrium [(16)] is disturbed, sound waves will be generated. When a plasma in equilibrium is disturbed, a great variety of waves may be generated. Some may even grow exponentially. These are subjects of primary interest in plasma studies.

6.3 Electrostatic Waves

Rewrite the Vlasov-Maxwell equations:

$$\begin{cases} \frac{\partial}{\partial t} f_\alpha + \mathbf{v} \cdot \nabla f_\alpha + \frac{q}{m}(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f_\alpha = 0 & (6.5) \\ \nabla \cdot \mathbf{B} = 0 & (7) \\ \nabla \cdot \mathbf{E} = 4\pi\rho & (8) \\ \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial}{\partial t}\mathbf{B} & (9) \\ \nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial}{\partial t}\mathbf{E} + \frac{4\pi}{c}\mathbf{J} & (10) \end{cases}$$

where $$\begin{cases} \rho(\mathbf{x}, t) = \sum_\alpha q_\alpha \int f_\alpha(\mathbf{x}, \mathbf{v}, t) d^3 v & (11) \\ \mathbf{J}(\mathbf{x}, t) = \sum_\alpha q_\alpha \int f_\alpha(\mathbf{x}, \mathbf{v}, t) \mathbf{v} d^3 v & (12) \end{cases}$$

Below, we present a kinetic treatment of the problem in Sec. 1.4. The electrons now have a velocity spread and, as a result, we will find that the electrostatic plasma oscillation becomes an electrostatic wave.

At high frequencies (e.g. $\omega \sim \omega_{pe}$), the ions cannot respond fast enough to play a significant role. So we assume that the ions form a *stationary* background of uniform density n_0. Since we consider only electron dynamics, the species subscript "α" in f_α will be dropped.

Equilibrium (Zero-Order) Solution:

Assume there is no field (including external field) at equilibrium. then, v_x, v_y, and v_z are constants of the motion and any fuction of $\mathbf{v}$, $f_0 = f_0(\mathbf{v}) = f_0(v_x, v_y, v_z)$, is an equilibrium solution for the electrons,

i.e. $$\underbrace{\frac{\partial}{\partial t} f_0(\mathbf{v})}_{0} + \mathbf{v} \cdot \underbrace{\nabla f_0(\mathbf{v})}_{0} - \frac{e}{m_e}(\underbrace{\mathbf{E}_0}_{0} + \frac{1}{c}\mathbf{v} \times \underbrace{\mathbf{B}_0}_{0}) \cdot \nabla_{\mathbf{v}} f_0(\mathbf{v}) = 0$$

provided $f_0(\mathbf{v})$, which represents a uniform distribution in real space, is normalized to the ion density n_0 and it gives rise to zero current,

i.e. $$\begin{cases} \int f_0(\mathbf{v}) d^3v = n_0 \\ \int f_0(\mathbf{v}) \mathbf{v} d^3v = 0 \end{cases} \tag{19}$$

For clarity, all equilibrium quantities are denoted by subscript "0". They are treated as zero-order quantities.

so that there are no fields at equilibrium ($\mathbf{E}_0 = \mathbf{B}_0 = 0$).

First-Order Solution (Linear Theory) of Electrostatic Waves by the Normal-Mode Method:

Consider small deviations from the equilibrium solution in (19) [$f = f_0(\mathbf{v})$, $\mathbf{E}_0 = \mathbf{B}_0 = 0$] and specialize to waves without a magnetic field (thus, $\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial}{\partial t}\mathbf{B} = 0 \Rightarrow \mathbf{E} = -\nabla \phi$). We may then write

$$\begin{cases} f(\mathbf{x}, \mathbf{v}, t) = f_0(\mathbf{v}) + f_1(\mathbf{x}, \mathbf{v}, t) \\ \phi(\mathbf{x}, t) = \phi_1(\mathbf{x}, t) \\ \mathbf{E}(\mathbf{x}, t) = \mathbf{E}_1(\mathbf{x}, t) = -\nabla \phi_1(\mathbf{x}, t) \end{cases} \tag{20}$$

For clarity, we denote all small quantities by subscript "1". They are treated as first-order quantities.

Sub. (20) into the Vlasov equation:

$$\frac{\partial}{\partial t} f + \mathbf{v} \cdot \nabla f - \frac{e}{m_e}(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \overbrace{\mathbf{B}}^{0}) \cdot \nabla_{\mathbf{v}} f = 0, \tag{6.5}$$

we find the zero-order terms vanish. Equating the first-order terms, we obtain

$$\frac{\partial}{\partial t} f_1 + \mathbf{v} \cdot \nabla f_1 = -\frac{e}{m_e} \nabla \phi_1 \cdot \nabla_{\mathbf{v}} f_0 \tag{21}$$

Sub. $\mathbf{E}_1 = -\nabla \phi_1$ and $\rho_1 = -e \int f_1 d^3v$ into the first-order field equation, $\nabla \cdot \mathbf{E}_1 = 4\pi \rho_1$, we obtain $$\nabla^2 \phi_1 = 4\pi e \int f_1 d^3v \tag{22}$$

Rewrite: $$\begin{cases} \dfrac{\partial}{\partial t} f_1 + \mathbf{v}\cdot\nabla f_1 = -\dfrac{e}{m_e}\nabla\phi_1\cdot\nabla_{\mathbf{v}} f_0 & (21) \\ \nabla^2\phi_1 = 4\pi e\int f_1 d^3v & (22) \end{cases}$$

Consider a <u>normal mode</u> (denoted by subscript "k") by letting

$$\begin{cases} f_1(\mathbf{x},\mathbf{v},t) = f_{1k}(\mathbf{v})e^{ik_z z - i\omega t} \quad [f_{1k}(\mathbf{v}): \text{a small function of } \mathbf{v}] & (23) \\ \phi_1(\mathbf{x},t) = \phi_{1k}e^{ik_z z - i\omega t} \quad [\phi_{1k}: \text{a small constant}] & (24) \end{cases}$$

where it is understood that the LHS is given by the real part of the RHS.

The normal-mode analysis is general because a complete solution can be expressed as a superposition of any number of normal modes.

(21), (23), and (24) give $(-i\omega + ik_z v_z) f_{1k}(\mathbf{v}) = -\frac{e}{m_e} ik_z\phi_{1k} \frac{\partial f_0(\mathbf{v})}{\partial v_z}$

$$\Rightarrow f_{1k}(\mathbf{v}) = \frac{e}{m_e}\frac{k_z}{\omega - k_z v_z}\phi_{1k}\frac{\partial f_0(\mathbf{v})}{\partial v_z} \tag{25}$$

(22)-(25) give $$-k_z^2\phi_{1k} = \underbrace{\frac{4\pi n_0 e^2}{m_e}}_{\omega_{pe}^2}\frac{1}{n_0}k_z\phi_{1k}\int\frac{\frac{\partial f_0(\mathbf{v})}{\partial v_z}}{\omega - k_z v_z}d^3v \tag{26}$$

Rewrite (26): $$-k_z^2\phi_{1k} = \omega_{pe}^2\frac{1}{n_0}k_z\phi_{1k}\int\frac{\frac{\partial f_0(\mathbf{v})}{\partial v_z}}{\omega - k_z v_z}d^3v \tag{26}$$

For $\phi_{1k} \neq 0$, we must have $$1 - \frac{\omega_{pe}^2}{k_z^2}\frac{1}{n_0}\int_{-\infty}^{\infty}\frac{\frac{\partial f_0(\mathbf{v})}{\partial v_z}}{v_z - \frac{\omega}{k_z}}d^3v \tag{27}$$

The v_x and v_y integrations in (27) may be immediately carried out to result in a one-dimensional distribution function $g_0(v_z)$ [which, by (19), is normalized to 1]: $$g_0(v_z) = \frac{1}{n_0}\int f_0(\mathbf{v})dv_x dv_y \tag{28}$$

Then, (27) becomes $$1 - \frac{\omega_{pe}^2}{k_z^2}\int_{-\infty}^{\infty}\frac{\frac{dg_0(\mathbf{v})}{dv_z}}{v_z - \frac{\omega}{k_z}}dv_z = 0 \quad \begin{bmatrix}\text{dispersion} \\ \text{relation}\end{bmatrix} \tag{29}$$

(29) has a singularity at $v_z = \omega / k_z$, which will be addressed later. For now, we circumvent this difficulty by assuming $v_z \ll \omega / k_z$ for the majority of electrons so that $g_0(v_z)$ is negligibly small at $v_z = \omega / k_z$.

Since $g_0(v_z \to \pm\infty) \to 0,$ integrating (29) by parts gives

$$1-\frac{\omega_{pe}^2}{k_z^2}\int_{-\infty}^{\infty}\frac{g_0(v_z)}{(v_z-\frac{\omega}{k_z})^2}dv_z = 1-\frac{\omega_{pe}^2}{\omega^2}\int_{-\infty}^{\infty}\frac{g_0(v_z)}{(1-\frac{k_z v_z}{\omega})^2}dv_z = 0 \qquad (6.25)$$

Note: The z-direction here is the x-direction in Nicholson. g_0, k_z, and v_z are, respectively, g, k, and u in Nicholson.

Expanding $(1-\frac{k_z v_z}{\omega})^{-2}$ and keeping terms up to second order in $\frac{k_z v_z}{\omega}$, we obtain $1-\frac{\omega_{pe}^2}{\omega^2}\int_{-\infty}^{\infty} g_0(v_z)[1+2\frac{k_z v_z}{\omega}+3(\frac{k_z v_z}{\omega})^2]dv_z = 0 \quad (6.26)$

A specific example: If the equilibrium soultion is Maxwellian:

$$f_0(\mathbf{v}) = \frac{n_0}{(2\pi)^{3/2}v_{Te}^3}\exp(-\frac{v_x^2+v_y^2+v_z^2}{2v_{Te}^2}) \qquad (6.23)$$

then, using the formula in (18): $\int_0^\infty e^{-ax^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$, we obtain

$$g_0(v_z) = \frac{1}{n_0}\int f_0(\mathbf{v})dv_x dv_y = \frac{1}{\sqrt{2\pi}v_{Te}}\exp(-\frac{v_z^2}{2v_{Te}^2}) \qquad (6.24), (30)$$

With $g_0(v_z) = \frac{1}{\sqrt{2\pi}v_{Te}}\exp(-\frac{v_z^2}{2v_{Te}^2})$ [(30)] and the formulae:

$\int_0^\infty e^{-ax^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$ and $\int_0^\infty x^2 e^{-ax^2}dx = \frac{1}{4a}\sqrt{\frac{\pi}{a}}$ [(18)], we obtain

$$\int_{-\infty}^{\infty} g_0(v_z)dv_z = 1; \quad \int_{-\infty}^{\infty} \underbrace{v_z g_0(v_z)}_{\text{an odd function of } v_z} dv_z = 0; \quad \int_{-\infty}^{\infty} v_z^2 g_0(v_z)dv_z = v_{Te}^2 \qquad (31)$$

Sub. (31) into $1-\frac{\omega_{pe}^2}{\omega^2}\int_{-\infty}^{\infty} g_0(v_z)[1+2\frac{k_z v_z}{\omega}+3(\frac{k_z v_z}{\omega})^2]dv_z = 0$ [(6.26)],

we obtain
$$1-\frac{\omega_{pe}^2}{\omega^2}-3\frac{k_z^2 v_{Te}^2 \omega_{pe}^2}{\omega^4} = 0, \qquad (6.27)$$

By assumption (30), $\omega \gg k_z v_{Te}$. Thus, to lowest order, (6.27) gives $\omega \approx \omega_{pe}$ and, to next order, we obtain the dispersion relation for the <u>Langmuir wave</u>: (See Sec. 7.3 for a fluid treatment)

$$\omega^2 = \omega_{pe}^2 + 3k_z^2 v_{Te}^2 \ [\Rightarrow \omega = \pm\omega_{pe}(1+\tfrac{3}{2}k_z^2 v_{Te}^2)] \qquad (6.28)$$

6.4 Landau Contour

We now address the singularity encountered in (29). First, a review of relevant definitions and theorems involving complex variables. (Reference: Mathews and Walker, "Math. Methods of Phys.," 2nd ed.)

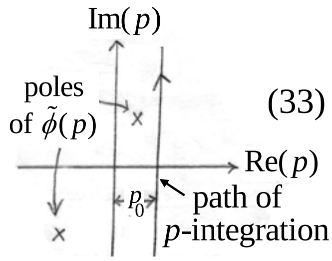
Laplace Transform : (M&W, Sec. 4.3)

A tilde " ~ " on top of a symbol indicates a p-space quantity.

$$\begin{cases} L[\phi(t)] \equiv \int_0^\infty \phi(t)e^{-pt}dt = \tilde{\phi}(p) \\ L[\phi'(t)] = p\tilde{\phi}(p) - \overbrace{\phi(t=0)}^{\text{initial value}} \\ L[\phi''(t)] = p^2\tilde{\phi}(p) - p\,\overbrace{\phi(t=0)}^{\text{initial value}} - \overbrace{\phi'(t=0)}^{\text{initial value}} \end{cases} \quad (32)$$

Inverse Laplace transform:

$$\phi(t) = \frac{1}{2\pi i}\int_{p_0-i\infty}^{p_0+i\infty} \tilde{\phi}(p)e^{pt}dp \quad (33)$$



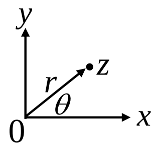
Note: $p_0 (> 0)$ is sufficiently large so that all the poles of $\tilde{\phi}(p)$ lie to the left of the path of p-integration. Hence, $\phi(t) = 0$ if $t < 0$. (why?)

Analytic Function : (M&W, Appendix A)

A function $f(z)$ in the complex z-plane ($z = x + iy = re^{i\theta}$) is said to be analytic at a point z if it has a derivative there and the derivative

$$f'(z) = \lim_{h\to 0}\frac{f(z+h)-f(z)}{h} \quad [h\text{: a complex number}]$$

is independent of the path by which h approaches 0.



The necessary and sufficient conditions for a function

$$W(z) = U(x,y) + iV(x,y)$$

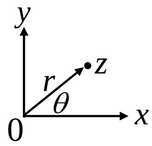
to be analytic are: $\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}$ and $\frac{\partial V}{\partial x} = -\frac{\partial U}{\partial y}$ (34)

Examples: $W = z^2$, $W = \sqrt{z}$, and $W = e^z$ are all analytic functions. $W = z^*$ (z^*: complex conjugate of z) is not an analytic function.

Single-Valued Function: (M&W, Appendix A)

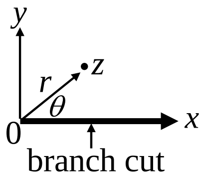
A function $W(z)$ in the complex z-plane is <u>single-valued</u> if

$$W(z = re^{i\theta}) = W(z = re^{i(\theta+2n\pi)}) \; [n = 1, 2, \cdots]$$



Examples: $\begin{cases} W = z^2 \text{ is a single-valued function.} \\ W = z^{1/2} \text{ is not a single-valued function.} \end{cases}$

For $W = z^{1/2}$, we may draw a <u>branch cut</u> from $z = 0$ to ∞ in the z-plane and forbid z to cross it. Then, $W = z^{1/2}$ is single-valued in the z-plane where θ is restricted to the range $0 \le \theta \le 2\pi$, i.e. the θ value would have a 2π jump if z were to cross the branch cut.

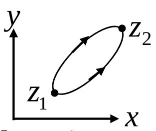


Regular Function: (M&W, Appendix A)

A function is said to be <u>regular</u> in a region R if it is both analytic and single-valued in R. Thus, $W = z^2$ is regular in the z-plane with an arbitray θ and $W = z^{1/2}$ is regular in the z-plane with a branch cut.

Cauchy' Theorem: (M&W, Appendix A)

If a function $f(z)$ is regular in a region R, then $\oint_C f(z)dz = 0$, where C is any closed path lying within R. Hence, the line integral $\int_{z_1}^{z_2} f(z)dz$ is independent of the path of integration from z_1 to z_2 if the path lies within region R.



Theorem of Residues: (M&W, Sec. 3.3 & Appendix A)

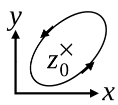
If $f(z)$ is regular in a region R, except for a finite number of poles, then,

$$\oint_C f(z)dz = 2\pi i \sum \text{residues inside } C, \tag{35}$$

where C is any closed path (in the *counterclockwise* direction) within R and

$$\begin{bmatrix} \text{the residue of a pole} \\ \text{of order } n \text{ at } z = z_0 \end{bmatrix} = \frac{1}{(n-1)!}\left\{\left(\frac{d}{dz}\right)^{n-1}\left[(z-z_0)^n f(z)\right]\right\}_{z=z_0} \tag{36}$$

Example: $\oint_C \frac{g(z)}{z-z_0}dz = 2\pi i g(z_0)$ $\begin{bmatrix} g(z) \text{ is regular} \\ \text{with no poles.} \end{bmatrix}$

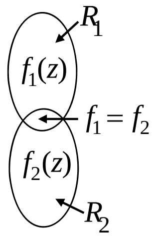


Principal Value: $P\int_a^b \frac{g(x)}{x-x_0}dx \equiv \lim_{\delta\to\infty}\left[\int_a^{x_0-\delta}\frac{g(x)}{x-x_0}dx + \int_{x_0+\delta}^b \frac{g(x)}{x-x_0}dx\right]$ (37)

Identity Theorem : (M&W, Appendix A)

If two functions are each regular in a region R, and having the same values for all points within some subregion or for all points along an arc of some curve within R, then the two functions are identical everywhere in R. For example, in the the z-plane, e^z is the unique function in the z-plane which equals e^x on the x-axis.

Analytic Continuation : (M&W, Appendix A)

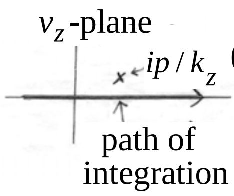


If $f_1(z)$ and $f_2(z)$ are analytic in regions R_1 and R_2, respectively, and $f_1 = f_2$ in a common region (or line), then $f_2(z)$ is the analytic continuation of $f_1(z)$ into R_2. By identity theorem, it is the *unique* analytic continuation.

Example 1: $f_1 = 1 + z + z^2 + z^3 + \cdots$ is analytic in the region $|z| < 1$. $f_2 = 1/(1-z)$ is analytic everywhere except at the pole $z = 1$. Since $f_1 = f_2$ in the common region $|z| < 1$, f_2 is the unique analytic continuation of f_1 into the $|z| \geq 1$ region (except for the pole at $z = 1$).

Example 2: Consider the following 2 analytic functions of p:

$$f_1(p) = \int_{-\infty}^{\infty} dv_z \frac{g(v_z)}{v_z - ip/k_z} \quad \begin{bmatrix} \text{analytic in the upper} \\ \text{half plane, } \mathrm{Re}(p) > 0 \end{bmatrix} \tag{38}$$



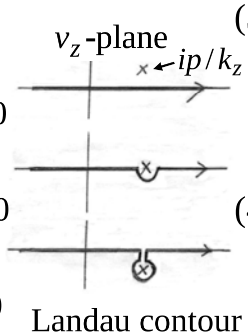
(k_z is real and positive.)

$$f_2(p) = \int_L dv_z \frac{g(v_z)}{v_z - ip/k_z} \quad \begin{bmatrix} \text{analytic in the} \\ \text{entire } p\text{-plane} \end{bmatrix} \tag{39}$$

Landau contour

$$\equiv \begin{cases} \displaystyle\int_{-\infty}^{\infty} dv_z \frac{g(v_z)}{v_z - ip/k_z}, & \mathrm{Re}(p) > 0 \\ \displaystyle P\int_{-\infty}^{\infty} dv_z \frac{g(v_z)}{v_z - ip/k_z} + \pi i g(\frac{ip}{k_z}), & \mathrm{Re}(p) = 0 \\ \displaystyle\int_{-\infty}^{\infty} dv_z \frac{g(v_z)}{v_z - ip/k_z} + 2\pi i g(\frac{ip}{k_z}), & \mathrm{Re}(p) < 0 \end{cases} \tag{40}$$

definition of Landau contour



Since $f_1(p) = f_2(p)$ in the upper half plane, $f_2(p)$ is the (unique) analytical continuation of $f_1(p)$ into the lower half plane.

Electrostatic Waves by the Method of Laplace Transform:

(Ref. Krall & Trivelpiece, Secs. 8.3 and 8.4)

Return to (21) and (22):
$$\begin{cases} \frac{\partial}{\partial t} f_1 + \mathbf{v}\cdot\nabla f_1 = -\frac{e}{m_e}\nabla\phi_1\cdot\nabla_{\mathbf{v}} f_0 & (21) \\ \nabla^2\phi_1 = 4\pi e\int f_1 d^3v & (22) \end{cases}$$

Direct substitution of the nomal mode [(23), (24)] into (21) and (22) results in a singularity in (29). Landau resolved this problem by treating (21) and (22) as an initial value problem in t, while analyzing a spatial Fourier component in z (denoted by subscript "k").

Let
$$\begin{cases} f_1(\mathbf{x},\mathbf{v},t) = f_{1k}(\mathbf{v},t)e^{ik_z z} & (41) \\ \phi_1(\mathbf{x},t) = \phi_{1k}(t)e^{ik_z z} & (42) \end{cases}$$
[By assumption, the wave has no x, y-variation.]

Sub. (41), (42) into (21), (22), we obtain
$$\begin{cases} \frac{\partial}{\partial t} f_{1k}(\mathbf{v},t) + ik_z v_z f_{1k}(\mathbf{v},t) = -\frac{e}{m_e} ik_z\phi_{1k}(t)\frac{\partial f_0(\mathbf{v})}{\partial v_z} & (43) \\ -k_z^2\phi_{1k}(t) = 4\pi e\int f_{1k}(\mathbf{v},t)d^3v & (44) \end{cases}$$

Rewrite
$$\begin{cases} \frac{\partial}{\partial t} f_{1k}(\mathbf{v},t) + ik_z v_z f_{1k}(\mathbf{v},t) = -\frac{e}{m_e} ik_z\phi_{1k}(t)\frac{\partial f_0(\mathbf{v})}{\partial v_z} & (43) \\ -k_z^2\phi_{1k}(t) = 4\pi e\int f_{1k}(\mathbf{v},t)d^3v & (44) \end{cases}$$

Perform a Laplace transform on (43) and (44) [see (32)], we obtain
$$\begin{cases} p\tilde{f}_{1k}(\mathbf{v},p) - f_{1k}(\mathbf{v},t=0) + ik_z v_z\tilde{f}_{1k}(\mathbf{v},p) = -\frac{e}{m_e} ik_z\tilde{\phi}_{1k}(p)\frac{\partial f_0(\mathbf{v})}{\partial v_z} & (45) \\ -k_z^2\tilde{\phi}_{1k}(p) = 4\pi e\int \tilde{f}_{1k}(\mathbf{v},p)d^3v & (46) \end{cases}$$

where
$$\begin{cases} \tilde{f}_{1k}(\mathbf{v},p) = \int_0^\infty f_{1k}(\mathbf{v},t)e^{-pt}dt & (47) \\ \tilde{\phi}_{1k}(p) = \int_0^\infty \phi_{1k}(t)e^{-pt}dt & (48) \end{cases}$$

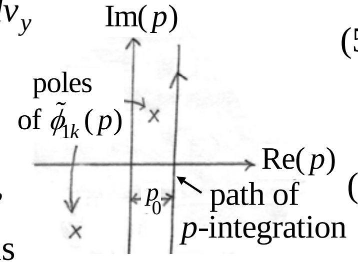
$$(45) \Rightarrow \tilde{f}_{1k}(\mathbf{v},p) = \frac{f_{1k}(\mathbf{v},t{=}0) - \frac{e}{m_e}ik_z\frac{\partial f_0(\mathbf{v})}{\partial v_z}\tilde{\phi}_{1k}(p)}{p+ik_z v_z} \qquad (49)$$

A note on notations: Subscripts "0" and "1" indicate, respectively, zero-order and first-order quantities. Subscript "k" indicates a Fourier component in z. Symbols with a " ~ " sign on top are p-space quantities.

Sub. (49) into (46), we obtain

$$\tilde{\phi}_{1k}(p)=\frac{-4\pi e\int_{-\infty}^{\infty}\dfrac{f_{1k}(\mathbf{v},t=0)}{p+ik_zv_z}d^3v}{k_z^2\left[1-\dfrac{\omega_{pe}^2}{k_z^2}\displaystyle\int_{-\infty}^{\infty}\dfrac{\frac{ik_z}{n_0}\frac{\partial f_0(\mathbf{v})}{\partial v_z}}{p+ik_zv_z}d^3v\right]}=\frac{-i\dfrac{4\pi n_0 e}{k_z^3}\displaystyle\int_{-\infty}^{\infty}\dfrac{g_{1k}(v_z,t=0)}{v_z-\frac{ip}{k_z}}dv_z}{1-\dfrac{\omega_{pe}^2}{k_z^2}\displaystyle\int_{-\infty}^{\infty}\dfrac{\frac{dg_0(v_z)}{dv_z}}{v_z-\frac{ip}{k_z}}dv_z} \qquad (50)$$

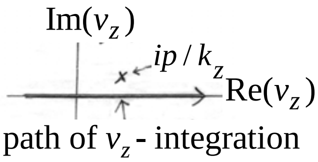
where
$$\begin{cases} g_{1k}(v_z,t=0)=\dfrac{1}{n_0}\int f_{1k}(\mathbf{v},t=0)dv_xdv_y \\ g_0(v_z)=\dfrac{1}{n_0}\int f_0(\mathbf{v})dv_xdv_y \end{cases} \qquad (51)$$



Inverse Laplace transform:

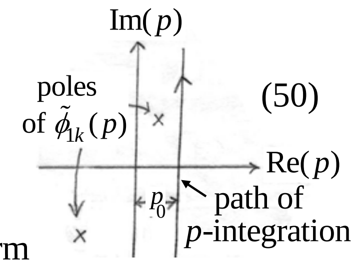
By (33), $\phi_{1k}(t)=\frac{1}{2\pi i}\int_{p_0-i\infty}^{p_0+i\infty}\tilde{\phi}_{1k}(p)e^{pt}dp$, (52)

where $p_0(>0)$ is a real number. In (50), k_z is real by assumption. If $k_z>0$, we see from (50) that the pole (ip/k_z) of the v_z-integrals lies above the Re(v_z) axis. Hence, $\tilde{\phi}_{1k}(p)$ & $\phi_{1k}(t)$ are valid solutions without any singularity.

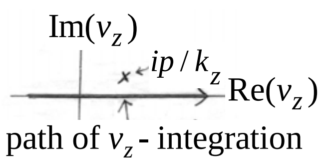


Rewrite (52): $\phi_{1k}(t)=\frac{1}{2\pi i}\int_{p_0-i\infty}^{p_0+i\infty}\tilde{\phi}_{1k}(p)e^{pt}dp$, (52)

where
$$\tilde{\phi}_{1k}(p)=\frac{-i\dfrac{4\pi n_0 e}{k_z^3}\displaystyle\int_{-\infty}^{\infty}\dfrac{g_{1k}(v_z,t=0)}{v_z-\frac{ip}{k_z}}dv_z}{1-\dfrac{\omega_{pe}^2}{k_z^2}\displaystyle\int_{-\infty}^{\infty}\dfrac{\frac{dg_0(v_z)}{dv_z}}{v_z-\frac{ip}{k_z}}dv_z} \qquad (50)$$

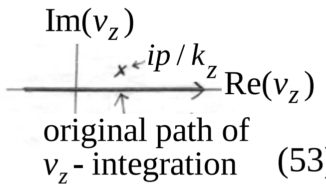


The *mathematical* solution (52) is in a form in which the physics (e.g. normal modes and dispersion relation, etc.) is not transparent. We need more work to obtain a *physics* solution.

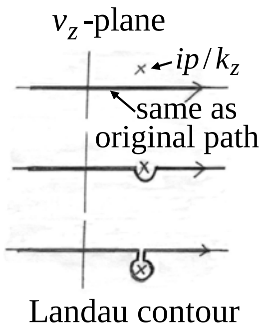


This will require a detour of the path of the p-integration into the Re(p) ≤ 0 region, which implies that the pole at ip/k_z will cross the Re(v_z) axis, on which $\tilde{\phi}_{1k}(p)$ is singular. Thus, the analytic region of $\tilde{\phi}_{1k}(p)$ is bounded by the Re(p) $=0$ line, and our first step is to analytically continue $\tilde{\phi}_{1k}(p)$ from Re(p) >0 into Re(p) ≤ 0.

Step 1 : Analytic continuation : By the method in (38)-(40), we may analytically continue $\tilde{\phi}_{1k}(p)$ from Re(p) > 0 into Re(p) ≤ 0 by by changing the path of v_z-integration in (50) from the straight line $\int_{-\infty}^{\infty} dv_z$ to the Landau contour: $\int_L dv_z$.



$$\text{Thus, } \tilde{\phi}_{1k}(p) = \frac{-i\dfrac{4\pi n_0 e}{k_z^3}\displaystyle\int_L \frac{g_{1k}(v_z,t=0)}{v_z-\frac{ip}{k_z}}dv_z}{1-\dfrac{\omega_{pe}^2}{k_z^2}\displaystyle\int_L \frac{\frac{dg_0(v_z)}{dv_z}}{v_z-\frac{ip}{k_z}}dv_z} \qquad (53)$$

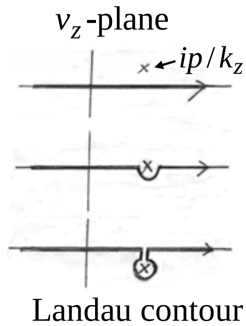


Note that this path change will not affect the value of $\phi_{1k}(t) = \dfrac{1}{2\pi i}\displaystyle\int_{p_0-i\infty}^{p_0+i\infty}\tilde{\phi}_{1k}(p)e^{pt}dp$ in (52) because $\tilde{\phi}_{1k}(p)$ in (52) is evaluated along the Re(p) = p_0(> 0) line, for which the Landau path is the same as the original v_z-path (see figures).

Write (53) as $\tilde{\phi}_{1k}(p) = \dfrac{S(p,k_z)}{D(p,k_z)}$, (54)

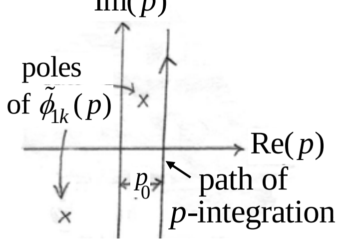
$$\text{where } \begin{cases} S(p,k_z) = -i\dfrac{4\pi n_0 e}{k_z^3}\displaystyle\int_L \frac{g_{1k}(v_z,t=0)}{v_z-\frac{ip}{k_z}}dv_z \\ D(p,k_z) = 1-\dfrac{\omega_{pe}^2}{k_z^2}\displaystyle\int_L \frac{\frac{dg_0(v_z)}{dv_z}}{v_z-\frac{ip}{k_z}}dv_z \end{cases} \qquad (55)$$

$\boxed{\text{implying "source"}}$



then, (52) ⇒ $\phi_{1k}(t) = \dfrac{1}{2\pi i}\displaystyle\int_{p_0-i\infty}^{p_0+i\infty}\frac{S(p,k_z)}{D(p,k_z)}e^{pt}dp$ (56)

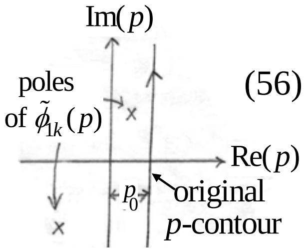
Since $\tilde{\phi}_{1k}(p)$ is now regular (analytic and single-valued) in the entire p-plane (except at poles), we are free to deform the path of p-integration in (56) (by Cauchy's theorem), provided the new path does not cross any pole.



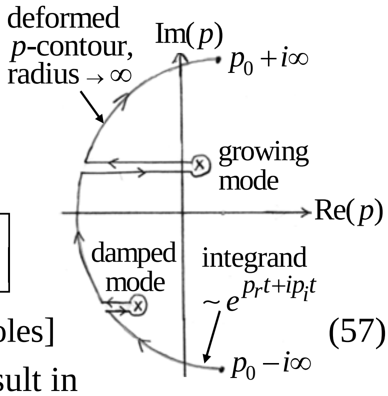
Note: We now have 2 complex planes: p-plane and v_z-plane, and there are integrals along complex paths in both planes.

Step 2 : Deformation of the p-contour in (56):

$$\phi_{1k}(t)=\frac{1}{2\pi i}\int_{p_0-i\infty}^{p_0+i\infty}\frac{S(p,k_z)}{D(p,k_z)}e^{pt}dp \qquad (56)$$



To bring out the physics in (56), we deform the p-contour as shown to the right. Cauchy's theorem requires the new path to encircle (rather than cross) the poles it encounters.



Assume that all the poles of $\tilde{\phi}_{1k}(p)$ $[=S(p,k_z)/D(p,k_z)]$ are at the (1st-order) roots $p_j\,(j=1,2,\cdots)$ of $D(p,k_z)=0$. Then,

$$\phi_{1k}(t)=\sum_j\left[(p-p_j)\frac{S(p,k_z)}{D(p,k_z)}\right]_{p=p_j}e^{p_j t}\left[\begin{array}{l}\text{residues}\\ \text{at poles}\end{array}\right]$$

+ transient effects [integrations away from poles] (57)

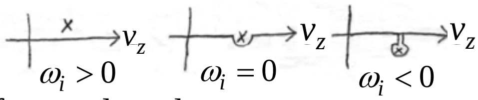
Path integrations away from the poles result in $\phi_{1k}(t)\to 0$ as $t\to\infty$ because e^{pt} oscillates rapidly with $p_i[=\mathrm{Im}(p)]$.

Normal modes: Rewrite:
$$1-\frac{\omega_{pe}^2}{k_z^2}\int_L\frac{\frac{dg_0(v_z)}{dv_z}}{v_z-\frac{ip}{k_z}}dv_z=0 \qquad (55)$$

Define $\omega\equiv ip$ so that $\phi_{1k}(t)\sim e^{-i\omega t}$ and $\phi_1(\mathbf{x},t)\sim e^{-i\omega t+ik_z z}$ (58)

then, (55) can be written
$$1-\frac{\omega_{pe}^2}{k_z^2}\int_L\frac{\frac{dg_0(v_z)}{dv_z}}{v_z-\frac{\omega}{k_z}}dv_z=0\quad\left[\begin{array}{l}\text{dispersion}\\ \text{relation}\end{array}\right] \qquad (59)$$

(corresponding Landau contour:



)

and (57) can be written as a sum of normal modes

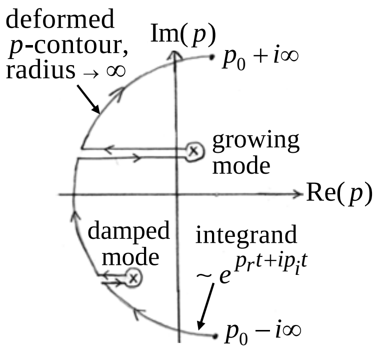
$$\phi_{1k}(t)=\sum_j\left[\tfrac{1}{i}(\omega-\omega_j)\frac{S(\omega,k_z)}{D(\omega,k_z)}\right]_{\omega=\omega_j}e^{-i\omega_j t}+\left[\begin{array}{l}\text{transient}\\ \text{effects}\end{array}\right] \qquad (60)$$

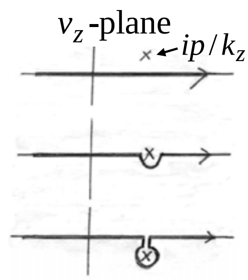
$$\Rightarrow\ \phi_1(\mathbf{x},t)=\sum_j\left[\tfrac{1}{i}(\omega-\omega_j)\frac{S(\omega,k_z)}{D(\omega,k_z)}\right]_{\omega=\omega_j}e^{-i\omega_j t+ik_z z}+\left[\begin{array}{l}\text{transient}\\ \text{effects}\end{array}\right] \qquad (61)$$

where frequencies $\omega_j\,(=ip_j,\ j=1,2,\cdots)$ of the normal modes (in gereral complex numbers) for a given k_z can be found from (59).

Summary of techniques and theorems used :

A *Laplace transform* brings us into the complex variable territory. *Cauchy's theorem* is then used to deform the p-contour (left figure). This requires the *analytic continuation* of $\tilde{\phi}_{1k}(p)$ to the entire p-plane, which in turn leads to the *Landau contour* for the v_z-integration (right figure). With the deformed p-contour, we are able to apply the *residue theorem* to extract the essential physics by isolating the normal modes from the (non-essential) transient effects.





Landau contour

A Recipe for Handling Singularities in Normal-Mode Method :

Rewrite the solution [(29)] obtained by the normal-mode method in Sec. 6.3 and the new solution [(59) and (61)] obtained by the Laplace transform method in this section.

$$1 - \frac{\omega_{pe}^2}{k_z^2}\int_{-\infty}^{\infty} \frac{\frac{dg_0(v_z)}{dv_z}}{v_z - \frac{\omega}{k_z}} dv_z = 0 \qquad (29)$$

By comparison, we find that the Laplace-transform method gives the additional information of mode amplitude in terms of the initial perturbation [see (61)].

$$\begin{cases} 1 - \dfrac{\omega_{pe}^2}{k_z^2}\displaystyle\int_L \frac{\frac{dg_0(v_z)}{dv_z}}{v_z - \frac{\omega}{k_z}} dv_z = 0 & (59) \\ \phi_1(\mathbf{x},t) = \displaystyle\sum_j \left[\frac{1}{i}(\omega - \omega_j)\frac{S(\omega,k_z)}{D(\omega,k_z)}\right]_{\omega=\omega_j} e^{-i\omega_j t + ik_z z} + \begin{bmatrix}\text{transient} \\ \text{effects}\end{bmatrix} & (61) \end{cases}$$

We find that (29) and (59) have the same form except for the path of the v_z-integration. This provides a simple recipe for removing the singularity in (29): replacing $\int_{-\infty}^{\infty} dv_z$ with the Landau contour: $\int_L dv_z$.

The above recipe is of general applicability; namely, we may solve a variety of problems by the (simpler) normal-mode method, then remove similar singularities in the solutions by replacing the $\int_{-\infty}^{\infty} dv_z$ contour with the Laudau contour $\int_L dv_z$.

A question may arise as to whether the pole should remain above or below the Landau contour. As just shown, this depends on whether the *original* position of the pole $\frac{ip}{k_z}$ is above or below the Re(v_z)-axis

in (50):
$$\tilde{\phi}_{1k}(p) = \frac{-i\frac{4\pi n_0 e}{k_z^3}\int_{-\infty}^{\infty}\frac{g_{1k}(v_z,t=0)}{v_z-\frac{ip}{k_z}}dv_z}{1-\frac{\omega_{pe}^2}{k_z^2}\int_{-\infty}^{\infty}\frac{\frac{dg_0(v_z)}{dv_z}}{v_z-\frac{ip}{k_z}}dv_z} \tag{50}$$

before we deform the p-contour in (52): $\phi_{1k}(t) = \frac{1}{2\pi i}\int_{p_0-i\infty}^{p_0+i\infty}\tilde{\phi}_{1k}(p)e^{pt}dp$

i.e. we determine the original position of $\frac{ip}{k_z}$ by setting $\text{Re}(p) = p_0 > 0$.

Rewrite
$$\begin{cases} \phi_{1k}(t) = \frac{1}{2\pi i}\int_{p_0-i\infty}^{p_0+i\infty}\tilde{\phi}_{1k}(p)e^{pt}dp, & (52) \\ \tilde{\phi}_{1k}(p) = \frac{-i\frac{4\pi n_0 e}{k_z^3}\int_{-\infty}^{\infty}\frac{g_{1k}(v_z,t=0)}{v_z-\frac{ip}{k_z}}dv_z}{1-\frac{\omega_{pe}^2}{k_z^2}\int_{-\infty}^{\infty}\frac{\frac{dg_0(v_z)}{dv_z}}{v_z-\frac{ip}{k_z}}dv_z} & (50) \end{cases}$$

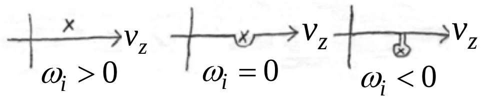
where $\text{Re}(p) = p_0 > 0$. We see, if $k_z > 0$, the original position of the pole $\frac{ip}{k_z}$ lies above the path of v_z-integration, and should remain so when the p-contour in (52) is deformed and the v_z-contour in (50) is changed to the Landau contour. If $k_z < 0$, then the original position of the pole $\frac{ip}{k_z}$ lies below, and should remain below, the v_z-contour.

If we convert the variable p to a new variable $\omega \equiv \pm ip$ so that the solution has the form of a normal mode: $\exp(\mp i\omega t + ik_z z)$, the original position of the pole can be similarly determined as follows:

Recipe for Landau path in normal-mode analysis:

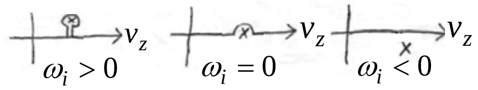
$e^{-i\omega t+ik_z z}$ dependence:

$k_z > 0$ (poles remain above v_z-contour):



(62)

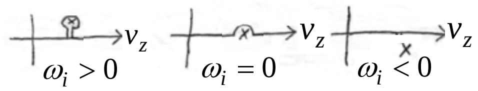
$k_z < 0$ (poles remain below v_z-contour):



(63)

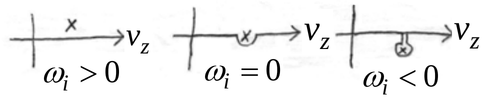
$e^{i\omega t-ik_z z}$ dependence:

$k_z > 0$ (poles remain below v_z-contour):



(64)

$k_z < 0$ (poles remain above v_z-contour):



(65)

6.5 Landau Damping

We have shown that a Laplace transform elegantly resolves the singularty problem in normal-mode analysis. The recipe is in the form of the Landau contour for the v_z-integration [(62)-(65)].

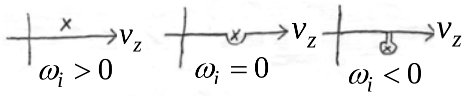
Waves considered so far are electrostatic in nature (i.e. without a **B**-field component). The Langmuir wave derived in Sec. 6.3 is only one example of such waves. In this section, we reconsider the Langmuir wave by properly accounting for the singularity through the use of the Landau contour. This leads to a very important new phenomenon known as Landau damping.

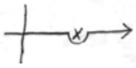
We will then go beyond the scope implied by the section title with an examination of two types of electostatic *instabilities*, the Landau growth and the two-stream instability, which occur in a plasma with a non-Maxwellian electron distribution. We will also consider a different type of (low-frequency) electrostatic wave, the ion sound wave, which involves both the electrons and ions.

Landau Damping in a Plasma with a Maxwellian $g_0(v_z)$:

Rewrite (59): $$1-\frac{\omega_{pe}^2}{k_z^2}\int_L \frac{1}{v_z-\frac{\omega}{k_z}}\frac{dg_0(v_z)}{dv_z}dv_z = 0, \qquad (59)$$

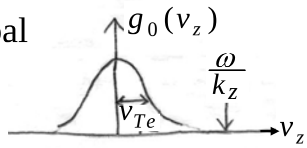
which was derived under the $e^{-i\omega t+ik_z z}$ dependence. Thus, for $k_z > 0$, the v_z-contour is [see (62)]



If $\omega_i \to 0,$ we may take the path  and write (see Nicholson, pp. 279-284)

$$\int_L \frac{1}{v_z-\frac{\omega}{k_z}}\frac{dg_0(v_z)}{dv_z}dv_z = P\int_{-\infty}^{\infty}\frac{1}{v_z-\frac{\omega}{k_z}}\frac{dg_0(v_z)}{dv_z}dv_z + \pi i\left.\frac{dg_0(v_z)}{dv_z}\right|_{v_z=\omega/k_z} \qquad (66)$$

If $\omega/k_z \gg v_{Te},$ we may expand the principal value term to obtain $\int_L \frac{1}{v_z-\frac{\omega}{k_z}}\frac{dg_0(v_z)}{dv_z}dv_z$



$$= -\frac{k_z}{\omega}\int_{-\infty}^{\infty}[1+\frac{k_z v_z}{\omega}+(\frac{k_z v_z}{\omega})^2+(\frac{k_z v_z}{\omega})^3+\cdots]dv_z + \pi i\left.\frac{dg_0}{dv_z}\right|_{v_z=\omega/k_z} \qquad (67)$$

Rewrite (67): $\int_L \frac{1}{v_z-\frac{\omega}{k_z}}\frac{dg_0(v_z)}{dv_z}dv_z$

$$= -\frac{k_z}{\omega}\int_{-\infty}^{\infty}\frac{dg_0}{dv_z}[1+\frac{k_z v_z}{\omega}+(\frac{k_z v_z}{\omega})^2+(\frac{k_z v_z}{\omega})^3+\cdots]dv_z + \pi i\left.\frac{dg_0}{dv_z}\right|_{v_z=\frac{\omega}{k_z}} \qquad (68)$$

For a Maxwellian g_0:
$$\begin{cases} g_0(v_z) = \frac{1}{\sqrt{2\pi}v_{Te}}\exp(-\frac{v_z^2}{2v_{Te}^2}) \ \text{[see (30)]} \\ \frac{dg_0(v_z)}{dv_z} = -\frac{v_z}{\sqrt{2\pi}v_{Te}^3}\exp(-\frac{v_z^2}{2v_{Te}^2}) \end{cases} \qquad (69)$$

we use $\int_0^\infty x^2 e^{-ax^2}dx = \frac{1}{4a}\sqrt{\frac{\pi}{a}}$ and $\int_0^\infty x^4 e^{-ax^2}dx = \frac{3}{8a^2}\sqrt{\frac{\pi}{a}}$ [(18)] to obtain

$$\int_L \frac{1}{v_z-\frac{\omega}{k_z}}\frac{dg_0(v_z)}{dv_z}dv_z = (\frac{k_z}{\omega})^2 + 3(\frac{k_z}{\omega})^4 v_{Te}^2 + \pi i\left.\frac{dg_0}{dv_z}\right|_{v_z=\frac{\omega}{k_z}} \qquad (70)$$

Sub. (70) into (59) and using (69), we obtain

$$1-\frac{\omega_{pe}^2}{\omega^2}(1+3\frac{k_z^2 v_{Te}^2}{\omega^2}) + i\sqrt{\frac{\pi}{2}}\frac{\omega_{pe}^2\omega}{k_z^3 v_{Te}^3}e^{\frac{-\omega^2}{2k_z^2 v_{Te}^2}} = 0 \qquad (71)$$

Rewrite (71): $1-\frac{\omega_{pe}^2}{\omega^2}(1+\underbrace{3\frac{k_z^2 v_{Te}^2}{\omega^2}}_{\ll 1})+\underbrace{i\sqrt{\frac{\pi}{2}}\frac{\omega_{pe}^2\omega}{k_z^3 v_{Te}^3}e^{\frac{-\omega^2}{2k_z^2 v_{Te}^2}}}_{\ll 1}=0$ (71)

Let $\omega=\omega_r+i\omega_i$ and $\frac{\omega_i}{\omega_r}\ll 1 \Rightarrow \frac{1}{\omega^2}\approx\frac{1}{\omega_r^2}(1+i\frac{\omega_i}{\omega_r})^{-2}\approx\frac{1}{\omega_r^2}(1-2i\frac{\omega_i}{\omega_r})$

Keeping terms up to first order, (71) gives

$$1-\frac{\omega_{pe}^2}{\omega_r^2}(1+3\frac{k_z^2 v_{Te}^2}{\omega_r^2})+2i\frac{\omega_{pe}^2}{\omega_r^2}\frac{\omega_i}{\omega_r}+i\sqrt{\frac{\pi}{2}}\frac{\omega_{pe}^2\omega_r}{k_z^3 v_{Te}^3}e^{\frac{-\omega_r^2}{2k_z^2 v_{Te}^2}}=0 \qquad (72)$$

We may solve (72) by the method of iteration. To the lowest order, (72) gives $\omega_r=\omega_{pe}$. To first order, the real part of (72) gives

$$\omega_r^2=\omega_{pe}^2(1+3\frac{k_z^2 v_{Te}^2}{\omega_{pe}^2}) \;\Rightarrow\; \omega_r=\omega_{pe}(1+\frac{3}{2}\frac{k_z^2 v_{Te}^2}{\omega_{pe}^2}) \qquad (73)$$

and the imaginary part of (72) gives

$$\omega_i=-\sqrt{\frac{\pi}{8}}\frac{\omega_{pe}^4}{k_z^3 v_{Te}^3}e^{-\frac{\omega_{pe}^2}{2k_z^2 v_{Te}^2}-\frac{3}{2}} \quad [k_z>0] \qquad (74)$$

The electron Debye length (λ_{De}) can be written

$$\lambda_{De}^2=\frac{kT_e}{4\pi n_0 e^2}=\frac{kT_e}{m_e}\frac{m_e}{4\pi n_0 e^2}=\frac{v_{Te}^2}{\omega_{pe}^2} \qquad (75)$$

Thus, (73) and (74) can be written

$$\begin{cases}\omega_r=\omega_{pe}(1+\frac{3}{2}\frac{k_z^2 v_{Te}^2}{\omega_{pe}^2})=\omega_{pe}(1+\frac{3}{2}k_z^2\lambda_{De}^2)\\ \omega_i=-\sqrt{\frac{\pi}{8}}\frac{\omega_{pe}^4}{k_z^3 v_{Te}^3}e^{-\frac{\omega_{pe}^2}{2k_z^2 v_{Te}^2}-\frac{3}{2}}=-\sqrt{\frac{\pi}{8}}\frac{\omega_{pe}}{k_z^3\lambda_{De}^3}e^{-\frac{1}{2k_z^2\lambda_{De}^2}-\frac{3}{2}}\end{cases} \qquad (76)$$

where ω_r and ω_i agree with (6.52) and (6.53) in Nicholson.

Discussion:

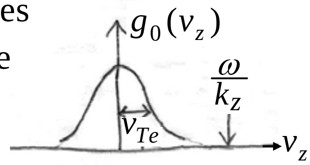
(i) (76) is derived under the $e^{-i\omega t+ik_z z}$ dependence with $k_z>0$. Hence, ω_i is a negative number and $\phi\sim e^{-|\omega_i|t}$, which implies that the wave is damped even though the plasma is assumed to be collisionless. This is known as Landau damping.

Rewrite (76):

$$\begin{cases} \omega_r = \omega_{pe}(1+\frac{3}{2}\frac{k_z^2 v_{Te}^2}{\omega_{pe}^2}) = \omega_{pe}(1+\frac{3}{2}k_z^2\lambda_{De}^2) \\ \omega_i = -\sqrt{\frac{\pi}{8}}\frac{\omega_{pe}^4}{k_z^3 v_{Te}^3} e^{-\frac{\omega_{pe}^2}{2k_z^2 v_{Te}^2}-\frac{3}{2}} = -\sqrt{\frac{\pi}{8}}\frac{\omega_{pe}}{k_z^3\lambda_{De}^3} e^{-\frac{1}{2k_z^2\lambda_{De}^2}-\frac{3}{2}} \end{cases} \quad (76)$$

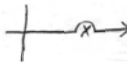
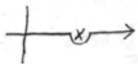
(ii) In the limit of $T_e = 0$, we have $\omega_r = \omega_{pe}$ and $\omega_i = 0$. Thus, (76) reduce to the (undamped) plasma oscillation of a cold plasma. discussed in Sec. 1.4. This shows that the plasma temperature is responsible for both the Landau damping and the change from an oscillation phenomenon to a wave phenomenon.

(iii) Mathematically, contribution to ω_i comes from the residue of the pole at $v_z = \omega / k_z$ in the v_z-integration. Physically, this implies that Landau damping is due to resonant electrons moving at the phase velocity (ω / k_z) of the wave.



Question: How will the result be changed if $k_z < 0$, while still assuming the $e^{-i\omega t + ik_z z}$ dependence?

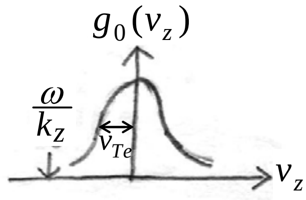
Answer:

In this case, from (63), we use the contour  v_z instead of  v_z. Thus, the only change is to replace "$+\pi i$" in (66):

$$\int_L \frac{1}{v_z - \frac{\omega}{k_z}} \frac{dg_0(v_z)}{dv_z} dv_z = P\int_{-\infty}^{\infty} \frac{1}{v_z - \frac{\omega}{k_z}} \frac{dg_0(v_z)}{dv_z} dv_z + \pi i \left. \frac{dg_0(v_z)}{dv_z} \right|_{v_z = \omega/k_z}$$

with "$-\pi i$". This will result in the same expression for ω_r, but with a sign change in ω_i i.e.

$$\omega_i = \sqrt{\frac{\pi}{8}}\frac{\omega_{pe}^4}{k_z^3 v_{Te}^3} e^{-\frac{\omega_{pe}^2}{2k_z^2 v_{Te}^2}-\frac{3}{2}} = \sqrt{\frac{\pi}{8}}\frac{\omega_{pe}}{k_z^3\lambda_{De}^3} e^{-\frac{1}{2k_z^2\lambda_{De}^2}-\frac{3}{2}}$$

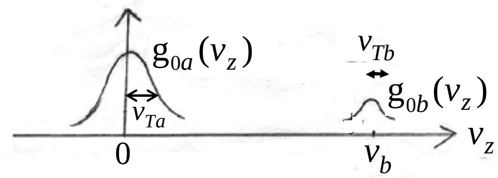


Since $k_z < 0$, ω_i is still a negative number and the wave will be *damped* at the same rate as is expected from symmetry considerations.

Landau Growth in a Plasma with a Bump-in-Tail Distribution :

Consider a plasma whose electrons consist of 2 spatially uniform components with densities n_{0a} and n_{0b}, and equilibrium distributions $g_{0a}(v_z)$ and $g_{0b}(v_z)$ (see figure below). Assume that (1) $n_{0a} \gg n_{0b}$; (2) $g_{0a}(v_z)$ has a v_z-spread of v_{Ta} centered at $v_z = 0$; and (3) $g_{0b}(v_z)$ has a v_z-spread of v_{Tb} centered at $v_z = v_b \gg v_{Ta}$. Thus, $g_{0b}(v_z)$ looks like a small "bump" in the "tail" portion of $g_{0a}(v_z)$.

To be self-consistent, we assume further that the ion density is equal to the total electron density, and the ions drift to the right with a current equal and opposite to the electron current. Thus, there is no electric or magnetic field at equilibrium.

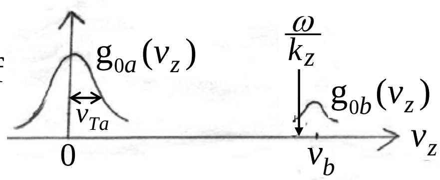


Assuming $e^{-i\omega t+ik_z z}$ dependence with $k_z > 0$ and treating each component separately as before, we obtain the dispersion relation:

$$1-\frac{\omega_{pa}^2}{k_z^2}\underbrace{\int_L \frac{1}{v_z-\frac{\omega}{k_z}}\frac{dg_{0a}(v_z)}{dv_z}dv_z}_{(A)}-\frac{\omega_{pb}^2}{k_z^2}\underbrace{\int_L \frac{1}{v_z-\frac{\omega}{k_z}}\frac{dg_{0b}(v_z)}{dv_z}dv_z}_{(B)}=0 \qquad (77)$$

For term (A), we assume $\omega/k_z \gg v_{Ta}$ (no resonant electrons). Thus, $g_{0a}(v_z) \approx \delta(v_z)$. An integration by parts gives: Term (A) $= k_z^2/\omega^2$ (78)

Since $g_{0b}(v_z) \ll g_{0a}(v_z)$, the real part of term (B) is negligible compared with term (A). However, we must keep the imaginary part of term (B) because it determines ω_i.



$$\text{Thus, Term } (B) = \pi i \left.\frac{dg_{0b}(v_z)}{dv_z}\right|_{v_z=\omega/k_z}. \qquad (79)$$

(77)-(79) give

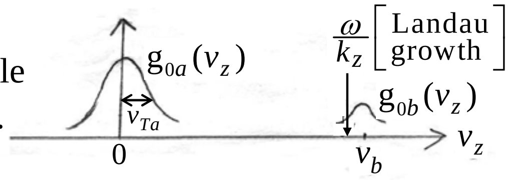
$$1-\frac{\omega_{pa}^2}{\omega^2}-\pi i\frac{\omega_{pb}^2}{k_z^2}\left.\frac{dg_{0b}(v_z)}{dv_z}\right|_{v_z=\omega/k_z}=0 \qquad (80)$$

Rewrite (80): $1-\dfrac{\omega_{pa}^2}{\omega^2}-\pi i\dfrac{\omega_{pb}^2}{k_z^2}\dfrac{dg_{0b}(v_z)}{dv_z}\Big|_{v_z=\omega/k_z}=0$ (80)

Writing $\omega=\omega_r+i\omega_i$ and assuming $\dfrac{\omega_i}{\omega_r}\ll 1,$ we obtain by expansion: $\dfrac{1}{\omega^2}=\dfrac{1}{\omega_r^2}(1+i\dfrac{\omega_i}{\omega_r})^{-2}\approx\dfrac{1}{\omega_r^2}(1-2i\dfrac{\omega_i}{\omega_r})$. Then (80) gives

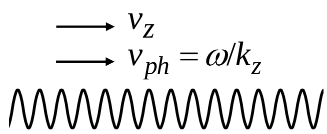
$$\begin{cases}\omega_r\approx\omega_{pa} & \boxed{\omega_r:\text{due to } g_{0a};\ \ \omega_r:\text{due to } g_{0b}}\\ \omega_i\approx\dfrac{\pi}{2}\dfrac{\omega_{pa}\omega_{pb}^2}{k_z^2}\dfrac{dg_{0b}(v_z)}{dv_z}\Big|_{v_z=\omega_r/k_z}=\dfrac{\pi}{2}\dfrac{\omega_{pa}^3}{k_z^2}\dfrac{n_{0b}}{n_{0a}}\dfrac{dg_{0b}(v_z)}{dv_z}\Big|_{v_z=\omega_r/k_z}\end{cases}\qquad(81)$$

(81) shows that the sign of ω_i depends on the sign of dg_{0b}/dv_z at $v_z=\omega_r/k_z$. Thus, $\omega_i>0$ (<u>Landau growth</u>) if ω_r/k_z falls on the positive slope of g_{0b}, and $\omega_i<0$ (Landau damping) if ω_r/k_z falls on the negative slope of g_{0b}. The Landau growth is our first example of *unstable* equilibrium solutions.



A Qualitative Interpretation of Landau Damping and Landau Growth:

Assume that an electrostatic wave with phase velocity ω/k_z is present in the plasma. An electron moving with velocity v_z sees the wave at the Doppler-shifted frequency ω':

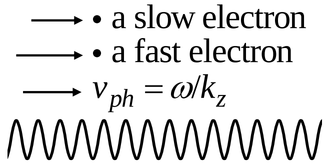


$$\omega'=\omega-k_zv_z\qquad(82)$$

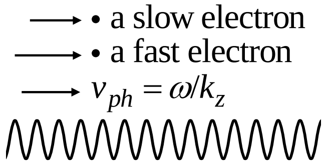
If $\omega'=\omega-k_zv_z\approx 0$ (i.e. electron velocity $\approx$ phase velocity), the electron experiences almost a DC electric field. In this field, it will gain or lose energy for an extended period of time ($\approx 2\pi/\omega'$). This phenomenon is known as <u>resonant interaction</u>.

Divide the electrons into

$$\begin{cases}\text{slow electrons: } \omega'=\omega-k_zv_z\geq 0\\ \text{fast electrons: } \ \ \omega'=\omega-k_zv_z\leq 0\end{cases}$$



For both slow and fast electrons, some will lose energy to the wave and some will gain energy from the wave, depending on the position of the electron relative to the phase of the wave.



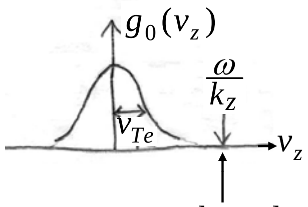
If a slow electron loses energy in the resonant interaction, its v_z decreases. Hence, its ω' $(= \omega - k_z v_z > 0)$, which is a positive number becomes greater. As a result, the time for sustained interaction $(\frac{2\pi}{\omega'})$ becomes shorter. This will result in *weaker* resonance.

On the other hand, if a slow electron gains energy in the resonant interaction, v_z increases and ω' becomes smaller. Hence, the time for sustained interaction becomes longer (*stronger* resonance). This give the electrons in the energy-gaining phase the advantage and, on average, slow electrons gain energy from the wave.

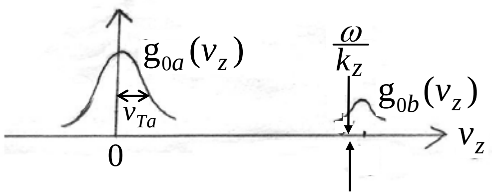
Similarly, fast electrons will, on average, lose energy to wave.

We have just concluded that, on average, slow electrons (relative to the phase velocity of the wave) gain energy from the wave and fast electrons lose energy to the wave. Thus, if the plasma contains more slow electrons than fast electrons (i.e. a negative slope of g_0 at ω/k_z, see left figure), the net effect is an energy transfer from the wave to the electrons (Landau damping).

By similar argument, if the plasma contains more fast electrons than slow electrons (see right figure), there will be a net energy transfer from the electrons to the wave (Landau growth).



more slow electrons than fast electrons



more fast electrons than slow electrons

Kinetic Treatment vs Fluid Treatment :

We have just considered a case in which details of the particle distribution function determine whether a wave grows or damps. On the other hand, fluid equations (derived in Sec. 1.4 of lecture notes by a simple method) are formally derived from the Vlasov equation (Sec. 7.2) by an integration procedure over the velocity space, in which details of the distribution function are lost. Hence, a fluid treatment will miss the Landau damping/growth (Sec. 7.3) and other effects sensitive to the distribution function, but results of fluid equations are implicit in kinetic equations.

Chapter 7 contains a fluid treatment of important plasma modes and instabilities. Here, as a supplement to Ch. 6, we will cover some of these topisc in the framework of the Vlasov equation.

Our first study of fluid modes is on the two-stream instability. More will be considered in subsequent sections of this chapter.

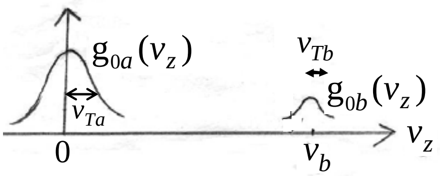
Two-Stream Instability I :

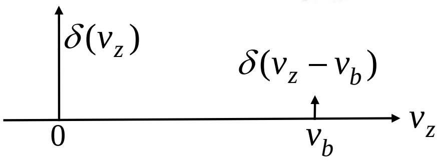
Consider again the bump-in-tail model for Landau growth (upper figure). The dispersion relation obtained by the kinetic approach is

$$1-\frac{\omega_{pa}^2}{k_z^2}\int_L \frac{1}{v_z-\frac{\omega}{k_z}}\frac{dg_{0a}(v_z)}{dv_z}dv_z-\frac{\omega_{pb}^2}{k_z^2}\int_L \frac{1}{v_z-\frac{\omega}{k_z}}\frac{dg_{0b}(v_z)}{dv_z}dv_z=0 \qquad (77)$$

Suppose the velocity spreads (v_{Ta}, v_{Tb}) of the 2 components vanish (lower figure). We then have a situation where one component streams through another component. Integrating (77) by parts and letting $g_{0a}=\delta(v_z)$, $g_{0b}=\delta(v_z-v_b)$, we obtain

$$1-\frac{\omega_{pa}^2}{\omega^2}-\frac{\omega_{pb}^2}{(\omega-k_z v_b)^2}=0 \qquad (83)$$

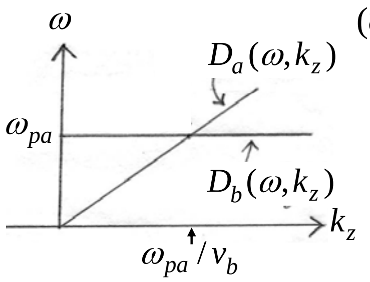




This example shows that the the kinetic result [(77)] can be reduced to the fluid result [(83)] in the proper limit.

The dispersion relation: $1-\frac{\omega_{pa}^2}{\omega^2}-\frac{\omega_{pb}^2}{(\omega-k_z v_b)^2}=0$ [(83)] can be

written $\underbrace{(1-\frac{\omega_{pa}^2}{\omega^2})}_{D_a(\omega,\,k_z)}\underbrace{(1-\frac{k_z v_b}{\omega})^2}_{D_b(\omega,\,k_z)}=\underbrace{\frac{\omega_{pb}^2}{\omega^2}}_{\ll 1}$ (84)



where $\begin{cases} D_a(\omega,k_z)\equiv 1-\frac{\omega_{pa}^2}{\omega^2} \\ D_b(\omega,k_z)\equiv (1-\frac{k_z v_b}{\omega})^2 \end{cases}$

(84) can be regarded as the coupling between the plasma mode ($D_a=0$) and the beam mode ($D_b=0$). The coupling is strongest near the intersection of the two modes (see Figure). The intersecting point is at $\omega=\omega_{pa}$ and $k_z=\frac{\omega_{pa}}{v_b}$, which are solutions of

$$\begin{cases} D_a(\omega,k_z)=0 \\ D_b(\omega,k_z)=0 \end{cases}$$

Rewrite (84): $(1-\frac{\omega_{pa}^2}{\omega^2})(1-\frac{k_z v_b}{\omega})^2=\frac{\omega_{pb}^2}{\omega^2}$ (84)

To show that there is an instability, we will only look for the ω value at $k_z=\omega_{pa}/v_b$, i.e. at the point of strongest interaction. Letting $\omega=\omega_{pa}+\Delta\omega$, we get $\frac{1}{\omega}\approx\frac{1}{\omega_{pa}}(1-\frac{\Delta\omega}{\omega_{pa}})$; $\frac{1}{\omega^2}\approx\frac{1}{\omega_{pa}^2}(1-2\frac{\Delta\omega}{\omega_{pa}})$ (85)

Sub. (85) and $k_z=\omega_{pa}/v_b$ into (84) and keeping terms up to first order in small quantities $\Delta\omega$ and ω_{pb}, we obtain

$$2\frac{\Delta\omega}{\omega_{pa}}\frac{\Delta\omega^2}{\omega_{pa}^2}=\frac{\omega_{pb}^2}{\omega_{pa}^2}=(\frac{n_{0b}}{n_{0a}})^2$$

$$\Rightarrow \Delta\omega^3=\tfrac{1}{2}\omega_{pa}^3(\frac{n_b}{n_a})^2 \Rightarrow \Delta\omega=\frac{1}{2^{1/3}}\omega_{pa}(\frac{n_{0b}}{n_{0a}})^{\frac{2}{3}}e^{\frac{2in\pi}{3}},\quad n=1,2,3$$

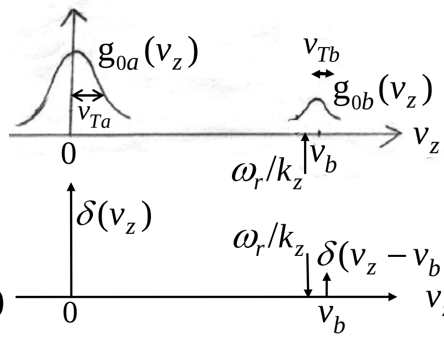
$$\Rightarrow \Delta\omega=\frac{1}{2^{1/3}}\omega_{pa}(\frac{n_{0b}}{n_{0a}})^{\frac{2}{3}}\begin{cases} 1 \\ -\frac{1}{2}-i\frac{\sqrt{3}}{2} \\ -\frac{1}{2}+i\frac{\sqrt{3}}{2} \longleftarrow \boxed{\text{Unstable mode}} \end{cases} \quad (86)$$

From (86), we find the frequency of the unstable mode:

$$\omega = \omega_{pa} + \Delta\omega = \omega_{pa}[1 + \frac{1}{2^{1/3}}(\frac{n_{0b}}{n_{0a}})^{\frac{2}{3}}(-\frac{1}{2} + i\frac{\sqrt{3}}{2})]$$

$$\Rightarrow \begin{cases} \omega_r = \omega_{pa}[1 - \frac{1}{2^{4/3}}(\frac{n_{0b}}{n_{0a}})^{\frac{2}{3}}] \\ \omega_i = \frac{\sqrt{3}}{2^{4/3}}\omega_{pa}(\frac{n_{0b}}{n_{0a}})^{\frac{2}{3}} \quad \text{[growth rate]} \end{cases} \tag{87}$$

Like the case of Landau growth (upper figure), the free energy available in a non-Maxwellian plasma drives a two-stream instability (lower figure) . However, in the latter case, there are no electrons at exactly the phase velocity of the wave. Thus, there is no singularity problem and a fluid treatment will also be adequate. (See Sec. 7.13 for a fluid treatment of a slightly different two-stream model.)



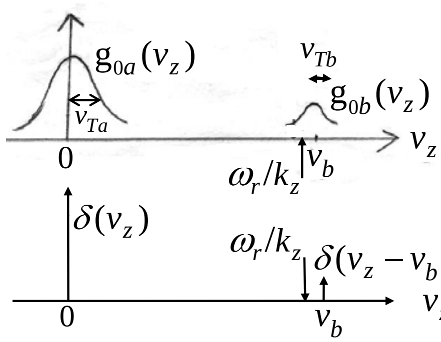
Comparing the Landau growth rate in (81) and the two-stream growth rate in (87):

$$\begin{cases} \omega_i = \frac{\pi}{2}\frac{\omega_{pa}^3}{k_z^2}\frac{n_{0b}}{n_{0a}}\frac{dg_{0b}(v_z)}{dv_z}\Big|_{v_z=\omega_r/k_z} & \text{[Landau growth]} \\ \omega_i = \frac{\sqrt{3}}{2^{4/3}}\omega_{pa}(\frac{n_{0b}}{n_{0a}})^{\frac{2}{3}} & \text{[Two-stream instability]} \end{cases}$$

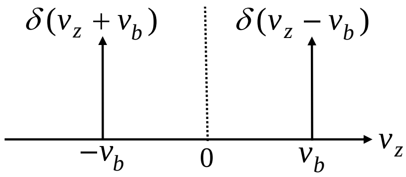
we may show that ω_i [Landau growth] $\ll \omega_i$ [Two-stream instability].

This is because, in the Landau growth (upper figure), electrons drifting slower than the wave absorb energy from the wave, whereas in the two-stream instability (lower figure), all electrons deliver energy to the wave because they all drift faster than the wave.

As the g_{0b} spread increases from 0 to a large value, the fluid instability will transition to a kinetic instability.



Two-Stream Instability II :



Consider two cold electron beams of equal density n_b streaming in an ion neutralizing background in opposite directions (v_b and $-v_b$). In (83), we have already obtained the term for the forward stream: $\dfrac{\omega_{pb}^2}{(\omega - k_z v_b)^2}$.

By symmetry, the backward beam will have a similar form, with v_b replaced by $-v_b$. Thus, following the same treatment leading to (83), we obtain the dispersion relation:

$$1 - \frac{\omega_{pb}^2}{(\omega - k_z v_b)^2} - \frac{\omega_{pb}^2}{(\omega + k_z v_b)^2} = 0, \tag{88}$$

which gives a quadratic equation in ω^2:

$$\omega^4 - 2(\omega_{pb}^2 + k_z^2 v_b^2)\omega^2 - 2\omega_{pb}^2 k_z^2 v_b^2 + k_z^4 v_b^4 = 0 \tag{89}$$

Rewrite $\omega^4 - 2(\omega_{pb}^2 + k_z^2 v_b^2)\omega^2 - 2\omega_{pb}^2 k_z^2 v_b^2 + k_z^4 v_b^4 = 0$ (89)

The solution for ω^2 as obtained from (89) is

$$\omega^2 = \omega_{pb}^2 + k_z^2 v_b^2 \pm \omega_{pb}(\omega_{pb}^2 + 4k_z^2 v_b^2)^{\frac{1}{2}} \tag{90}$$

It can be shown from (90) that for, $k_z^2 > 2\omega_{pb}^2 / v_b^2$, all value of ω are real (no instability). However, for $k_z^2 < 2\omega_{pb}^2 / v_b^2$, two values of ω will be a pair of complex conjugates and one of them gives rise to an instability.

We may find the wave number ($k_z^{\max}$) for which the growth rate maximizes by finding the value of k_z^2 for which $d\omega^2 / dk_z^2 = 0$. The result is $k_z^{\max} = \pm \dfrac{\sqrt{3}\omega_{pb}}{2v_b}$, which corresponds to a maximum growth rate of $\omega_i = \omega_{pb} / 2$. We will return to this problem again in Special Topic II in connection with a new subject: the absolute instability.

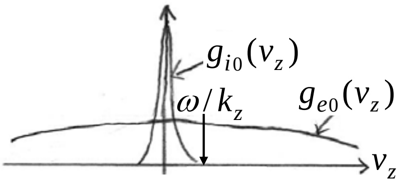
Ion-Acoustic Waves: (See Sec. 7.3 for a fluid treatment)

At low frequencies, the ion contribution to the dispersion relation cannot be ignored. For low frequency electrostatic waves, we simply add to (59) the ion term (the species subscript "*e*" or "*i*" is also added),

$$1-\frac{\omega_{pe}^2}{k_z^2}\int_L \frac{1}{v_z-\frac{\omega}{k_z}}\frac{dg_{e0}(v_z)}{dv_z}dv_z-\frac{\omega_{pi}^2}{k_z^2}\int_L \frac{1}{v_z-\frac{\omega}{k_z}}\frac{dg_{i0}(v_z)}{dv_z}dv_z=0 \qquad (91)$$

Assume a Maxwellian distribution for both the electrons and ions:

$$\begin{cases} g_{e0}(v_z)=\dfrac{1}{\sqrt{2\pi}v_{Te}}\exp(-\dfrac{v_z^2}{2v_{Te}^2}) & (92) \\ g_{i0}(v_z)=\dfrac{1}{\sqrt{2\pi}v_{Ti}}\exp(-\dfrac{v_z^2}{2v_{Ti}^2}) & (93) \end{cases}$$



with $v_{Te} \gg v_{Ti}$. Assume further $v_{Te} \gg \omega/k_z \gg v_{Ti}$ (see figure) so that there is negligible electron Landau damping (because $dg_{e0}/dv_z \to 0$) and negligible ion Landau damping (because $\omega/k_z \gg v_{Ti}$). We will therefore neglect the imaginary part of both integrals in (91).

For the electrons, sub. $\frac{dg_{e0}(v_z)}{dv_z}=-\frac{v_z}{v_{Te}^2}g_0$ into $\int_L \frac{1}{v_z-\frac{\omega}{k_z}}\frac{dg_{e0}(v_z)}{dv_z}dv_z$ and neglect $\frac{\omega}{k_z}$ ($\ll v_{Te}$) in the denominator, we obtain

$$\int_L \frac{1}{v_z-\frac{\omega}{k_z}}\frac{dg_{e0}(v_z)}{dv_z}dv_z \approx -\frac{1}{v_{Te}^2}\int_{-\infty}^{\infty} g_0(v_z)dv_z=-\frac{1}{v_{Te}^2}\overset{(75)}{=}-\frac{1}{\omega_{pe}^2\lambda_{De}^2} \qquad (94)$$

For the ion integral: $\int_L \frac{1}{v_z-\frac{\omega}{k_z}}\frac{dg_{i0}(v_z)}{dv_z}dv_z$, we assume $\frac{\omega}{k_z} \gg v_z$ and follow the same steps leading to (70). This gives the cold ion limit:

$$\int_L \frac{1}{v_z-\frac{\omega}{k_z}}\frac{dg_{i0}(v_z)}{dv_z}dv_z \approx (\frac{k_z}{\omega})^2 \qquad (95)$$

Sub. (94) and (95) into (91), we obtain $\quad 1+\frac{1}{k_z^2\lambda_{De}^2}-\frac{\omega_{pi}^2}{\omega^2}=0, \quad$ (96)

which is the most basic form of the dispersion relation because ion thermal effects and electron Landau damp have all been neglected. From (96), we find $\omega < \omega_{pi}$, i.e. this is indeed a low-frequency wave.

The dispersion relation $$1+\frac{1}{k_z^2\lambda_{De}^2}-\frac{\omega_{pi}^2}{\omega^2}=0 \qquad (94)$$

gives $$\omega^2=\frac{k_z^2\lambda_{De}^2\omega_{pi}^2}{1+k_z^2\lambda_{De}^2}=\frac{k_z^2}{1+k_z^2\lambda_{De}^2}\frac{kT_e}{4\pi n_0 e^2}\frac{4\pi n_0 e^2}{m_i}=\frac{k_z^2}{1+k_z^2\lambda_{De}^2}\frac{kT_e}{m_i}$$

$$\Rightarrow \qquad \omega=\frac{k_z C_s}{\sqrt{1+k_z^2\lambda_{De}^2}} \quad \text{with } C_s\equiv\sqrt{\frac{kT_e}{m_i}} \qquad (95)$$

Physically, when the ions are perturbed, electrons tend to follow the ions to shield their electric field, thereby reducing the restoring forces on the ions. So the wave has a frequency lower than ω_{pi} (ω_{pi} would be the ion oscillation frequency if the electrons were immobile). However, the electrons cannot effectively shield the ion electric field if λ (wavelength) $\le \lambda_{De}$ (see Sec. 1.2). Thus, when $\lambda \ll \lambda_{De}$, or $k_z\lambda_{De} \gg 1$, (94) shows that ω will approach ω_{pi}.

Since the plasma motion is longitudinal and C_s is similar to the sound speed of a neutral gas, the wave is called an <u>ion acoustic wave</u>.

6.10 General Theory of Linear Vlasov Waves   Part 2

We have so far treated only electrostatic waves in the absence of an external field. In this section, we lay the groundwork for a general theory of linear waves, both electrostatic and electromagnetic, in an *infinite* and *uniform* plasma. We assume that the plasma is immersed in a uniform *external* magnetic field along the z-axis: $\mathbf{B}_0 = B_0\mathbf{e}_z$, but there is no external electric field ($\mathbf{E}_0 = 0$).

Equilibrium (Zero-Order) Solution:

An equilibrium solution $f_{\alpha 0}(\mathbf{v})$ must satisfy the zero-order Vlasov equation: $\underbrace{\frac{\partial}{\partial t} f_{\alpha 0}(\mathbf{v})}_{0} + \mathbf{v}\cdot\underbrace{\nabla f_{\alpha 0}(\mathbf{v})}_{0} - \frac{q_\alpha}{m_\alpha}(\underbrace{\mathbf{E}_0}_{0} + \frac{1}{c}\mathbf{v}\times B_0\mathbf{e}_z)\cdot\nabla_{\mathbf{v}} f_{\alpha 0}(\mathbf{v}) = 0$

$$\Rightarrow \qquad (\mathbf{v}\times B_0\mathbf{e}_z)\cdot\nabla_{\mathbf{v}} f_{\alpha 0}(\mathbf{v}) = 0 \qquad (101)$$

Thus, any function of the form $f_{\alpha 0}(v_\perp, v_z)$ satisfies (101), provided that the total charge and current densities of all species vanish so that there is no net self field at equilibrium. This in turn makes $v_\perp$ and v_z constants of the motion in the only field present: $B_0\mathbf{e}_z$.

Examples of equilibrium solutions (normalized to $n_{\alpha 0}$) are:

$$\begin{cases} f_0 = \dfrac{n_0}{(2\pi)^{3/2} v_T^3}\exp(-\dfrac{v^2}{2v_T^2}) \quad \text{[Maxwellian]} & (102) \\[2ex] f_0 = \dfrac{n_0}{(2\pi)^{3/2} v_{T\perp}^2 v_{Tz}}\exp(-\dfrac{v_\perp^2}{2v_{T\perp}^2} - \dfrac{v_z^2}{2v_{Tz}^2}) \quad \text{[bi-Maxwellian]} & (103) \\[2ex] f_0 = \dfrac{n_0}{2\pi v_\perp}\delta(v_\perp - v_{\perp 0})\delta(v_z) & (104) \end{cases}$$

In (103), the particles have two temperatures, $v_{T\perp}$ and v_{Tz}. In (104), all particles have the same $v_\perp (= v_{\perp 0})$ and $v_z (= 0)$. (104) is approximately self-consistent if the self magnetic field due to the gyrating particles is negligible.

First-Order Equations:

The linear properties of a plasma is contained in the dispersion relation. To obtain the dispersion relation, we first linearize the set of Vlasov/Maxwell equations by writing

$$\begin{cases} f_\alpha(\mathbf{x},\mathbf{v},t) = f_{\alpha 0}(\mathbf{v}) + f_{\alpha 1}(\mathbf{x},\mathbf{v},t) & (105) \\ \mathbf{E}(\mathbf{x},t) = \mathbf{E}_1(\mathbf{x},t) & (106) \\ \mathbf{B}(\mathbf{x},t) = B_0\mathbf{e}_z + \mathbf{B}_1(\mathbf{x},t) & (107) \\ \rho(\mathbf{x},t) = \rho_1(\mathbf{x},t) & (108) \\ \mathbf{J}(\mathbf{x},t) = \mathbf{J}_1(\mathbf{x},t) & (109) \end{cases}$$

As before, first-order quantities are denoted by subscript "1".

Sub. (105)-(109) into the Vlasov/Maxwell equations. Zero-order terms give the equilibrium solution. Equating the first-order terms,

we obtain
$$\begin{cases} \frac{\partial}{\partial t} f_{\alpha 1} + \mathbf{v}\cdot\nabla f_{\alpha 1} + \frac{q_\alpha}{m_\alpha c}(\mathbf{v}\times B_0\mathbf{e}_z)\cdot\nabla_{\mathbf{v}} f_{\alpha 1} \\ \qquad = -\frac{q_\alpha}{m_\alpha}(\mathbf{E}_1 + \frac{1}{c}\mathbf{v}\times\mathbf{B}_1)\cdot\nabla_{\mathbf{v}} f_{\alpha 0} & (110) \\ \nabla\cdot\mathbf{B}_1 = 0 & (111) \\ \nabla\cdot\mathbf{E}_1 = 4\pi\rho_1 & (112) \\ \nabla\times\mathbf{E}_1 = -\frac{1}{c}\frac{\partial}{\partial t}\mathbf{B}_1 & (113) \\ \nabla\times\mathbf{B}_1 = \frac{1}{c}\frac{\partial}{\partial t}\mathbf{E}_1 + \frac{4\pi}{c}\mathbf{J}_1 & (114) \end{cases}$$

where
$$\begin{cases} \rho_1(\mathbf{x},t) = \sum_\alpha q_\alpha \int f_{\alpha 1}(\mathbf{x},\mathbf{v},t)d^3v & (115) \\ \mathbf{J}_1(\mathbf{x},t) = \sum_\alpha q_\alpha \int f_{\alpha 1}(\mathbf{x},\mathbf{v},t)\mathbf{v}d^3v & (116) \end{cases}$$

Note: In Nicholson (6.145)-(6.153), zero-order fields depends on $\mathbf{x}$ and t. But from (6.154) on, $\mathbf{E}_0 = 0$ and $\mathbf{B}_0 = B_0\mathbf{e}_z$, as in our model.

Particle dynamics: These first-order equations (110)-(116) are coupled. To examine the particle dynamics, we start from (110):

$$\frac{\partial}{\partial t} f_{\alpha 1} + \mathbf{v}\cdot\nabla f_{\alpha 1} + \frac{q_\alpha}{m_\alpha c}(\mathbf{v}\times B_0\mathbf{e}_z)\cdot\nabla_{\mathbf{v}} f_{\alpha 1} = -\frac{q_\alpha}{m_\alpha}(\mathbf{E}_1 + \frac{1}{c}\mathbf{v}\times\mathbf{B}_1)\cdot\nabla_{\mathbf{v}} f_{\alpha 0} \quad (110)$$

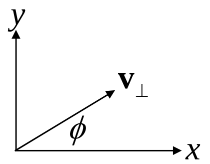
The LHS is a total time derivative [$\frac{d}{dt} f_{\alpha 1}$] along the zero-order orbit because the acceleration force is $\frac{q_\alpha}{c}\mathbf{v}\times B_0\mathbf{e}_z$. Thus, (110) can be written

$$\frac{d}{dt} f_{\alpha 1} = -\frac{q_\alpha}{m_\alpha}(\mathbf{E}_1 + \frac{1}{c}\mathbf{v}\times\mathbf{B}_1)\cdot\nabla_{\mathbf{v}} f_{\alpha 0} \quad (117)$$

Rewrite (117):

$$\frac{d}{dt} f_{\alpha 1}(\mathbf{x},\mathbf{v},t) = -\frac{q_\alpha}{m_\alpha}[\mathbf{E}_1(\mathbf{x},t) + \tfrac{1}{c}\mathbf{v}\times\mathbf{B}_1(\mathbf{x},t)]\cdot\nabla_{\mathbf{v}} \overbrace{f_{\alpha 0}(\mathbf{v})}^{\boxed{= f_{\alpha 0}(v_\perp, v_z)}}, \qquad (117)$$

where $\frac{d}{dt}$ follows the zero-order orbit of a particle, which we denote by $\mathbf{x}'(t')$ and $\mathbf{v}'(t')$. Under the conditions: $\mathbf{x}'(t'=t) = \mathbf{x}$ and $\mathbf{v}'(t'=t) = \mathbf{v}$,



we have
$$\begin{cases} v'_x(t') = v_\perp \cos[\phi - \Omega_\alpha(t'-t)] \\ v'_y(t') = v_\perp \sin[\phi - \Omega_\alpha(t'-t)] \\ v'_z(t') = v_z \\ x'(t') = x - \frac{v_\perp}{\Omega_\alpha}\sin[\phi - \Omega_\alpha(t'-t)] + \frac{v_\perp}{\Omega_\alpha}\sin\phi \\ y'(t') = y + \frac{v_\perp}{\Omega_\alpha}\cos[\phi - \Omega_\alpha(t'-t)] - \frac{v_\perp}{\Omega_\alpha}\cos\phi \\ z'(t') = v_z(t'-t) + z \end{cases} \qquad (118)$$

where $\Omega_\alpha = \frac{q_\alpha B_0}{m_\alpha c}$. (118) reduces to (1.25) if wes set ϕ (polar angle of $\mathbf{v}_\perp$) $= \pi/2$ and $t = 0$. At $t' = t$, (118) gives $\mathbf{x}' = \mathbf{x}$ and $\mathbf{v}' = \mathbf{v}$ as required

Change the variable t in (117) to t' and note that $\mathbf{x}'(t'=t) = \mathbf{x}$ and $\mathbf{v}'(t'=t) = \mathbf{v}$. A t'-integration of (117) from $-\infty$ to t gives

$$\int_{-\infty}^{t} \frac{d}{dt'} f_{\alpha 1}[\mathbf{x}'(t'),\mathbf{v}'(t'),t']dt' = f_{\alpha 1}(\mathbf{x},\mathbf{v},t) - f_{\alpha 1}[\mathbf{x}'(t'),\mathbf{v}'(t'),t']_{t'=-\infty}$$
$$= -\frac{q_\alpha}{m_\alpha}\int_{-\infty}^{t} dt'\{\mathbf{E}_1[\mathbf{x}'(t'),t'] + \tfrac{1}{c}\mathbf{v}'(t')\times\mathbf{B}_1[\mathbf{x}'(t'),t']\}\cdot\nabla_{\mathbf{v}'} f_{\alpha 0}(\mathbf{v}') \qquad (119)$$

We now consider a normal mode by assuming

$$\begin{Bmatrix} \mathbf{E}_1(\mathbf{x},t) \\ \mathbf{B}_1(\mathbf{x},t) \\ \mathbf{J}_1(\mathbf{x},t) \\ f_{\alpha 1}(\mathbf{x},\mathbf{v},t) \end{Bmatrix} = \begin{Bmatrix} \mathbf{E}_{1k} \\ \mathbf{B}_{1k} \\ \mathbf{J}_{1k} \\ f_{\alpha 1k}(\mathbf{v}) \end{Bmatrix} e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}} \qquad (120)$$

As before, subscript "k" denotes a normal mode.

where $\mathbf{E}_{1k}, \mathbf{B}_{1k}, \mathbf{J}_{1k}$ are complex constants, and $f_{\alpha 1k}(\mathbf{v})$ is a complex function of $\mathbf{v}$. $\mathbf{J}_{1k}$ can be expressed in terms $f_{\alpha 1k}(\mathbf{v})$ as

$$\mathbf{J}_{1k} = \sum_\alpha q_\alpha \int f_{\alpha 1k}(\mathbf{v})\mathbf{v}d^3v \qquad (121)$$

Question: Why is $\rho_1(\mathbf{x},t)$ not included in (120)? (see note below)

Sub. (120) into

$$f_{\alpha 1}(\mathbf{x},\mathbf{v},t) - f_{\alpha 1}[\mathbf{x}'(t'),\mathbf{v}'(t'),t']_{t'=-\infty}$$

$$= -\frac{q_\alpha}{m_\alpha}\int_{-\infty}^{t} dt'\{\mathbf{E}_1[\mathbf{x}'(t'),t'] + \tfrac{1}{c}\mathbf{v}'(t')\times\mathbf{B}_1[\mathbf{x}'(t'),t']\}\cdot\nabla_{\mathbf{v}'} f_{\alpha 0}(\mathbf{v}') \quad (119)$$

we obtain

$$f_{\alpha 1k}(\mathbf{v})e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}} - f_{\alpha 1k}[\mathbf{v}'(t')]e^{-i\omega t' + i\mathbf{k}\cdot\mathbf{x}'}\Big|_{t'=-\infty}$$

$$= -\frac{q_\alpha}{m_\alpha}\int_{-\infty}^{t} dt'[\mathbf{E}_{1k} + \tfrac{1}{c}\mathbf{v}'(t')\times\mathbf{B}_{1k}]e^{\overbrace{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}^{\boxed{= -i\omega t - i\omega(t'-t) + i\mathbf{k}\cdot\mathbf{x} + i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})}}}\cdot\nabla_{\mathbf{v}'} f_{\alpha 0}(\mathbf{v}')$$

$$= -\frac{q_\alpha}{m_\alpha}e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}\int_{-\infty}^{t} dt'[\mathbf{E}_{1k} + \tfrac{1}{c}\mathbf{v}'(t')\times\mathbf{B}_{1k}]\cdot\nabla_{\mathbf{v}'} f_{\alpha 0}(\mathbf{v}') \cdot e^{-i\omega(t'-t) + i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})} \quad (122)$$

Note: In (122), the opertaor operates on $f_{\alpha 0}(\mathbf{v}')$ only.

We assume $\omega_i (= \mathrm{Im}\,\omega) > 0$, then $f_{\alpha 1k}[\mathbf{v}'(t')]e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}\Big|_{t'=-\infty} = 0$ and (122) becomes

$$f_{\alpha 1k}(\mathbf{v})e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}$$

$$= -\frac{q_\alpha}{m_\alpha}e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}\int_{-\infty}^{t} dt'[\mathbf{E}_{1k} + \tfrac{1}{c}\mathbf{v}'(t')\times\mathbf{B}_{1k}]\cdot\nabla_{\mathbf{v}'} f_{\alpha 0}(\mathbf{v}') \cdot e^{-i\omega(t'-t) + i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})}$$

Factoring out the $e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}$ dependence, we obtain

$$f_{\alpha 1k}(\mathbf{v}) = -\frac{q_\alpha}{m_\alpha}\int_{-\infty}^{t} dt'[\mathbf{E}_{1k} + \tfrac{1}{c}\mathbf{v}'(t')\times\mathbf{B}_{1k}]\cdot\nabla_{\mathbf{v}'} f_{\alpha 0}(\mathbf{v}') \cdot e^{-i\omega(t'-t) + i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})} \quad (123)$$

Note:

(i) $\rho_1(\mathbf{x},t)$ is not in (120) because, with the assumed $e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}$ dependence, ρ_{1k} is implicit in $\mathbf{J}_{1k}$ though the continuity equation:

$$\nabla\cdot\mathbf{J} + \frac{\partial\rho}{\partial t} = 0 \;\Rightarrow\; i\mathbf{k}\cdot\mathbf{J}_{1k} - i\omega\rho_{1k} = 0 \;\Rightarrow\; \rho_{1k} = \tfrac{1}{\omega}\mathbf{k}\cdot\mathbf{J}_{1k}$$

(ii) Although (123) is derived under the assumption of $\omega_i > 0$, the dispersion relation to be obtained from (123) can be analytically continued to an arbitrary ω by the use of Landau contour for the v_z-integral [see (135)]. The argument follows the treatment of Landau damping.

(iii) $\mathbf{v}'$ as a vector is a function of t'. However, it is understood that $f_{\alpha 0}(\mathbf{v}')$ is a function of scalars $v_\perp$ and v_z, both being constants of the motion. Hence, in writing $f_{\alpha 0}(\mathbf{v}')$ in (119), (122), and (123), $\mathbf{v}'$ is not displayed as a function of t'.

(iv) The method we employed to obtain (123) is called "method of characteristics" or "integrating over unperturbed orbit".

Field equation: From the linearized Maxwell equations:

$$\begin{cases} \nabla \times \mathbf{E}_1 = -\frac{1}{c}\frac{\partial}{\partial t}\mathbf{B}_1 & (113) \\ \nabla \times \mathbf{B}_1 = \frac{1}{c}\frac{\partial}{\partial t}\mathbf{E}_1 + \frac{4\pi}{c}\mathbf{J}_1 & (114) \end{cases}$$

we obtain $$\nabla \times (\nabla \times \mathbf{E}_1) = -\frac{1}{c}\frac{\partial}{\partial t}\nabla \times \mathbf{B}_1 = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E}_1 - \frac{4\pi}{c^2}\frac{\partial}{\partial t}\mathbf{J}_1 \qquad (124)$$

For a normal mode: $$\begin{Bmatrix} \mathbf{E}_1(\mathbf{x},t) \\ \mathbf{B}_1(\mathbf{x},t) \\ \mathbf{J}_1(\mathbf{x},t) \\ f_{\alpha 1}(\mathbf{x},\mathbf{v},t) \end{Bmatrix} = \begin{Bmatrix} \mathbf{E}_{1k} \\ \mathbf{B}_{1k} \\ \mathbf{J}_{1k} \\ f_{\alpha 1k}(\mathbf{v}) \end{Bmatrix} e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}} \qquad (120)$$

(113) and (114) give $$\mathbf{k} \times (\mathbf{k} \times \mathbf{E}_{1k}) + \frac{\omega^2}{c^2}\mathbf{E}_{1k} = -\frac{4\pi i\omega}{c^2}\mathbf{J}_{1k} \qquad (125)$$

A note on notations: Subscripts "0" and "1" indicate, respectively, zero and first order quantities. Subscript "k" indicates a normal mode. Subscript "α" indicates particle species.

(123) and (125) together with the orbit equations (118) form the basis for our treatment of linear plasma waves in Secs. 6.11 and 6.12.

6-11 Linear Vlasov Waves in Unmagnetized Plasma

Rewrite (123): $f_{\alpha 1k}(\mathbf{v}) = -\frac{q_\alpha}{m_\alpha}\int_{-\infty}^{t} dt'[\mathbf{E}_{1k} + \frac{1}{c}\mathbf{v}'(t')\times\mathbf{B}_{1k}]\cdot\nabla_{\mathbf{v}'} f_{\alpha 0}(\mathbf{v}')$

$$\cdot e^{-i\omega(t'-t)+i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})} \quad (123)$$

Assume the absence of an external magnetic field ($\Omega_\alpha = 0$). Then,

(118) reduces to $\begin{cases} \mathbf{v}'(t') = \mathbf{v} = const \\ \mathbf{x}'(t') = \mathbf{x} + \mathbf{v}(t'-t') \end{cases}$, and (123) can be written

$$f_{\alpha 1k}(\mathbf{v}) = -\frac{q_\alpha}{m_\alpha}(\mathbf{E}_{1k} + \frac{1}{c}\mathbf{v}\times\mathbf{B}_{1k})\cdot\nabla_{\mathbf{v}} f_{\alpha 0}(\mathbf{v})\int_{-\infty}^{t} dt' e^{-i\omega(t'-t)+i\mathbf{k}\cdot\mathbf{v}(t'-t)} \quad (126)$$

Again, assuming $\omega_i > 0,$ we obtain from (126)

$$f_{\alpha 1k}(\mathbf{v}) = \frac{q_\alpha}{m_\alpha}\frac{(\mathbf{E}_{1k} + \frac{1}{c}\mathbf{v}\times\mathbf{B}_{1k})\cdot\nabla_{\mathbf{v}} f_{\alpha 0}(\mathbf{v})}{i(\omega - \mathbf{k}\cdot\mathbf{v})} \quad (127)$$

Note: (127) can be readily derived by setting $B_0 = 0$ and sub. the normal mode (120) into the linearized Vlasov equation (110):

$$\frac{\partial}{\partial t} f_{\alpha 1} + \mathbf{v}\cdot\nabla f_{\alpha 1} + \frac{q_\alpha}{m_\alpha c}(\mathbf{v}\times B_0\mathbf{e}_z)\cdot\nabla_{\mathbf{v}} f_{\alpha 1} = -\frac{q_\alpha}{m_\alpha}(\mathbf{E}_1 + \frac{1}{c}\mathbf{v}\times\mathbf{B}_1)\cdot\nabla_{\mathbf{v}} f_{\alpha 0}$$

but in the presence of $B_0\mathbf{e}_z$, we must use (123) (see next section).

Rewrite $f_{\alpha 1k}(\mathbf{v}) = \frac{q_\alpha}{m_\alpha}\frac{(\mathbf{E}_{1k} + \frac{1}{c}\mathbf{v}\times\mathbf{B}_{1k})\cdot\nabla_{\mathbf{v}} f_{\alpha 0}(\mathbf{v})}{i(\omega - \mathbf{k}\cdot\mathbf{v})}$ (127)

Assume $f_{\alpha 0}(\mathbf{v})$ is an isotropic function, i.e. $f_{\alpha 0}(\mathbf{v}) = f_{\alpha 0}(v)$. Then, $(\mathbf{v}\times\mathbf{B}_{1k})\cdot\nabla_{\mathbf{v}} f_{\alpha 0}(v) = 0$. Since there is no external magnetic field and $f_{\alpha 0}(v)$ is isotropic, the plasma properties are also isotropic. Without loss of generality, we assume $\mathbf{k} = k_z\mathbf{e}_z$. Thus, (127) becomes

$$f_{\alpha 1k}(\mathbf{v}) = \frac{q_\alpha}{m_\alpha}\frac{\mathbf{E}_{1k}\cdot\nabla_{\mathbf{v}} f_{\alpha 0}(v)}{i(\omega - k_z v_z)} \quad (128)$$

Sub. (128) into $\mathbf{J}_{1k} = \sum_\alpha q_\alpha \int f_{\alpha 1k}(\mathbf{v})\mathbf{v} d^3v$ [(121)], we obtain

$$\mathbf{J}_{1k} = \sum_\alpha \frac{q_\alpha^2}{m_\alpha}\int \mathbf{v}\frac{\mathbf{E}_{1k}\cdot\nabla_{\mathbf{v}} f_{\alpha 0}(v)}{i(\omega - k_z v_z)} d^3v \quad (129)$$

The field equation: $\mathbf{k}\times(\mathbf{k}\times\mathbf{E}_{1k}) + \frac{\omega^2}{c^2}\mathbf{E}_{1k} = -\frac{4\pi i\omega}{c^2}\mathbf{J}_{1k}$ [(125)]

may be written $\quad k_z^2 E_{1kz}\mathbf{e}_z - (k_z^2 - \frac{\omega^2}{c^2})\mathbf{E}_{1k} = -\frac{4\pi i\omega}{c^2}\mathbf{J}_{1k} \quad (130)$

Electrostatic waves:

Rewrite
$$\begin{cases} \mathbf{J}_{1k} = \sum\limits_\alpha \frac{q_\alpha^2}{m_\alpha} \int \mathbf{v} \frac{\mathbf{E}_{1k} \cdot \nabla_{\mathbf{v}} f_{\alpha 0}(v)}{i(\omega - k_z v_z)} d^3v & (129) \\ -k_z^2 E_{1kz}\mathbf{e}_z + (k_z^2 - \frac{\omega^2}{c^2})\mathbf{E}_{1k} - \frac{4\pi i\omega}{c^2}\mathbf{J}_{1k} = 0 & (130) \end{cases}$$

For electrostatic waves, $\mathbf{E}_{1k} \parallel \mathbf{k} (= k_z \mathbf{e}_z)$. So we set

$$\mathbf{E}_{1k} = E_{1kz}\mathbf{e}_z \qquad (131)$$

(129) then gives

$$\mathbf{J}_{1k} = -\sum_\alpha i \frac{q_\alpha^2}{m_\alpha} E_{1kz} \int (v_x\mathbf{e}_x + v_y\mathbf{e}_y + v_z\mathbf{e}_z) \frac{\frac{\partial f_{\alpha 0}(v)}{\partial v_z}}{\omega - k_z v_z} d^3v \qquad (132)$$

In (132), $f_{\alpha 0}(v) = f_{\alpha 0}[(v_x^2 + v_y^2 + v_z^2)^{1/2}]$ is an even function of v_x and v_y. Hence, the x and y components vanish upon, respectively, v_x and v_y integrations, and we have

$$\mathbf{J}_{1k} = -\sum_\alpha i \frac{q_\alpha^2}{m_\alpha} E_{1kz}\mathbf{e}_z \int \frac{v_z \frac{\partial f_{\alpha 0}(v)}{\partial v_z}}{\omega - k_z v_z} d^3v \qquad (133)$$

Rewrite $$\mathbf{J}_{1k} = -\sum_\alpha i \frac{q_\alpha^2}{m_\alpha} E_{1kz}\mathbf{e}_z \int \frac{v_z \frac{\partial f_{\alpha 0}(v)}{\partial v_z}}{\omega - k_z v_z} d^3v \qquad (133)$$

Defining $g_{\alpha 0}(v_z) = \frac{1}{n_{\alpha 0}} \int f_{\alpha 0}(v) dv_x dv_y$ [as in (28)], we obtain

$$\mathbf{J}_{1k} = -\sum_\alpha i \frac{n_{\alpha 0} q_\alpha^2}{m_\alpha} E_{1kz}\mathbf{e}_z \int_{-\infty}^{\infty} \frac{v_z \frac{dg_{\alpha 0}(v_z)}{dv_z}}{\omega - k_z v_z} dv_z$$

Writing $\frac{v_z}{\omega - k_z v_z} = \frac{1}{k_z}(-1 + \frac{\omega}{\omega - k_z v_z})$, we have

$$\mathbf{J}_{1k} = \sum_\alpha i \frac{n_{\alpha 0} q_\alpha^2}{m_\alpha} E_{1kz}\mathbf{e}_z \frac{1}{k_z} \int_{-\infty}^{\infty} (1 - \frac{\omega}{\omega - k_z v_z}) \frac{dg_{\alpha 0}(v_z)}{dv_z} dv_z,$$

The first term vanishes upon v_z-integration because $g_{\alpha 0}(v_z) = 0$ at $v_z = \pm\infty$. Thus,

$$\mathbf{J}_{1k} = \sum_\alpha i \frac{n_{\alpha 0} q_\alpha^2}{m_\alpha} \frac{\omega}{k_z^2} E_{1kz}\mathbf{e}_z \int_{-\infty}^{\infty} \frac{\frac{dg_{\alpha 0}(v_z)}{dv_z}}{v_z - \frac{\omega}{k_z}} dv_z, \qquad (134)$$

Sub. $\mathbf{E}_{1k} = E_{1kz}\mathbf{e}_z$ and $\mathbf{J}_{1k} = \sum_\alpha i \frac{n_{\alpha 0} q_\alpha^2}{m_\alpha} \frac{\omega}{k_z^2} E_{1kz}\mathbf{e}_z \int_{-\infty}^{\infty} \frac{\frac{dg_{\alpha 0}(v_z)}{dv_z}}{v_z - \frac{\omega}{k_z}} dv_z$

into the field equation: $k_z^2 E_{1kz}\mathbf{e}_z - (k_z^2 - \frac{\omega^2}{c^2})\mathbf{E}_{1k} = -\frac{4\pi i\omega}{c^2}\mathbf{J}_{1k}$ (130)

we obtain the dispersion relation for electrostatic waves:

$$1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{k_z^2} \int_L \frac{\frac{dg_{\alpha 0}(v_z)}{dv_z}}{v_z - \frac{\omega}{k_z}} dv_z = 0, \tag{135}$$

where, by the recipe in (62), we have replaced $\int_{-\infty}^{\infty} dv_z$ with $\int_L dv_z$.

In deriving (123) and (127), we have assumed $\omega_i > 0$. With the Landau contour, ω in (135) can have any value provided the pole ω/k_z does not cross the Landau contour.

(135) agrees with the electrostatic disperson relations in Sec. 6.5 for the Langmuir wave, Landau damping/growth, two-stream instabilities, and ion acoustic waves.

Discussion: Rewrite $\nabla \times \mathbf{B}_1 = \frac{1}{c}\frac{\partial}{\partial t}\mathbf{E}_1 + \frac{4\pi}{c}\mathbf{J}_1$ (114)

For the normal mode in (120), the RHS of (114) gives

$$\frac{1}{c}\frac{\partial}{\partial t}\mathbf{E}_1 + \frac{4\pi}{c}\mathbf{J}_1 = (\frac{-i\omega}{c}\mathbf{E}_{1k} + \frac{4\pi}{c}\mathbf{J}_{1k})e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}} \tag{136}$$

Inserting $\mathbf{J}_{1k} = \sum_\alpha i \frac{n_{\alpha 0} q_\alpha^2}{m_\alpha} \frac{\omega}{k_z^2} E_{1kz} \int_{-\infty}^{\infty} \frac{\frac{dg_{\alpha 0}(v_z)}{dv_z}}{v_z - \frac{\omega}{k_z}} dv_z \mathbf{e}_z$ [(134)] and $\mathbf{E}_{1k} = E_{1kz}\mathbf{e}_z$ [(131)] into (136), we find

$$\frac{-i\omega}{c}\mathbf{E}_{1k} + \frac{4\pi}{c}\mathbf{J}_{1k} = \frac{-i\omega}{c}E_{1kz}\mathbf{e}_z + \frac{4\pi}{c}\sum_\alpha i \frac{n_{\alpha 0} q_\alpha^2}{m_\alpha} \frac{\omega}{k_z^2} E_{1kz}\mathbf{e}_z \int_{-\infty}^{\infty} \frac{\frac{dg_{\alpha 0}(v_z)}{dv_z}}{v_z - \frac{\omega}{k_z}} dv_z$$

$$= \frac{-i\omega}{c}E_{1kz}\mathbf{e}_z \underbrace{\{1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{k_z^2}\int_L \frac{\frac{dg_{\alpha 0}(v_z)}{dv_z}}{v_z - \frac{\omega}{k_z}} dv_z\}}_{=0 \text{ by } (135)} = 0$$

This shows that, the displacement current and particle current exactly cancel out. Hence, we have an electrostatic wave.

Electromagnetic waves:

Rewrite
$$\begin{cases} \mathbf{J}_{1k} = \sum_\alpha \frac{q_\alpha^2}{m_\alpha} \int \mathbf{v} \frac{\mathbf{E}_{1k} \cdot \nabla_\mathbf{v} f_{\alpha 0}(v)}{i(\omega - k_z v_z)} d^3 v & (129) \\ -k_z^2 E_{1kz} \mathbf{e}_z + (k_z^2 - \frac{\omega^2}{c^2}) \mathbf{E}_{1k} - \frac{4\pi i \omega}{c^2} \mathbf{J}_{1k} = 0 & (130) \end{cases}$$

For electromagnetic waves, $\mathbf{E}_{1k} \perp \mathbf{k}(= k_z \mathbf{e}_z)$. So, without loss of generality (because the plasma is isotropic), we set $\mathbf{E}_{1k} = E_{1ky} \mathbf{e}_y$ (137)

Then, (129) and (130) give

$$\begin{cases} \mathbf{J}_{1k} = \sum_\alpha -i \frac{q_\alpha^2}{m_\alpha} E_{1ky} \int (v_x \mathbf{e}_x + v_y \mathbf{e}_y + v_z \mathbf{e}_z) \frac{\frac{\partial f_{\alpha 0}(v)}{\partial v_y}}{\omega - k_z v_z} d^3 v & (138) \\ (k_z^2 - \frac{\omega^2}{c^2}) E_{1ky} \mathbf{e}_y - \frac{4\pi i \omega}{c^2} \mathbf{J}_{1k} = 0 & (139) \end{cases}$$

$f_{\alpha 0}(v)$ is an even function of v_x. Hence, the x-component of (138) vanishes upon v_x-integration. $\frac{\partial}{\partial v_y} f_{\alpha 0}(v)$ is an odd function of v_y. Hence, the z-component of (138) vanishes upon v_y-integration.

We are then left with only the y component of (138):

$$\mathbf{J}_{1k} = \sum_\alpha -i \frac{q_\alpha^2}{m_\alpha} E_{1ky} \mathbf{e}_y \int \frac{v_y \frac{\partial f_{\alpha 0}(v)}{\partial v_y}}{\omega - k_z v_z} d^3 v \qquad (140)$$

Integrating (140) by parts of over v_y yields

$$\mathbf{J}_{1k} = \sum_\alpha i \frac{q_\alpha^2}{m_\alpha} E_{1ky} \mathbf{e}_y \int \frac{f_{\alpha 0}(v)}{\omega - k_z v_z} d^3 v$$

Using the one-dimensional equilibrium distribution function:

$$g_{\alpha 0}(v_z) = \frac{1}{n_{\alpha 0}} \int f_{\alpha 0}(v) dv_x dv_y,$$

we may write (140) as $\mathbf{J}_{1k} = \sum_\alpha i \frac{n_{\alpha 0} q_\alpha^2}{m_\alpha} E_{1ky} \mathbf{e}_y \int \frac{g_{\alpha 0}(v_z)}{\omega - k_z v_z} dv_z$ (141)

Sub. (141) and $\mathbf{E}_{1k} = E_{1ky} \mathbf{e}_y$ into

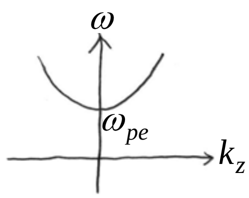
$$-k_z^2 E_{1kz} \mathbf{e}_z + (k_z^2 - \frac{\omega^2}{c^2}) \mathbf{E}_{1k} - \frac{4\pi i \omega}{c^2} \mathbf{J}_{1k} = 0 \qquad (130)$$

we obtain $k_z^2 - \frac{\omega^2}{c^2} + \omega \sum_\alpha \omega_{p\alpha}^2 \int \frac{g_{\alpha 0}(v_z)}{\omega - k_z v_z} dv_z = 0$ (142)

Rewrite $k_z^2 - \frac{\omega^2}{c^2} + \omega \sum_\alpha \omega_{p\alpha}^2 \int \frac{g_{\alpha 0}(v_z)}{\omega - k_z v_z} dv_z = 0$ (142)

EM waves in a plasma have a phase velocity $> c$ [see (143) below]. Hence, we may assume $\omega / k_z \gg v_z$ and neglect the the $k_z v_z$ term in the demonamitor of the integral in (142).

$$\Rightarrow \int \frac{g_{\alpha 0}(v_z)}{\omega - k_z v_z} dv_z \approx \frac{1}{\omega} \int g_{\alpha 0}(v_z) dv_z = \frac{1}{\omega}$$



This results in the dispersion relation:

$$\omega^2 = k_z^2 c^2 + \omega_{pe}^2 \quad (143)$$

where we have neglected the small ion contribution.

The ω vs k_z plot (see figure) is similar to that of the waveguide. There is a cutoff frequency ω_{pe}, below which EM waves can not propagate. Short radio waves (~ 10 MHz) are hence reflected from the ionosphere. This has been exploited for long-range broadcasting.

By comparison, the free space is non-dispersive with $\omega = k_z c$.

6-12 Linear Vlasov Waves in Magnetized Plasma

(Ref.: Krall and Trivelpiece, Sec. 8.10)

Dispersion Relation : We begin this section with a derivation of the general dispersion relation for waves in an infinite, uniform, and *magnetized* plasma on the basis of the following linearized equations derived in Sec. 6.10 for a normal mode with $e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}$ dependence:

$$\begin{cases} f_{\alpha 1k}(\mathbf{v}) = -\frac{q_\alpha}{m_\alpha} \int_{-\infty}^{t} dt' [\mathbf{E}_{1k} + \frac{1}{c} \mathbf{v}'(t') \times \mathbf{B}_{1k}] \cdot \nabla_{\mathbf{v}'} f_{\alpha 0}(\mathbf{v}') \\ \qquad \cdot e^{-i\omega(t'-t) + i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})} & (123) \\ \mathbf{k} \times (\mathbf{k} \times \mathbf{E}_{1k}) + \frac{\omega^2}{c^2} \mathbf{E}_{1k} = -\frac{4\pi i \omega}{c^2} \mathbf{J}_{1k} & (125) \end{cases}$$

where we have assumed a uniform external magnetic field $\mathbf{B}_0 = B_0 \mathbf{e}_z$, and shown that the equilibrium distribution function in such a field is $f_{\alpha 0}(\mathbf{v}) = f_{\alpha 0}(v_\perp, v_z)$. So the plasma is isotropic in the x, y-dimensions, but it is 3-dimensional anisotropic. Thus, we expect the conductivity to be in the form of a tensor $\overleftrightarrow{\sigma}$: $\mathbf{J}_{1k} = \overleftrightarrow{\sigma} \cdot \mathbf{E}_{1k}$.

General form of the dispersion ralation:

Write $$\mathbf{J}_{1k}=\overleftrightarrow{\boldsymbol{\sigma}}\cdot\mathbf{E}_{1k}=\begin{bmatrix}\sigma_{xx} & \sigma_{xy} & \sigma_{xz}\\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz}\\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz}\end{bmatrix}\begin{bmatrix}E_{1kx}\\ E_{1ky}\\ E_{1kz}\end{bmatrix}\qquad(144)$$

Without loss of generality (for a plasma isotropic in $x,\ y$), we let

$$\mathbf{k}=k_\perp\mathbf{e}_x+k_z\mathbf{e}_z\qquad(145)$$

Sub. (144) and (145) into the field equation:

$$\mathbf{k}\times(\mathbf{k}\times\mathbf{E}_{1k})+\frac{\omega^2}{c^2}\mathbf{E}_{1k}=-\frac{4\pi i\omega}{c^2}\mathbf{J}_{1k}\qquad(125)$$

the x,y,z components are

$$\begin{cases}(1-\frac{k_z^2c^2}{\omega^2}+\frac{4\pi i}{\omega}\sigma_{xx})E_{1kx}+\frac{4\pi i}{\omega}\sigma_{xy}E_{1ky}+(\frac{k_\perp k_z c^2}{\omega^2}+\frac{4\pi i}{\omega}\sigma_{xz})E_{1kz}=0\\ \frac{4\pi i}{\omega}\sigma_{yx}E_{1kx}+(1-\frac{k^2c^2}{\omega^2}+\frac{4\pi i}{\omega}\sigma_{yy})E_{1ky}+\frac{4\pi i}{\omega}\sigma_{yz}E_{1kz}=0\\ (\frac{k_\perp k_z c^2}{\omega^2}+\frac{4\pi i}{\omega}\sigma_{zx})E_{1kx}+\frac{4\pi i}{\omega}\sigma_{zy}E_{1ky}+(1-\frac{k_\perp^2c^2}{\omega^2}+\frac{4\pi i}{\omega}\sigma_{zz})E_{1kz}=0\end{cases}\qquad(146)$$

(146) can be written

$$\overleftrightarrow{\mathbf{D}}\cdot\mathbf{E}_{1k}=\begin{bmatrix}D_{xx} & D_{xy} & D_{xz}\\ D_{yx} & D_{yy} & D_{yz}\\ D_{zx} & D_{zy} & D_{zz}\end{bmatrix}\begin{bmatrix}E_{1kx}\\ E_{1ky}\\ E_{1kz}\end{bmatrix}=0\quad\text{or}\qquad(147)$$

$$\begin{bmatrix}1-\frac{k_z^2c^2}{\omega^2}+\frac{4\pi i}{\omega}\sigma_{xx} & \frac{4\pi i}{\omega}\sigma_{xy} & \frac{k_\perp k_z c^2}{\omega^2}+\frac{4\pi i}{\omega}\sigma_{xz}\\ \frac{4\pi i}{\omega}\sigma_{yx} & 1-\frac{k^2c^2}{\omega^2}+\frac{4\pi i}{\omega}\sigma_{yy} & \frac{4\pi i}{\omega}\sigma_{yz}\\ \frac{k_\perp k_z c^2}{\omega^2}+\frac{4\pi i}{\omega}\sigma_{zx} & \frac{4\pi i}{\omega}\sigma_{zy} & 1-\frac{k_\perp^2c^2}{\omega^2}+\frac{4\pi i}{\omega}\sigma_{zz}\end{bmatrix}\begin{bmatrix}E_{1kx}\\ E_{1ky}\\ E_{1kz}\end{bmatrix}=0\quad(148)$$

For (147) or (148) to be solvable, the determinent of $\overleftrightarrow{\mathbf{D}}$ must vanish:

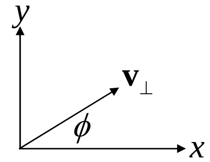
$$\left|\overleftrightarrow{\boldsymbol{D}}\right|=\begin{vmatrix}D_{xx} & D_{xy} & D_{xz}\\ D_{yx} & D_{yy} & D_{yz}\\ D_{zx} & D_{zy} & D_{zz}\end{vmatrix}=0\qquad(149)$$

(149) is the most comprehensive form of the dispersion relation.

Particle dynamics: In (149), the conductivity tensor $\vec{\vec{\sigma}}$ is still unknown. To obtain the specific expression of the dispersion relation, we need to work on the equations for particle dynamics.

Define $\tau = t - t'$ and rewrite (123) and (118) in terms of τ

$$f_{\alpha 1k}(\mathbf{v}) = -\frac{q_\alpha}{m_\alpha}\int_{-\infty}^{0} d\tau[\mathbf{E}_{1k} + \frac{1}{c}\mathbf{v}'(\tau)\times\mathbf{B}_{1k}]\cdot\nabla_{\mathbf{v}'} f_{\alpha 0}(\mathbf{v}') \cdot e^{-i\omega\tau + i\mathbf{k}\cdot[\mathbf{x}'(\tau)-\mathbf{x}]} \tag{150}$$



$$\begin{cases} v_x'(\tau) = v_\perp \cos(\phi - \Omega_\alpha \tau) \\ v_y'(\tau) = v_\perp \sin(\phi - \Omega_\alpha \tau) \\ v_z'(\tau) = v_z \\ x'(\tau) = x - \dfrac{v_\perp}{\Omega_\alpha}\sin(\phi - \Omega_\alpha\tau) + \dfrac{v_\perp}{\Omega_\alpha}\sin\phi \\ y'(\tau) = y + \dfrac{v_\perp}{\Omega_\alpha}\cos(\phi - \Omega_\alpha\tau) - \dfrac{v_\perp}{\Omega_\alpha}\cos\phi \\ z'(\tau) = v_z\tau + z \end{cases} \tag{151}$$

Using $\mathbf{k} = k_\perp \mathbf{e}_x + k_z \mathbf{e}_z$ [(145)] and the orbit equations in (151), we may write $\mathbf{k}\cdot[\mathbf{x}'(\tau) - \mathbf{x}] = k_\perp[x'(\tau) - x] + k_z[z'(\tau) - z]$

$$= -\frac{k_\perp v_\perp}{\Omega_\alpha}[\sin(\phi - \Omega_\alpha\tau) - \sin\phi] + k_z v_z \tau$$

$$\Rightarrow \quad e^{-i\omega\tau + i\mathbf{k}\cdot[\mathbf{x}'(\tau)-\mathbf{x}]} = e^{-i(\omega - k_z v_z)\tau + i\frac{k_\perp v_\perp}{\Omega_\alpha}[\sin(\phi-\Omega_\alpha\tau)-\sin\phi]}$$

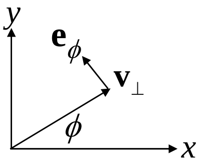
Using the Bessel function identity: $e^{\pm ix\sin\theta} = \sum_{s=-\infty}^{\infty} J_s(x) e^{\pm is\theta}$,

we obtain $\begin{cases} e^{-i\frac{k_\perp v_\perp}{\Omega_\alpha}\sin(\phi-\Omega_\alpha\tau)} = \sum_s J_s(\frac{k_\perp v_\perp}{\Omega_\alpha}) e^{-is(\phi-\Omega_\alpha\tau)} \\ e^{i\frac{k_\perp v_\perp}{\Omega_\alpha}\sin\phi} = \sum_{s'} J_{s'}(\frac{k_\perp v_\perp}{\Omega_\alpha}) e^{is'\phi} \end{cases}$

$$\Rightarrow \quad e^{-i\omega\tau + i\mathbf{k}\cdot[\mathbf{x}'(\tau)-\mathbf{x}]} = e^{-i(\omega - k_z v_z)\tau + i\frac{k_\perp v_\perp}{\Omega_\alpha}[\sin(\phi-\Omega_\alpha\tau)-\sin\phi]}$$

$$= \sum_s \sum_{s'} J_s(\frac{k_\perp v_\perp}{\Omega_\alpha}) J_{s'}(\frac{k_\perp v_\perp}{\Omega_\alpha}) e^{-i(\omega - k_z v_z - s\Omega_\alpha)\tau + i(s'-s)\phi} \tag{152}$$

Since $f_{\alpha 0}(\mathbf{v}') = f_{\alpha 0}(v'_\perp, v'_z)$ and $v'_\perp (= v_\perp)$, $v'_z (= v_z)$ are constants of the motion, we have $\nabla_{\mathbf{v}'} f_{\alpha 0}(\mathbf{v}') = \nabla_{\mathbf{v}} f_{\alpha 0}(v_\perp, v_z)$



$$= \frac{\partial f_{\alpha 0}}{\partial v_\perp} \overbrace{\mathbf{e}_\perp}^{\mathbf{v}_\perp / v_\perp} + \frac{1}{v_\perp} \cancel{\frac{\partial f_{\alpha 0}}{\partial \phi}} \mathbf{e}_\phi + \frac{\partial f_{\alpha 0}}{\partial v_z} \mathbf{e}_z$$

$$= 2 \frac{\partial f_{\alpha 0}}{\partial v_\perp^2} \mathbf{v}_\perp + 2 v_z \frac{\partial f_{\alpha 0}}{\partial v_z^2} \mathbf{e}_z \qquad (153)$$

Thus,

$$\mathbf{E}_{1k} \cdot \nabla_{\mathbf{v}'} f_{\alpha 0}(\mathbf{v}') = 2(E_{1kx}\mathbf{e}_x + E_{1ky}\mathbf{e}_y + E_{1kz}\mathbf{e}_z) \cdot \left(\frac{\partial f_{\alpha 0}}{\partial v_\perp^2} \mathbf{v}_\perp + v_z \frac{\partial f_{\alpha 0}}{\partial v_z^2} \mathbf{e}_z\right)$$

$$= 2(E_{1kx} v_x + E_{1ky} v_y) \frac{\partial f_{\alpha 0}}{\partial v_\perp^2} + 2 E_{1kz} v_z \frac{\partial f_{\alpha 0}}{\partial v_z^2} \qquad (154)$$

From (113) and (120), we obtain $\mathbf{B}_{1k} = \frac{c}{\omega} \mathbf{k} \times \mathbf{E}_{1k}$. Then,

$$\mathbf{v} \times \mathbf{B}_{1k} = \frac{c}{\omega} \mathbf{v} \times (\mathbf{k} \times \mathbf{E}_{1k}) = \frac{c}{\omega}[(\mathbf{v} \cdot \mathbf{E}_{1k})\mathbf{k} - (\mathbf{k} \cdot \mathbf{v})\mathbf{E}_{1k}] \qquad (155)$$

(153) and (155) give

$$\frac{1}{c}(\mathbf{v} \times \mathbf{B}_{1k}) \cdot \nabla_{\mathbf{v}'} f_{\alpha 0} = \frac{2}{\omega}[(\mathbf{v} \cdot \mathbf{E}_{1k})\mathbf{k} - (\mathbf{k} \cdot \mathbf{v})\mathbf{E}_{1k}] \cdot \left(\frac{\partial f_{\alpha 0}}{\partial v_\perp^2} \mathbf{v}_\perp + v_z \frac{\partial f_{\alpha 0}}{\partial v_z^2} \mathbf{e}_z\right)$$

$$= \frac{2}{\omega}\Big\{[(\mathbf{v} \cdot \mathbf{E}_{1k})(\mathbf{k} \cdot \mathbf{v}_\perp) - (\mathbf{k} \cdot \mathbf{v})(\mathbf{v}_\perp \cdot \mathbf{E}_{1k})] \frac{\partial f_{\alpha 0}}{\partial v_\perp^2} + [(\mathbf{v} \cdot \mathbf{E}_{1k}) k_z v_z - (\mathbf{k} \cdot \mathbf{v}) v_z E_{1k}] \frac{\partial f_{\alpha 0}}{\partial v_z^2}\Big\}$$

$$= \frac{2}{\omega}\Big\{[(v_x E_{1kx} + v_y E_{1ky} + v_z E_{1kz}) k_\perp v_x - (k_\perp v_x + k_z v_z)(v_x E_{1kx} + v_y E_{1ky})] \frac{\partial f_{\alpha 0}}{\partial v_\perp^2}$$
$$+ [(v_x E_{1kx} + v_y E_{1ky} + v_z E_{1kz}) k_z v_z - (k_\perp v_x + k_z v_z) v_z E_{1kz}] \frac{\partial f_{\alpha 0}}{\partial v_z^2}\Big\}$$

$$= \frac{2}{\omega}\Big[(-k_z v_x E_{1kx} - k_z v_y E_{1ky} + k_\perp v_x E_{1kz}) v_y \frac{\partial f_{\alpha 0}}{\partial v_\perp^2} + (k_z v_x E_{1kx} + k_z v_y E_{1ky} - k_\perp v_x E_{1kz}) v_z \frac{\partial f_{\alpha 0}}{\partial v_z^2}\Big] \qquad (156)$$

Combining (154) and (156), we obtain

$$(\mathbf{E}_{1k} + \frac{1}{c}\mathbf{v}'\times\mathbf{B}_{1k})\cdot\nabla_{\mathbf{v}'} f_{\alpha 0} = 2v'_x X + 2v'_y Y + 2\overbrace{v'_z}^{v_z} Z$$

$$= v_\perp[e^{i(\phi-\Omega_\alpha\tau)} + e^{-i(\phi-\Omega_\alpha\tau)}]X$$

$$- iv_\perp[e^{i(\phi-\Omega_\alpha\tau)} - e^{-i(\phi-\Omega_\alpha\tau)}]Y + 2v_z Z \qquad (157)$$

$$\text{where } \begin{cases} X = E_{1kx}\dfrac{\partial f_{\alpha 0}}{\partial v_\perp^2} + \dfrac{v_z}{\omega}(k_z E_{1kx} - k_\perp E_{1kz})(\dfrac{\partial f_{\alpha 0}}{\partial v_z^2} - \dfrac{\partial f_{\alpha 0}}{\partial v_\perp^2}) \\ Y = E_{1ky}\dfrac{\partial f_{\alpha 0}}{\partial v_\perp^2} + \dfrac{v_z}{\omega}k_z E_{1ky}(\dfrac{\partial f_{\alpha 0}}{\partial v_z^2} - \dfrac{\partial f_{\alpha 0}}{\partial v_\perp^2}) \\ Z = E_{1kz}\dfrac{\partial f_{\alpha 0}}{\partial v_z^2} \end{cases} \qquad (158)$$

Note: (i) $v_\perp$, v_z, and v are constants of the motion, but $\mathbf{v}_\perp, v_x$ and v_y are time dependent.

(ii) X, Y, and Z are functions of constants of the motion.

Combining (152) and (157) gives

$$(\mathbf{E}_{1k} + \frac{1}{c}\mathbf{v}'\times\mathbf{B}_{1k})\cdot\nabla_{\mathbf{v}'} f_{\alpha 0} e^{-i\omega\tau + i\mathbf{k}\cdot[\mathbf{x}'(\tau)-\mathbf{x}]}$$

$$= v_\perp X[e^{i(\phi-\Omega_\alpha\tau)} + e^{-i(\phi-\Omega_\alpha\tau)}]\sum_s\sum_{s'} J_s J_{s'} e^{-i(\omega-k_z v_z - s\Omega_\alpha)\tau + i(s'-s)\phi}$$

$$- iv_\perp Y[e^{i(\phi-\Omega_\alpha\tau)} - e^{-i(\phi-\Omega_\alpha\tau)}]\sum_s\sum_{s'} J_s J_{s'} e^{-i(\omega-k_z v_z - s\Omega_\alpha)\tau + i(s'-s)\phi}$$

$$+ 2v_z Z\sum_s\sum_{s'} J_s J_{s'} e^{-i(\omega-k_z v_z - s\Omega_\alpha)\tau + i(s'-s)\phi} \qquad (159)$$

Write

$$\sum_s\sum_{s'} J_s J_{s'} \begin{Bmatrix} e^{i(\phi-\Omega_\alpha\tau)} \\ e^{-i(\phi-\Omega_\alpha\tau)} \end{Bmatrix} e^{-i(\omega-k_z v_z - s\Omega_\alpha)\tau + i(s'-s)\phi}$$

$$= \sum_s\sum_{s'} J_s J_{s'} \begin{Bmatrix} e^{-i[\omega-k_z v_z-(s-1)\Omega_\alpha]\tau + i(s'-s+1)\phi} \\ e^{-i[\omega-k_z v_z-(s+1)\Omega_\alpha]\tau + i(s'-s-1)\phi} \end{Bmatrix}$$

$$\boxed{\begin{Bmatrix} n = s-1 \\ n = s+1 \end{Bmatrix}} \longrightarrow$$

$$= \sum_n\sum_{s'} \begin{Bmatrix} J_{n+1}J_{s'} \\ J_{n-1}J_{s'} \end{Bmatrix} e^{-i(\omega-k_z v_z - n\Omega_\alpha)\tau + i(s'-n)\phi} \qquad (160)$$

Sub. (160) into (159), we obtain

$$(\mathbf{E}_{1k}+\tfrac{1}{c}\mathbf{v}'\times\mathbf{B}_{1k})\cdot\nabla_{\mathbf{v}'} f_{\alpha 0}\, e^{-i\omega\tau+i\mathbf{k}\cdot[\mathbf{x}'(\tau)-\mathbf{x}]}$$

$$=\sum_n\sum_{s'}[v_\perp X(J_{n+1}+J_{n-1})-iv_\perp Y(J_{n+1}-J_{n-1})+2v_z Z J_n]J_{s'}$$

$$\cdot\underbrace{e^{-i(\omega-k_zv_z-n\Omega_\alpha)\tau}}_{\boxed{\text{the only factor that depends on } \tau}} e^{i(s'-n)\phi} \qquad (161)$$

Sub. (161) into $f_{\alpha 1k}(\mathbf{v})=-\frac{q_\alpha}{m_\alpha}\int_{-\infty}^{0} d\tau[\mathbf{E}_{1k}+\frac{1}{c}\mathbf{v}'(\tau)\times\mathbf{B}_{1k}]$

$$\cdot\nabla_{\mathbf{v}'} f_{\alpha 0}(\mathbf{v}')e^{-i\omega\tau+i\mathbf{k}\cdot[\mathbf{x}'(\tau)-\mathbf{x}]} \qquad (150)$$

and carrying out the τ-integration, we obtain

$$f_{\alpha 1k}(\mathbf{v})=\frac{q_\alpha}{m_\alpha}\sum_n\sum_{s'}\frac{v_\perp X(J_{n+1}+J_{n-1})-iv_\perp Y(J_{n+1}-J_{n-1})+2v_zZJ_n}{i(\omega-k_zv_z-n\Omega_\alpha)}J_{s'}e^{i(s'-n)\phi} \qquad (162)$$

This is Eq. (8.10.8) in Krall & Trivelpiece. Note that all Bessel functions have the same argument: $\frac{k_\perp v_\perp}{\Omega_\alpha}$.

The conductivity tensor: The perturbed current [(121)] can be written:

$$\mathbf{J}_{1k}=\sum_\alpha q_\alpha\int f_{\alpha 1k}(\mathbf{v})\mathbf{v}d^3v=\sum_\alpha q_\alpha\int_0^\infty v_\perp dv_\perp\int_0^{2\pi}d\phi\int_{-\infty}^{\infty}dv_z f_{\alpha 1k}(\mathbf{v})\mathbf{v}$$

First consider the x-component of $\mathbf{J}_{1k}$: $\boxed{v_\perp\cos\phi\mathbf{e}_x+v_\perp\sin\phi\mathbf{e}_y+v_z\mathbf{e}_z}$

$$J_{1kx}=\sum_\alpha q_\alpha\int_0^\infty v_\perp dv_\perp\int_0^{2\pi}d\phi\int_{-\infty}^{\infty}dv_z f_{\alpha 1k}(\mathbf{v})\tfrac{1}{2}v_\perp(e^{i\phi}+e^{-i\phi}) \qquad (163)$$

By (144), J_{1kx} can be expressed in terms of the conductivity tensor as

$$J_{1kx}=\sigma_{xx}E_{1kx}+\sigma_{xy}E_{1ky}+\sigma_{xz}E_{1kz} \qquad (164)$$

Then, σ_{xx} is the coefficient of the sum of all E_{1kx} terms in (163) [which can be found from (162) and (158)]:

$$\sigma_{xx}=\sum_\alpha\frac{q_\alpha^2}{m_\alpha}\int_0^\infty v_\perp dv_\perp\int_0^{2\pi}d\phi\int_{-\infty}^{\infty}dv_z\,\tfrac{1}{2}v_\perp(e^{i\phi}+e^{-i\phi})$$

$$\cdot\sum_n\sum_{s'}\frac{v_\perp[\frac{\partial f_{\alpha 0}}{\partial v_\perp^2}(1-\frac{k_zv_z}{\omega})+\frac{k_zv_z}{\omega}\frac{\partial f_{\alpha 0}}{\partial v_z^2}]}{i(\omega-k_zv_z-n\Omega_\alpha)}(J_{n+1}+J_{n-1})J_{s'}e^{i(s'-n)\phi} \qquad (165)$$

Using the Bessel function identities:

$$\begin{cases} J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x) \\ J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x) \end{cases} \tag{166}$$

we may write $$J_{n+1}\left(\frac{k_\perp v_\perp}{\Omega_\alpha}\right) + J_{n-1}\left(\frac{k_\perp v_\perp}{\Omega_\alpha}\right) = \frac{2n\Omega_\alpha}{k_\perp v_\perp} J_n\left(\frac{k_\perp v_\perp}{\Omega_\alpha}\right) \tag{167}$$

Sub. (167) into (165), we obtain

$$\sigma_{xx} = \sum_\alpha \frac{q_\alpha^2}{m_\alpha} \int_0^\infty v_\perp dv_\perp \int_0^{2\pi} d\phi \int_{-\infty}^\infty dv_z \, \frac{n v_\perp \Omega_\alpha}{k_\perp}$$

$$\cdot \sum_n \sum_{s'} \frac{\frac{\partial f_{\alpha 0}}{\partial v_\perp^2}(1 - \frac{k_z v_z}{\omega}) + \frac{k_z v_z}{\omega} \frac{\partial f_{\alpha 0}}{\partial v_z^2}}{i(\omega - k_z v_z - n\Omega_\alpha)} J_n J_{s'} \left[e^{i(s'-n+1)\phi} + e^{i(s'-n-1)\phi} \right], \tag{168}$$

where we see that only the $s' = n \pm 1$ terms in the s' sum will survive the ϕ-integration.

Carrying out the ϕ-integration in (168), we obtain

$$\sigma_{xx} = \sum_\alpha \frac{2\pi q_\alpha^2}{m_\alpha} \sum_n \int_0^\infty v_\perp dv_\perp \int_{-\infty}^\infty dv_z \, \frac{n v_\perp \Omega_\alpha}{k_\perp}$$

$$\cdot \frac{\frac{\partial f_{\alpha 0}}{\partial v_\perp^2}(1 - \frac{k_z v_z}{\omega}) + \frac{k_z v_z}{\omega} \frac{\partial f_{\alpha 0}}{\partial v_z^2}}{i(\omega - k_z v_z - n\Omega_\alpha)} J_n \overbrace{(J_{n-1} + J_{n+1})}^{\boxed{= \frac{2n\Omega_\alpha}{k_\perp v_\perp} J_n}}$$

$$= \sum_\alpha \frac{2\pi q_\alpha^2}{m_\alpha} \sum_n \int_0^\infty 2v_\perp dv_\perp \int_{-\infty}^\infty dv_z \, \frac{n^2 \Omega_\alpha^2}{k_\perp^2} J_n^2 \frac{\frac{\partial f_{\alpha 0}}{\partial v_\perp^2}(1 - \frac{k_z v_z}{\omega}) + \frac{k_z v_z}{\omega} \frac{\partial f_{\alpha 0}}{\partial v_z^2}}{i(\omega - k_z v_z - n\Omega_\alpha)}$$

$$= -\sum_\alpha \frac{\omega_{p\alpha}^2}{2} \sum_n \int_0^\infty 2v_\perp dv_\perp \int_{-\infty}^\infty dv_z \, \frac{n^2 \Omega_\alpha^2}{k_\perp^2} J_n^2 \frac{\frac{\partial \bar{f}_{\alpha 0}}{\partial v_\perp^2}(1 - \frac{k_z v_z}{\omega}) + \frac{k_z v_z}{\omega} \frac{\partial \bar{f}_{\alpha 0}}{\partial v_z^2}}{i(k_z v_z + n\Omega_\alpha - \omega)}, \tag{169}$$

where $\bar{f}_{\alpha 0} \equiv \frac{1}{n_{\alpha 0}} f_{\alpha 0}$ [hence $\int \bar{f}_{\alpha 0} d^3 v = 1$].

The general dispersion relation: Rewrite the dispersion relation:

$$\left|\overleftrightarrow{\mathbf{D}}\right| = \begin{vmatrix} D_{xx} & D_{xy} & D_{xz} \\ D_{yx} & D_{yy} & D_{yz} \\ D_{zx} & D_{zy} & D_{zz} \end{vmatrix}$$

$$= \begin{vmatrix} 1-\frac{k_z^2c^2}{\omega^2}+\frac{4\pi i}{\omega}\sigma_{xx} & \frac{4\pi i}{\omega}\sigma_{xy} & \frac{k_\perp k_z c^2}{\omega^2}+\frac{4\pi i}{\omega}\sigma_{xz} \\ \frac{4\pi i}{\omega}\sigma_{yx} & 1-\frac{k^2c^2}{\omega^2}+\frac{4\pi i}{\omega}\sigma_{yy} & \frac{4\pi i}{\omega}\sigma_{yz} \\ \frac{k_\perp k_z c^2}{\omega^2}+\frac{4\pi i}{\omega}\sigma_{zx} & \frac{4\pi i}{\omega}\sigma_{zy} & 1-\frac{k_\perp^2c^2}{\omega^2}+\frac{4\pi i}{\omega}\sigma_{zz} \end{vmatrix} = 0 \quad (149)$$

Sub. σ_{xx} from (169) into $D_{xx} = 1-\frac{k_z^2c^2}{\omega^2}+\frac{4\pi i}{\omega}\sigma_{xx}$, we obtain

$$D_{xx} = 1-\frac{k_z^2c^2}{\omega^2}-\frac{2\pi}{\omega}\sum_\alpha \omega_{p\alpha}^2 \sum_n \int_0^\infty 2v_\perp dv_\perp \int_{-\infty}^\infty dv_z$$

$$\cdot \frac{n^2\Omega_\alpha^2}{k_\perp^2} J_n^2\left(\frac{k_\perp v_\perp}{\Omega_\alpha}\right) \frac{\frac{\partial \bar{f}_{\alpha 0}(v_\perp,v_z)}{\partial v_\perp^2}\left(1-\frac{k_z v_z}{\omega}\right)+\frac{k_z v_z}{\omega}\frac{\partial \bar{f}_{\alpha 0}(v_\perp,v_z)}{\partial v_z^2}}{k_z v_z+n\Omega_\alpha-\omega} \quad (170)$$

6.12 Linear Vlasov Waves in Magnetized Plasma (*continued*)

By similar method, we obtain the other elements of the dispersion tensor. The complete results are (see Krall & Trivelpiece, pp. 405-406)

$$D_{xx} = 1-\frac{k_z^2c^2}{\omega^2}-\frac{2\pi}{\omega}\sum_\alpha\sum_n \omega_{p\alpha}^2 \left\langle \frac{n^2\Omega_\alpha^2}{k_\perp^2} J_n^2\left(\frac{k_\perp v_\perp}{\Omega_\alpha}\right)\chi_\alpha \right\rangle \quad (171)$$

$$D_{xy} = -\frac{2\pi i}{\omega}\sum_\alpha\sum_n \omega_{p\alpha}^2 \left\langle \frac{n\Omega_\alpha v_\perp}{k_\perp} J_n\left(\frac{k_\perp v_\perp}{\Omega_\alpha}\right)\frac{dJ_n(k_\perp v_\perp/\Omega_\alpha)}{d(k_\perp v_\perp/\Omega_\alpha)}\chi_\alpha \right\rangle \quad (172)$$

$$D_{xz} = \frac{k_z k_\perp c^2}{\omega^2}-\frac{2\pi}{\omega}\sum_\alpha\sum_n \omega_{p\alpha}^2 \left\langle \frac{n\Omega_\alpha v_z}{k_\perp} J_n^2\left(\frac{k_\perp v_\perp}{\Omega_\alpha}\right)\Lambda_\alpha \right\rangle \quad (173)$$

$$D_{yx} = -D_{xy} \quad (174)$$

$$D_{yy} = 1-\frac{(k_\perp^2+k_z^2)c^2}{\omega^2}-\frac{2\pi}{\omega}\sum_\alpha\sum_n \omega_{p\alpha}^2 \left\langle v_\perp^2\left[\frac{dJ_n(k_\perp v_\perp/\Omega_\alpha)}{d(k_\perp v_\perp/\Omega_\alpha)}\right]^2\chi_\alpha \right\rangle \quad (175)$$

$$D_{yz} = \frac{2\pi i}{\omega}\sum_\alpha\sum_n \omega_{p\alpha}^2 \left\langle v_z v_\perp J_n\left(\frac{k_\perp v_\perp}{\Omega_\alpha}\right)\frac{dJ_n(k_\perp v_\perp/\Omega_\alpha)}{d(k_\perp v_\perp/\Omega_\alpha)}\Lambda_\alpha \right\rangle \quad (176)$$

$$D_{zx} = \frac{k_z k_\perp c^2}{\omega^2} - \frac{2\pi}{\omega}\sum_\alpha \sum_n \omega_{p\alpha}^2 \left\langle v_z \frac{n\Omega_\alpha}{k_\perp} J_n^2\left(\frac{k_\perp v_\perp}{\Omega_\alpha}\right)\chi_\alpha \right\rangle \quad (177)$$

$$D_{zy} = -\frac{2\pi i}{\omega}\sum_\alpha \sum_n \omega_{p\alpha}^2 \left\langle v_z v_\perp J_n\left(\frac{k_\perp v_\perp}{\Omega_\alpha}\right)\frac{dJ_n(k_\perp v_\perp/\Omega_\alpha)}{d(k_\perp v_\perp/\Omega_\alpha)}\chi_\alpha \right\rangle \quad (178)$$

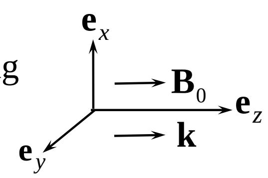
$$D_{zz} = 1 - \frac{k_\perp^2 c^2}{\omega^2} - \frac{2\pi}{\omega}\sum_\alpha \sum_n \omega_{p\alpha}^2 \left\langle v_z^2 J_n^2\left(\frac{k_\perp v_\perp}{\Omega_\alpha}\right)\Lambda_\alpha \right\rangle \quad (179)$$

where

$$\begin{cases} \left\langle F(v_\perp, v_z)\right\rangle \equiv \int_0^\infty 2v_\perp dv_\perp \int_{-\infty}^\infty dv_z \dfrac{F(v_\perp, v_z)}{k_z v_z + n\Omega_\alpha - \omega} & (180) \\ \chi_\alpha \equiv \dfrac{\partial \bar{f}_{\alpha 0}(v_\perp, v_z)}{\partial v_\perp^2}\left(1 - \dfrac{k_z v_z}{\omega}\right) + \dfrac{k_z v_z}{\omega}\dfrac{\partial \bar{f}_{\alpha 0}(v_\perp, v_z)}{\partial v_z^2} & (181) \\ \Lambda_\alpha \equiv \dfrac{\partial \bar{f}_{\alpha 0}(v_\perp, v_z)}{\partial v_z^2} - \dfrac{n\Omega_\alpha}{\omega}\left[\dfrac{\partial \bar{f}_{\alpha 0}(v_\perp, v_z)}{\partial v_z^2} - \dfrac{\partial \bar{f}_{\alpha 0}(v_\perp, v_z)}{\partial v_\perp^2}\right] & (182) \end{cases}$$

Question: The plasma is isotropic in x and y. Why are D_{xx} and D_{yy} unequal?

Waves Propagating Along $\mathrm{B}_0\mathbf{e}_z$ ($\mathbf{k} = k_z\mathbf{e}_z$):



The dispersion relation for waves propagating along $\mathbf{B}_0 = \mathrm{B}_0\mathbf{e}_z$ may be obtained by letting $k_\perp \to 0$ in (171)-(179).

For a small argument, the Bessel functions $J_n(x)$ and $J_{-n}(x)$ can be approximately written

$$\begin{cases} J_n(x) \overset{\boxed{x\to 0}}{\approx} \dfrac{1}{n!}\left(\dfrac{x}{2}\right)^n \\ J_{-n}(x) \overset{\boxed{\text{all } x}}{=} (-1)^n J_n(x) \overset{\boxed{x\to 0}}{\approx} -\dfrac{(-1)^n}{n!}\left(\dfrac{x}{2}\right)^n \end{cases} \quad (183)$$

For $n = 0$ and $n = 1$, we have in the limit $x \to 0$,

$$\begin{cases} J_0(x) \approx 1,\ J_1(x) \approx \dfrac{x}{2},\ J_{-1}(x) \approx -\dfrac{x}{2} \\ J_0'(x) \approx -\dfrac{x}{2},\ J_1'(x) \approx \dfrac{1}{2},\ J_{-1}'(x) \approx -\dfrac{1}{2} \end{cases} \quad (184)$$

In (171)-(180), the argument of all Bessel functions is $\dfrac{k_\perp v_\perp}{\Omega_\alpha}$.
Using (183) and (184), we find that in the limit $k_\perp \to 0$,

$$D_{xz} = D_{zx} = D_{zy} = D_{zy} = 0 \qquad (185)$$

and the other elements become

$$\begin{cases} D_{xx} = 1 - \dfrac{k_z^2 c^2}{\omega^2} - \dfrac{2\pi}{\omega}\sum_\alpha \omega_{p\alpha}^2 \dfrac{1}{4}\left[\left\langle v_\perp^2 \chi_\alpha \right\rangle_{n=1} + \left\langle v_\perp^2 \chi_\alpha \right\rangle_{n=-1}\right] & (186) \\ D_{yy} = D_{xx} & (187) \\ D_{xy} = -D_{yx} = -\dfrac{2\pi i}{\omega}\sum_\alpha \omega_{p\alpha}^2 \dfrac{1}{4}\left[\left\langle v_\perp^2 \chi_\alpha \right\rangle_{n=1} - \left\langle v_\perp^2 \chi_\alpha \right\rangle_{n=-1}\right] & (188) \\ D_{zz} = 1 - \dfrac{2\pi}{\omega}\sum_\alpha \omega_{p\alpha}^2 \left\langle v_z^2 \Lambda_\alpha \right\rangle_{n=0} & (189) \end{cases}$$

Thus, the dispersion relation (149) reduces to

$$\begin{vmatrix} D_{xx} & D_{xy} & 0 \\ D_{yx} & D_{yy} & 0 \\ 0 & 0 & D_{zz} \end{vmatrix} = D_{zz}(D_{xx}D_{yy} - D_{xy}D_{yx}) = D_{zz}(D_{xx}^2 + D_{xy}^2) = 0 \quad (190)$$

Rewrite the dispersion relation: $\quad D_{zz}(D_{xx}^2 + D_{xy}^2) = 0 \qquad (190)$

Several modes are contained in (190). To find these modes, we assume, for simplicity, that the plasma is isotropic in all 3 dimensions, i.e. $f_{\alpha 0}(v_\perp, v_z) = f_{\alpha 0}(v)$.

Electrostatic waves:

One of the solutions of (190) is $D_{zz} = 0$, which, by (189), (180), and (182), can be written

$$1 - \frac{2\pi}{\omega}\sum_\alpha \omega_{p\alpha}^2 \int_0^\infty 2v_\perp dv_\perp \int_{-\infty}^\infty dv_z \frac{v_z^2 \dfrac{\partial}{\partial v_z^2}\bar{f}_{\alpha 0}}{k_z v_z - \omega} = 0$$

or

$$1 - \frac{1}{\omega}\sum_\alpha \omega_{p\alpha}^2 \int_{-\infty}^\infty dv_z \frac{v_z \dfrac{d}{dv_z} g_{\alpha 0}(v_z)}{k_z v_z - \omega} = 0, \qquad (191)$$

where $\quad g_{\alpha 0}(v_z) \equiv \int_0^\infty 2\pi v_\perp dv_\perp \bar{f}_{\alpha 0}(v_\perp, v_z)$.

The integral in (191) can be written

$$\int_{-\infty}^{\infty} dv_z \frac{v_z \frac{d}{dv_z} g_{\alpha 0}(v_z)}{k_z v_z - \omega} = \frac{1}{k_z}\int_{-\infty}^{\infty} dv_z \frac{[\omega - (\omega - k_z v_z)]\frac{d}{dv_z} g_{\alpha 0}(v_z)}{k_z v_z - \omega}$$

$$= \frac{1}{k_z}\int_{-\infty}^{\infty} dv_z \frac{\omega \frac{d}{dv_z} g_{\alpha 0}(v_z)}{k_z v_z - \omega} - \frac{1}{k_z}\underbrace{\int_{-\infty}^{\infty} dv_z \frac{d}{dv_z} g_{\alpha 0}(v_z)}_{0}$$

$$= \frac{\omega}{k_z^2}\int_{-\infty}^{\infty} dv_z \frac{\frac{d}{dv_z} g_{\alpha 0}(v_z)}{v_z - \frac{\omega}{k_z}} \qquad (192)$$

Sub. (192) into (191), we get $1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{k_z^2}\int_{-\infty}^{\infty} dv_z \frac{\frac{d}{dv_z} g_{\alpha 0}(v_z)}{v_z - \frac{\omega}{k_z}} = 0,$

which agrees with the electrostatic dispersion relation [(135)] for an unmagnetized plasma. This is because electrostatic waves involve particle motion along $\mathbf{B}_0$, and hence are unaffected by the magnetic field. The mode considered below will provide an opposite example.

Electromagnetic waves: Rewrite $D_{zz}(D_{xx}^2 + D_{xy}^2) = 0 \qquad (190)$

(190) gives two other solutions: $\begin{cases} D_{xx} - iD_{xy} = 0 & (193) \\ D_{xx} + iD_{xy} = 0 & (194) \end{cases}$

We recall that the dispersion relation $|\overleftrightarrow{\mathbf{D}}| = 0$ [(149)] is based on the condition for the solvability of the field equations in $\overleftrightarrow{\mathbf{D}} \cdot \mathbf{E}_{1k} = 0$ [(147)]. For $k_\perp = 0$, we have $D_{xz} = D_{zx} = D_{zy} = D_{zy} = 0,\ D_{xx} = D_{yy}$, and $D_{yx} = -D_{xy}$ [see (187)]. Then, for solutions (193) and (194),

(147) gives $\begin{bmatrix} D_{xx} & D_{xy} \\ D_{yx} & D_{yy} \end{bmatrix}\begin{bmatrix} E_{1kx} \\ E_{1ky} \end{bmatrix} = \begin{bmatrix} D_{xx} & D_{xy} \\ -D_{xy} & D_{xx} \end{bmatrix}\begin{bmatrix} E_{1kx} \\ E_{1ky} \end{bmatrix} = 0 \qquad (195)$

$$\Rightarrow \begin{cases} D_{xx}E_{1kx} + D_{xy}E_{1ky} = 0 \\ -D_{xy}E_{1kx} + D_{xx}E_{1ky} = 0 \end{cases} \Rightarrow \begin{cases} E_{1kx} = -\frac{D_{xy}}{D_{xx}} E_{1ky} \\ E_{1kx} = \frac{D_{xx}}{D_{xy}} E_{1ky} \end{cases} \qquad (196)$$

$$\text{Rewrite:} \quad \begin{cases} E_{1kx} = -\dfrac{D_{xy}}{D_{xx}} E_{1ky} \\ E_{1kx} = \dfrac{D_{xx}}{D_{xy}} E_{1ky} \end{cases} \qquad (196)$$

The following information about the modes in (193) and (194) can be immediately learned from (196):

(1) The 2 equations in (196) are consistent only when $D_{xx}^2 = -D_{xy}^2$, or when (190) is satisfied. This is a specific example which shows the dispersion relation as the condition for solvability of field equations. We also find that (196) gives the *relative* amplitude (not the absolute values) of the field components, as is typical of *linear* solutions.

(2) The fields in (196) are in the x-y plane. With $\mathbf{k} = k_z \mathbf{e}_z$ and $\mathbf{E}_1(\mathbf{x},t) = \mathbf{E}_{1k} e^{-i\omega t + ik_z z}$ [see (120)], we find

$$\nabla \cdot \mathbf{E}_1(\mathbf{x},t) = ik_z \mathbf{e}_z \cdot \mathbf{E}_{1k} e^{-i\omega t + ik_z z} = 0.$$

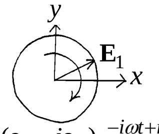
Thus, the two solutions represent electromagnetic waves.

(3) Either of the equations in (196) gives $E_{1kx} = -\dfrac{D_{xy}}{D_{xx}} E_{1ky}$. Hence,

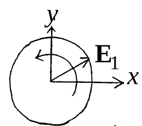
$$E_{1kx} = \begin{cases} iE_{1ky} & \text{for } D_{xx} - iD_{xy} = 0 \\ -iE_{1ky} & \text{for } D_{xx} + iD_{xy} = 0 \end{cases} \Rightarrow \mathbf{E}_{1k} = \begin{cases} E_{1k}[\mathbf{e}_x - i\mathbf{e}_y] & (198) \\ E_{1k}[\mathbf{e}_x + i\mathbf{e}_y] & (199) \end{cases}$$

where $E_{1k} = (E_{1kx}^2 + E_{1ky}^2)^{1/2}$.

Thus, both waves are circularly polarized*. Without loss of generality (explained later), we assume positive ω and B_0. At any fixed position z, the field in (198) rotates opposite to the gyration of electrons in $B_0\mathbf{e}_z$. In another view, it rotates in the direction of left-hand fingers if the thumb points to the direction of $B_0\mathbf{e}_z$, hence the name "left circularly polarized wave". In contrast, (199) gives a "right circularly polarized wave" rotating in the same sense as the electrons.



$E_{1k}(\mathbf{e}_x - i\mathbf{e}_y)e^{-i\omega t + ik_z z}$
left circularly polarized



$E_{1k}(\mathbf{e}_x + i\mathbf{e}_y)e^{-i\omega t + ik_z z}$
right circularly polarized

**Note*: The *circular* polarization is due to $D_{xx} = D_{yy}$.

Turning to the dispersion relations: $\begin{cases} D_{xx} - iD_{xy} = 0 & (193) \\ D_{xx} + iD_{xy} = 0 & (194) \end{cases}$

D_{xx} and D_{xy} are given by (186) and (188), respectively.

$$\begin{cases} D_{xx} = 1 - \dfrac{k_z^2 c^2}{\omega^2} - \dfrac{2\pi}{\omega}\sum_\alpha \omega_{p\alpha}^2 \dfrac{1}{4}\left[\left\langle v_\perp^2 \chi_\alpha \right\rangle_{n=1} + \left\langle v_\perp^2 \chi_\alpha \right\rangle_{n=-1}\right] & (186) \\ D_{xy} = -\dfrac{2\pi i}{\omega}\sum_\alpha \omega_{p\alpha}^2 \dfrac{1}{4}\left[\left\langle v_\perp^2 \chi_\alpha \right\rangle_{n=1} - \left\langle v_\perp^2 \chi_\alpha \right\rangle_{n=-1}\right] & (188) \end{cases}$$

$$\Rightarrow D_{xx} - iD_{xy} = 1 - \frac{k_z^2 c^2}{\omega^2} - \frac{\pi}{2\omega}\sum_\alpha \omega_{p\alpha}^2 [\left\langle v_\perp^2 \chi_\alpha \right\rangle_{n=1} + \left\langle v_\perp^2 \chi_\alpha \right\rangle_{n=-1} - \left\langle v_\perp^2 \chi_\alpha \right\rangle_{n=1} + \left\langle v_\perp^2 \chi_\alpha \right\rangle_{n=-1}]$$

$$= 1 - \frac{k_z^2 c^2}{\omega^2} - \frac{\pi}{\omega}\sum_\alpha \omega_{p\alpha}^2 \left\langle v_\perp^2 \chi_\alpha \right\rangle_{n=-1} = 0 \quad (200)$$

Similarly,

$$D_{xx} + iD_{xy} = 1 - \frac{k_z^2 c^2}{\omega^2} - \frac{\pi}{\omega}\sum_\alpha \omega_{p\alpha}^2 \left\langle v_\perp^2 \chi_\alpha \right\rangle_{n=1} = 0 \quad (201)$$

Using (181) and (182), (200) and (201) can be written

$$\begin{cases} \omega^2 - k_z^2 c^2 + 2\pi\omega \sum_\alpha \omega_{p\alpha}^2 \displaystyle\int \frac{\frac{\partial \bar{f}_{\alpha 0}}{\partial v_\perp^2}(1 - \frac{k_z v_z}{\omega}) + \frac{k_z v_z}{\omega}\frac{\partial \bar{f}_{\alpha 0}}{\partial v_z^2}}{\omega - k_z v_z - \Omega_\alpha} v_\perp^3 dv_\perp dv_z = 0 & (202) \\ \text{[left circularly polarized wave]} & \\ \omega^2 - k_z^2 c^2 + 2\pi\omega \sum_\alpha \omega_{p\alpha}^2 \displaystyle\int \frac{\frac{\partial \bar{f}_{\alpha 0}}{\partial v_\perp^2}(1 - \frac{k_z v_z}{\omega}) + \frac{k_z v_z}{\omega}\frac{\partial \bar{f}_{\alpha 0}}{\partial v_z^2}}{\omega - k_z v_z + \Omega_\alpha} v_\perp^3 dv_\perp dv_z = 0 & (203) \\ \text{[right circularly polarized wave]} & \end{cases}$$

These two dispersion relations in their present forms allow an anisotropic $\bar{f}_{\alpha 0}$ [e.g. (103) and (104)], which may lead to an instability (an example will be provided at the end of this section). For an isotropic plasma, $\bar{f}_{\alpha 0} = \bar{f}_{\alpha 0}(v_\perp^2 + v_z^2)$ [e.g. (102)], we have $\dfrac{\partial \bar{f}_{\alpha 0}}{\partial v_\perp^2} = \dfrac{\partial \bar{f}_{\alpha 0}}{\partial v_z^2}$. Then, (202) and (203) reduce to

$$\begin{cases} \omega^2 - k_z^2 c^2 + \pi\omega \sum_\alpha \omega_{p\alpha}^2 \int \dfrac{\frac{\partial \bar{f}_{\alpha 0}}{\partial v_\perp}}{\omega - k_z v_z - \Omega_\alpha} v_\perp^2 dv_\perp dv_z = 0 & (204) \\ \text{[left circularly polarized wave]} & \\ \omega^2 - k_z^2 c^2 + \pi\omega \sum_\alpha \omega_{p\alpha}^2 \int \dfrac{\frac{\partial \bar{f}_{\alpha 0}}{\partial v_\perp}}{\omega - k_z v_z + \Omega_\alpha} v_\perp^2 dv_\perp dv_z = 0 & (205) \\ \text{[right circularly polarized wave]} & \end{cases}$$

Integrating by parts with respect to $v_\perp$, we obtain

$$\begin{cases} \omega^2 - k_z^2 c^2 - 2\pi\omega \sum_\alpha \omega_{p\alpha}^2 \int \dfrac{\bar{f}_{\alpha 0}}{\omega - k_z v_z - \Omega_\alpha} v_\perp dv_\perp dv_z = 0 & (206) \\ \text{[left circularly polarized wave]} & \\ \omega^2 - k_z^2 c^2 - 2\pi\omega \sum_\alpha \omega_{p\alpha}^2 \int \dfrac{\bar{f}_{\alpha 0}}{\omega - k_z v_z + \Omega_\alpha} v_\perp dv_\perp dv_z = 0 & (207) \\ \text{[right circularly polarized wave]} & \end{cases}$$

The basic properties of the waves can be most clearly seen in a cold plasma. So, we let

$$\bar{f}_{\alpha 0} = \frac{1}{2\pi v_\perp} \delta(v_\perp)\delta(v_z) \qquad (208)$$

Note: $\int \bar{f}_{\alpha 0} d^3v = \int_0^\infty v_\perp dv_\perp \int_0^{2\pi} d\phi \int_{-\infty}^\infty dv_z f_{\alpha 0} = 1$

Then, (206) and (207) give

$$\begin{cases} \omega^2 - k_z^2 c^2 - \sum_\alpha \dfrac{\omega\omega_{p\alpha}^2}{\omega - \Omega_\alpha} = 0 & \text{[left circularly polarized]} \quad (209) \\ \omega^2 - k_z^2 c^2 - \sum_\alpha \dfrac{\omega\omega_{p\alpha}^2}{\omega + \Omega_\alpha} = 0 & \text{[right circularly polarized]} \quad (210) \end{cases}$$

As an exercise in kinetic treatment of plasma waves, we have gone through great length to arrive at the above dispersion relations for a cold plasma. In fact, (209) and (210) can be readily derived from the fluid equations [see Nicholson, Sec. 7.10; Krall & Trivelpiece, Sec. 4.10].

Assume the plasma contains only one ion species of charge e and mass m_i. In all equations, $\Omega_\alpha \, (= \frac{q_\alpha B_0}{m_\alpha c})$ carries the sign of q_α and B_0.
To be more explicit, we define the notations: $\Omega_e = \frac{|e|B_0}{m_e c}$; $\Omega_i = \frac{|e|B_0}{m_i c}$

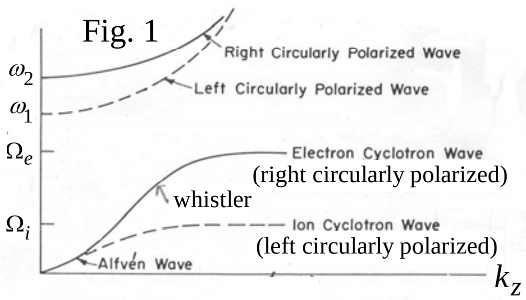
Then, (209) and (210) can be written

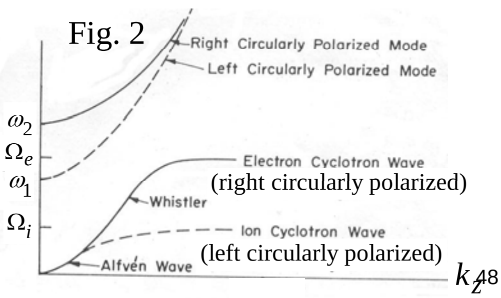
$$\begin{cases} \omega^2 - k_z^2 c^2 - \omega\left[\dfrac{\omega_{pe}^2}{\omega+\Omega_e} + \dfrac{\omega_{pi}^2}{\omega-\Omega_i}\right] = 0 & \text{[left circularly polarized]} \quad (211) \\ \omega^2 - k_z^2 c^2 - \omega\left[\dfrac{\omega_{pe}^2}{\omega-\Omega_e} + \dfrac{\omega_{pi}^2}{\omega+\Omega_i}\right] = 0 & \text{[right circularly polarized]} \quad (212) \end{cases}$$

Each equation can be put in the form of a 4th order polynomial in ω. So, for a given k_z, there are 4 solutions for ω. However, with ω changed to $-\omega$, one equation become the other equation. Thus, a negaitive-ω solution of one equation is identical to a positive-ω solution of the other equation. So there must be 2 positive-ω and 2 negative-ω solutions for each equation. This results in a total of 4 independent solutions.

Furthermore, with a change of the sign of B_0, the two equations also reverse. So, without loss of generality [confirming the statement following (199)], we may restrict our consideration to positive-ω solutions for a positive B_0 (i.e. $\mathbf{B}_0$ is in the positive z direction).

The 4 independent, positive-ω solutions of (211) and (212) in a positive B_0 are shown in Fig. 1 or 2 in four branches, ranging from very low to very high frequencies. Various waves in these branches will be classified below according to their frequency range.
(See Nicholson, Sec. 7.10 & 7.11; Krall & Trivelpiece, Sec. 4.10).





A. High frequency electromagnetic waves - Faraday rotation

For high frequency waves ($\omega \gg \Omega_i$), the ion terms in (211) and (212) can be neglected. Thus,

$$\begin{cases} \omega^2 - k_z^2 c^2 - \dfrac{\omega\omega_{pe}^2}{\omega+\Omega_e} = 0 \quad \text{[left circularly polarized]} & (213) \\ \omega^2 - k_z^2 c^2 - \dfrac{\omega\omega_{pe}^2}{\omega-\Omega_e} = 0 \quad \text{[right circularly polarized]} & (214) \end{cases}$$

The high frequency branches are plotted in the top two curves in Figs. 1 and 2.

Fig. 1

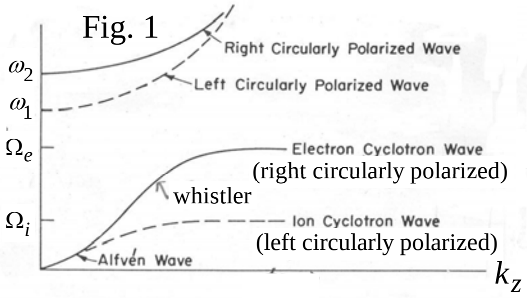
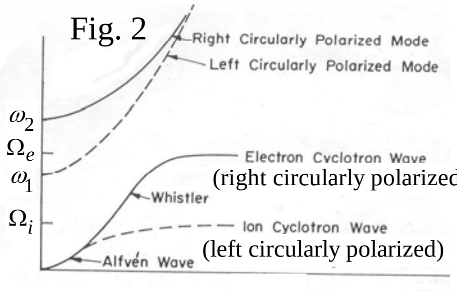


Fig. 2



Setting $k_z = 0$, we find the cut-off frequencies of the two branches:

$$\begin{Bmatrix} \omega_1 \\ \omega_2 \end{Bmatrix} = \sqrt{\omega_{pe}^2 + \frac{\Omega_e^2}{4}} \mp \frac{\Omega_e}{2} \qquad (215)$$

The 2 figures differ in plasma densities. When $\omega_{pe} \geq \sqrt{2}\Omega_e$, we have $\omega_1 \geq \Omega_e$ (Fig. 1). When $\omega_{pe} \leq \sqrt{2}\Omega_e$, we have $\omega_1 \leq \Omega_e$ (Fig. 2).

As $B_0 \to 0$ ($\Omega_e \to 0$), the 2 branches coalesce with the same cutoff frequency ω_{pe} and the same dispersion relation $\omega^2 - k_z^2 c^2 - \omega_{pe}^2 = 0$, consistent with (143).

Fig. 1

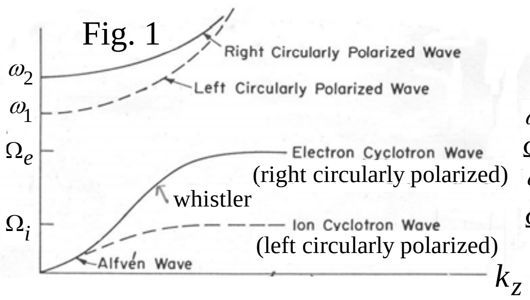
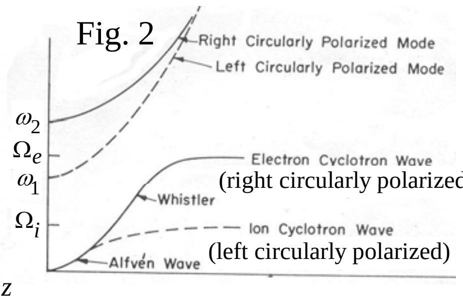


Fig. 2

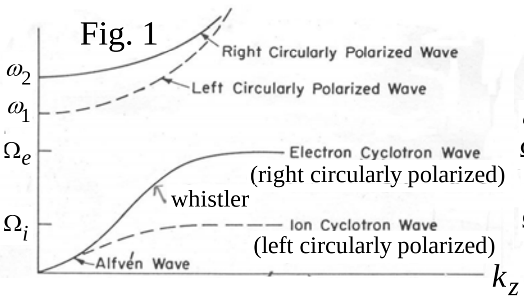


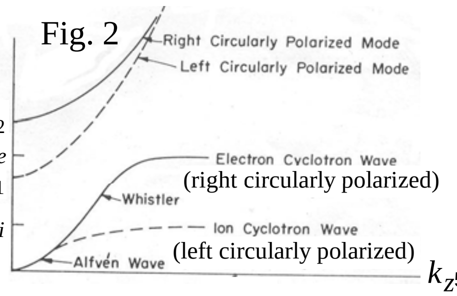
As shown in Figs. 1 and 2, at a given frequency, the right circularly polarized wave has a greater phase velocity than the left ciucularly polarized wave. Hence, if a linearly polarized wave is injected into the plasma, it may be regarded as the superposition of a right circularly polarized wave and a left circularly polarized wave of equal amplitude, each traveling at a different phase velocity. The combined wave is still linearly polarized but its **E** field (i.e. its polarization) will rotate as the wave propagates. This is called the Faraday rotation and is exploited for plasma density measurement because the degree of polarization rotaion depends on the plasma density.

In an unmagnetized plasma, there is no electromagnetic wave below the cutoff frequency ω_{pe}. A magnetized plasma, however, can support other branches of electromagnetic waves at frequencies below the cutoff frequencyies of the top two branches, as discussed below.

B. Intermediate frequency electromagnetic waves - whistler wave and electron cyclotron wave

In the intermediate frequency range, we still have $\omega \gg \Omega_i$, hence (213) and (214) still apply. (213) has no other solution in this range. (214) has a solution marked as "whistler" & "electron cyclotron wave" in Figs. 1 and 2. The electron cyclotron wave can be exploited for electron cyclotron resonance heating since it has the same frequency as the electron cyclotron frequency and it rotates in the same sense as the electrons.





For the whistler wave, the group velocity $v_g \,(= d\omega / dk_z)$ increases as ω increases. When a lightning stroke on earth generates a pluse of EM waves containing many frequencies, the pulse may reach the ionosphere and propagate along the earth magnetic field as a whistler wave. Some the wave will eventually leave the ionosphere to impinge on the earth, where it can be received by a radio and generate a sound like that of a whistle, hence the name whistler wave. The radio signal has a duration longer than the original pulse because components at different frequencies travel at differnet v_g in the ionosphere.

Fig. 1

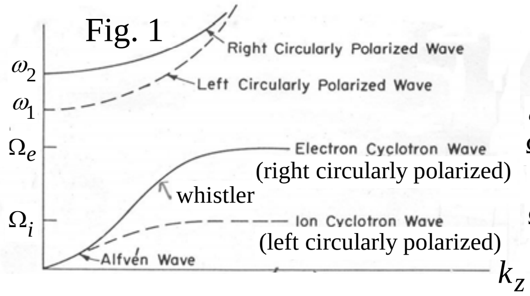
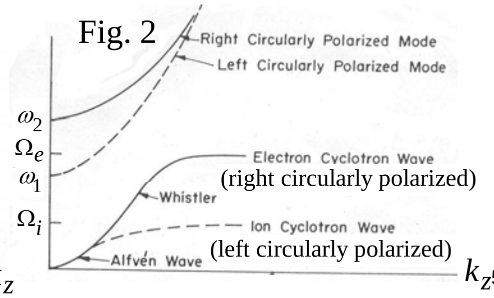


Fig. 2



C. Low frequency electromagnetic waves - Alfven wave and ion cyclotron wave

As the frequency gets lower, the ions participate more and more. At frequencies near or below Ω_i, the ions play a major role and we must use (211) and (212). In the vicinity of Ω_i, (211) gives the low-frequency end of the whistler wave (slightly modified by the ions), and (212) gives a new wave called the ion cyclotron wave.

Fig. 1

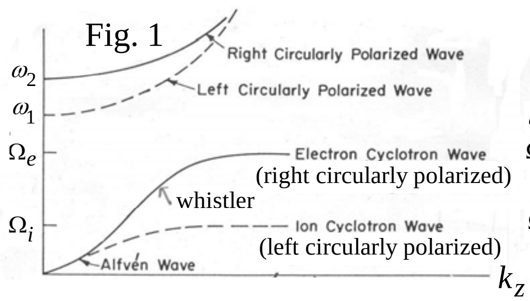
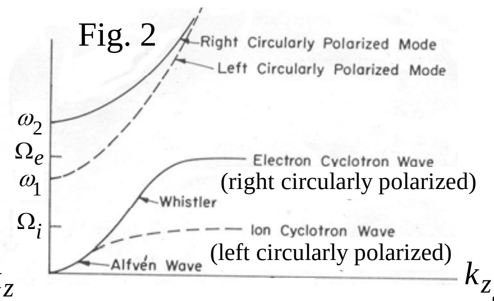


Fig. 2



When $\omega \ll \Omega_i$, we have

$$\frac{1}{\omega+\Omega_{e,i}} = \frac{1}{\Omega_{e,i}}\left(1+\frac{\omega}{\Omega_{e,i}}\right)^{-1} \approx \frac{1}{\Omega_{e,i}}\left(1-\frac{\omega}{\Omega_{e,i}}\right) \tag{216}$$

$$\frac{1}{\omega-\Omega_{e,i}} = \frac{1}{\Omega_{e,i}}\left(1-\frac{\omega}{\Omega_{e,i}}\right)^{-1} \approx \frac{1}{\Omega_{e,i}}\left(1+\frac{\omega}{\Omega_{e,i}}\right) \tag{217}$$

Sub. (216) and (217) into either (211) or (212), we get the same results (i.e the two low-frequency branches merge into one):

$$\omega^2 - k_z^2 c^2 + \omega^2\left(\frac{\omega_{pe}^2}{\Omega_e^2} + \frac{\omega_{pi}^2}{\Omega_i^2}\right) = 0, \tag{218}$$

Since $\frac{\omega_{pe}^2}{\Omega_e^2} = \frac{\omega_{pi}^2}{\Omega_e \Omega_i} \ll \frac{\omega_{pi}^2}{\Omega_i^2}$, we neglect the electron term to get

$$\omega^2 - k_z^2 c^2 + \omega^2 \frac{\omega_{pi}^2}{\Omega_i^2} = \omega^2 - k_z^2 c^2 + \omega^2 \underbrace{\frac{4\pi n_{i0} e^2}{m_i} \frac{m_i^2 c^2}{e^2 B_0^2}}_{\frac{4\pi n_{i0} m_i c^2}{B_0^2}} = 0 \tag{219}$$

Defining a speed V_A (called the Alfven speed) in terms of B_0 and the ion mass density $\rho_i = n_{i0} m_i$:

$$V_A = \frac{B_0}{\sqrt{4\pi\rho_i}}, \tag{220}$$

we obtain from (219) the diepersion relation of the <u>Alfven wave</u>:

$$\omega^2 - k_z^2 c^2 + \frac{\omega^2 c^2}{V_A^2} = 0 \quad \text{or} \quad \omega^2 = \frac{k_z^2 V_A^2}{1+V_A^2/c^2} \tag{221}$$

Fig. 1

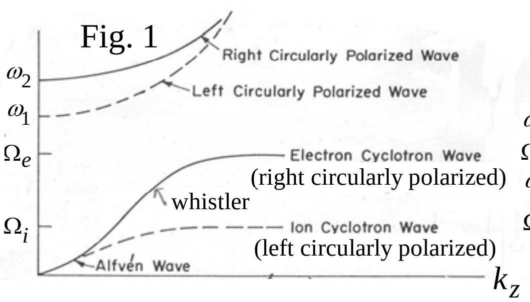
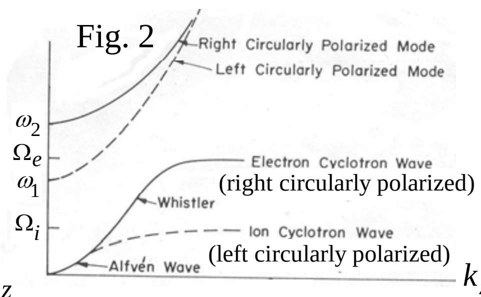
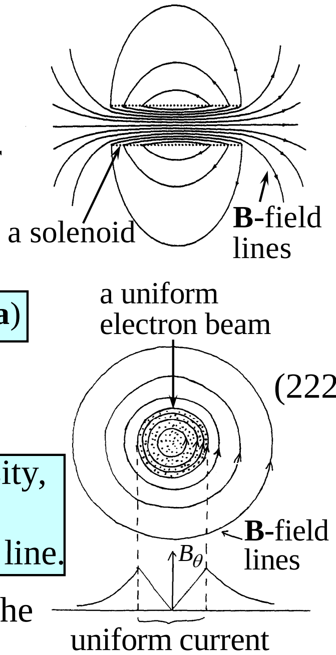


Fig. 2



Physics of the Alfven wave: We first develop the useful concept of magnetic pressure and magnetic tension. Using the static law: $\nabla\times\mathbf{B}=\frac{4\pi}{c}\mathbf{J}$ (approximately applicable at very low frequencies), we may express the magnetic force density $\mathbf{f}$ (force per unit volume) entirely in terms of the **B**-field:



$$\mathbf{f}=\frac{1}{c}\mathbf{J}\times\mathbf{B}=\frac{1}{4\pi}(\nabla\times\mathbf{B})\times\mathbf{B}$$

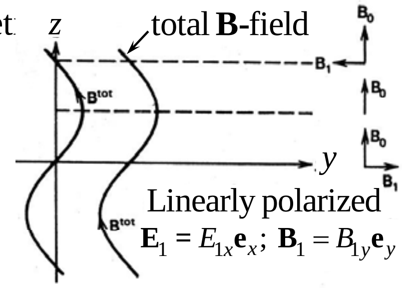
$$\boxed{\nabla(\mathbf{a}\cdot\mathbf{b})=(\mathbf{a}\cdot\nabla)\mathbf{b}+(\mathbf{b}\cdot\nabla)\mathbf{a}+\mathbf{a}\times(\nabla\times\mathbf{b})+\mathbf{b}\times(\nabla\times\mathbf{a})}$$

$$=\underbrace{-\nabla\frac{B^2}{8\pi}}_{\text{magnetic pressure force density}}+\underbrace{\frac{1}{4\pi}(\mathbf{B}\cdot\nabla)\mathbf{B}}_{\text{magnetic tension force density, as if a curved }\mathbf{B}\text{-field line tended to become a straight line.}} \quad (222)$$

In regions where $\mathbf{J}=0$, we have $\mathbf{f}=0$, i.e. the pressure and tension force densities cancel out.

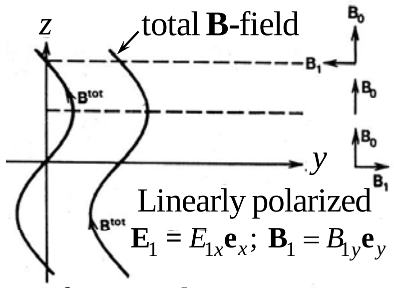
Return to the Alfven wave. Since the two lower branches merge at $\omega\ll\Omega_i$, the left and right circularly polarized waves have the same phase velocity. So a linearly polarized wave will remain linearly polarized (no Faraday rotation). The figure below shows a linearly polarized wave with $\mathbf{B}_1$ in the y-direction and $\mathbf{E}_1$ in the x-direction.

Since $\omega\ll\Omega_i,\ \Omega_e,$ the electron and ion behavior can be described by their $\mathbf{E}_1\times\mathbf{B}_0$ drift motion (same speed and same direction). The wave electric field $E_{1x}\mathbf{e}_x$ cause both the electrons and ions to drift in the $\pm y$-direction, while the wave magnetic field $B_{1y}\mathbf{e}_y$ bends the external $\mathbf{B}_0$ in the direction of the plasma drift (see figure). A quantitative analysis (Nicholson, p. 163) shows that the field lines and the plasma move together as if the field lines were "frozen" to the plasma (or the plasma frozen to field lines).



On the other hand, when the magnetic field lines are bent, there is a "tension force density" on the plasma, which acts as a restoring force to drive the plasma back so that the field lines (which are frozen to the plasma) become straight. As the field lines are straightened, the momentum of the plasma carries the field lines further back, thus bending the field lines again, in the opposite direction. The tension force then acts again to start another oscillation cycle.

Note that we have assumed $\mathbf{k} = k_z \mathbf{e}_z$; hence, all quantities vary only with the z-variable. This implies that, at a given time, the $E_{1x}\mathbf{e}_x \times \mathbf{B}_0$ drift in the y-direction has the same speed at all points along y. Thus, the drift motion will not compress/decompress the plasma to produce a density variation. The plasma remains uniform in the processes.



Alternative derivation of the Alfven wave dispersion relation:

The $\mathbf{E}_1 \times \mathbf{B}$ drifts cause the plasma electron and ions to move in the same direction with the same speed, hence generating no current. However, there is another drift motion due to the time variation of $\mathbf{E}_1$, which results in a polarization drift current given by (2.43) of Sec. 2.5:

$$\mathbf{J}_p = \frac{\rho_m c^2}{B_0^2}\frac{\partial \mathbf{E}_1}{\partial t} = \frac{-i\omega\rho_m c^2}{B_0^2}\mathbf{E}_{1k} e^{-i\omega t + ik_z z}$$

or $$\mathbf{J}_{pk} = \frac{-i\omega\rho_m c^2}{B_0^2}\mathbf{E}_{1k}, \qquad (223)$$

where $\rho_m = n_{i0}m_i + n_{e0}m_e$ is the plasma mass density. Note that the polarization drift speed is much greater for the ions than electrons.

$\mathbf{J}_{pk}$ plays a critical role in the Alfven wave. It generates the wave magnetic field $\mathbf{B}_{1k}$ and hence the magnetic tension force density. In fact, we may derive the dispersion relation based on (223).

Sub. $\mathbf{J}_{1k} = \mathbf{J}_{pk} = \dfrac{-i\omega\rho_m c^2}{B_0^2}\mathbf{E}_{1k}$ [(223)], $\mathbf{E}_{1k} = E_{1kx}\mathbf{e}_x$, and $\mathbf{k} = k_z\mathbf{e}_z$

into the field equation derived earlier:

$$\mathbf{k}\times(\mathbf{k}\times\mathbf{E}_{1k}) + \frac{\omega^2}{c^2}\mathbf{E}_{1k} = -\frac{4\pi i\omega}{c^2}\mathbf{J}_{1k} \qquad (125)$$

we obtain

$$(k_z^2 - \frac{\omega^2}{c^2} - \frac{4\pi\rho_m\omega^2}{B_0^2})E_{1kx}\mathbf{e}_x = 0,$$

which gives the same dispersion relation as (221):

$$\omega^2 - k_z^2c^2 + \frac{\omega^2c^2}{V_A^2} = 0$$

Waves propagating perpendicular to $B_0\mathbf{e}_z$ $(k_z = 0)$:

Assume isotropic distribution, $\bar{f}_{\alpha 0} = \bar{f}_{\alpha 0}(v_\perp^2 + v_z^2)$, we have $\dfrac{\partial\bar{f}_{\alpha 0}}{\partial v_z^2} = \dfrac{\partial\bar{f}_{\alpha 0}}{\partial v_\perp^2}$. Then, with $k_z = 0$, we obtain from (173)-(178)

$$D_{xz} = -\frac{2\pi}{\omega}\sum_\alpha\sum_n \omega_{p\alpha}^2 \frac{n\Omega_\alpha}{k_\perp}\int_0^\infty 2v_\perp dv_\perp \int_{-\infty}^\infty dv_z \frac{v_z J_n^2 \dfrac{\partial\bar{f}_{\alpha 0}}{\partial v_z^2}}{n\Omega_\alpha - \omega} \qquad (224)$$

$$D_{yz} = \frac{2\pi i}{\omega}\sum_\alpha\sum_n \omega_{p\alpha}^2 \int_0^\infty 2v_\perp dv_\perp \int_{-\infty}^\infty dv_z \frac{v_z v_\perp J_n \dfrac{dJ_n}{d(k_\perp v_\perp/\Omega_\alpha)}\dfrac{\partial\bar{f}_{\alpha 0}}{\partial v_z^2}}{n\Omega_\alpha - \omega} \qquad (225)$$

$$D_{zx} = -\frac{2\pi}{\omega}\sum_\alpha\sum_n \omega_{p\alpha}^2 \frac{n\Omega_\alpha}{k_\perp}\int_0^\infty 2v_\perp dv_\perp \int_{-\infty}^\infty dv_z \frac{v_z J_n^2 \dfrac{\partial\bar{f}_{\alpha 0}}{\partial v_\perp^2}}{n\Omega_\alpha - \omega} \qquad (226)$$

$$D_{zy} = -\frac{2\pi i}{\omega}\sum_\alpha\sum_n \omega_{p\alpha}^2 \int_0^\infty 2v_\perp dv_\perp \int_{-\infty}^\infty dv_z \frac{v_z v_\perp J_n \dfrac{dJ_n}{d(k_\perp v_\perp/\Omega_\alpha)}\dfrac{\partial\bar{f}_{\alpha 0}}{\partial v_\perp^2}}{n\Omega_\alpha - \omega} \qquad (227)$$

Factoring out the v_z-integrals from (224) and (225), we have

$$D_{yz},\ D_{xz} \propto \int_{-\infty}^{\infty} dv_z v_z \frac{\partial \bar{f}_{\alpha 0}}{\partial v_z^2}$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} dv_z \frac{\partial \bar{f}_{\alpha 0}}{\partial v_z} = \frac{1}{2}\, \bar{f}_{\alpha 0}(v_z)\Big|_{-\infty}^{\infty} = 0$$

Factoring out the v_z-integrals from (226) and (227), we have

$$D_{zx},\ D_{zy} \propto \int_{-\infty}^{\infty} dv_z v_z \frac{\partial \bar{f}_{\alpha 0}}{\partial v_\perp^2} = 0$$

because $\bar{f}_{\alpha 0}$ is an even function of v_z.

Thus, $D_{yz} = D_{xz} = D_{zx} = D_{zy}$ and (147) reduces to

$$\begin{bmatrix} D_{xx} & D_{xy} & 0 \\ D_{yx} & D_{yy} & 0 \\ 0 & 0 & D_{zz} \end{bmatrix} \begin{bmatrix} E_{1kx} \\ E_{1ky} \\ E_{1kz} \end{bmatrix} = 0 \qquad (228)$$

where

$$D_{xx} = 1 - \frac{2\pi}{\omega}\sum_\alpha \sum_n \frac{\omega_{p\alpha}^2}{n\Omega_\alpha - \omega}\, \omega_{p\alpha}^2 \int_0^\infty 2v_\perp dv_\perp \int_{-\infty}^{\infty} dv_z \frac{n^2\Omega_\alpha^2}{k_\perp^2} J_n^2 \frac{\partial \bar{f}_{\alpha 0}}{\partial v_\perp^2} \qquad (229)$$

$$D_{xy} = -D_{yx} = -\frac{2\pi i}{\omega}\sum_\alpha \sum_n \frac{\omega_{p\alpha}^2}{n\Omega_\alpha - \omega}\int_0^\infty 2v_\perp dv_\perp \int_{-\infty}^{\infty} dv_z \frac{n\Omega_\alpha v_\perp}{k_\perp}$$

$$\cdot J_n \frac{dJ_n}{d(k_\perp v_\perp/\Omega_\alpha)} \frac{\partial \bar{f}_{\alpha 0}}{\partial v_\perp^2} \qquad (230)$$

$$D_{yy} = 1 - \frac{k_\perp^2 c^2}{\omega^2} - \frac{2\pi}{\omega}\sum_\alpha \sum_n \frac{\omega_{p\alpha}^2}{n\Omega_\alpha - \omega}\int_0^\infty 2v_\perp dv_\perp \int_{-\infty}^{\infty} dv_z v_\perp^2$$

$$\cdot \left[\frac{dJ_n}{d(k_\perp v_\perp/\Omega_\alpha)}\right]^2 \frac{\partial \bar{f}_{\alpha 0}}{\partial v_\perp^2} \qquad (231)$$

$$D_{zz} = 1 - \frac{k_\perp^2 c^2}{\omega^2} - \frac{2\pi}{\omega}\sum_\alpha \sum_n \frac{\omega_{p\alpha}^2}{n\Omega_\alpha - \omega}\int_0^\infty 2v_\perp dv_\perp \int_{-\infty}^{\infty} dv_z v_z^2 J_n^2 \frac{\partial \bar{f}_{\alpha 0}}{\partial v_z^2} \qquad (232)$$

In (229)-(232), the argument of all Bessel functions is $k_\perp v_\perp / \Omega_\alpha$. In the limit of a cold plasma ($v_\perp, v_z \to 0$), we only need to keep the lowest-order, non-vanishing terms in the sum over *n*.

Using (183): $\lim_{x\to 0} J_n(x) = \frac{1}{n!}(\frac{x}{2})^n$; $\lim_{x\to 0} J_{-n}(x) = -\frac{(-1)^n}{n!}(\frac{x}{2})^n$,
we find that the lowest-order, non-vanishing terms for D_{xx} are the $n = \pm 1$ terms. Thus, the sum over n in D_{xx} is

$$\sum_n \frac{\omega_{p\alpha}^2}{n\Omega_\alpha - \omega} \int_0^\infty 2v_\perp dv_\perp \int_{-\infty}^\infty dv_z \frac{n^2\Omega_\alpha^2}{k_\perp^2} J_n^2 \frac{\partial \bar{f}_{\alpha 0}}{\partial v_\perp^2}$$

$$= [\frac{\omega_{p\alpha}^2}{\Omega_\alpha - \omega} - \frac{\omega_{p\alpha}^2}{\Omega_\alpha + \omega}] \int_0^\infty 2v_\perp dv_\perp \int_{-\infty}^\infty dv_z \frac{\Omega_\alpha^2}{k_\perp^2} \frac{k_\perp^2 v_\perp^2}{4\Omega_\alpha^2} \frac{\partial \bar{f}_{\alpha 0}}{\partial v_\perp^2}$$

$$= \frac{\omega\omega_{p\alpha}^2}{\Omega_\alpha^2 - \omega^2} \int_0^\infty dv_\perp^2 \int_{-\infty}^\infty dv_z v_\perp^2 \frac{\partial \bar{f}_{\alpha 0}}{\partial v_\perp^2}$$

$\leftarrow$ integration by parts over $v_\perp^2$

$$= -\frac{\omega\omega_{p\alpha}^2}{\Omega_\alpha^2 - \omega^2} \int_0^\infty dv_\perp^2 \int_{-\infty}^\infty dv_z \bar{f}_{\alpha 0}$$

$\leftarrow$ for a cold plasma: $\bar{f}_{\alpha 0} = \frac{1}{2\pi v_\perp}\delta(v_\perp)\delta(v_z)$

$$= \frac{\omega\omega_{p\alpha}^2}{2\pi(\omega^2 - \Omega_\alpha^2)} \tag{233}$$

6.12 Linear Vlasov Waves in Magnetized Plasma (*continued*)

Sub. (233) into (229), we obtain

$$D_{xx} = 1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2 - \Omega_\alpha^2} \tag{234}$$

Similarly, the lowest-order, non-vanishing terms for D_{xy}, D_{yx}, and D_{yy} are also the $n = \pm 1$ terms, and we obtain

$$D_{xy} = -D_{yx} = -i\sum_\alpha \frac{\omega_{p\alpha}^2 \Omega_\alpha}{\omega(\omega^2 - \Omega_\alpha^2)} \tag{235}$$

$$D_{yy} = 1 - \frac{k_\perp^2 c^2}{\omega^2} - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2 - \Omega_\alpha^2} \tag{236}$$

The lowest-order, non-vanishing term for D_{zz} is the $n = 0$ term,

which gives $$D_{zz} = 1 - \frac{k_\perp^2 c^2}{\omega^2} - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2} \tag{237}$$

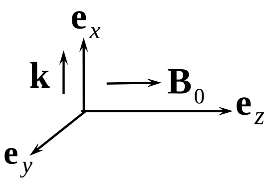
(228) gives $\begin{bmatrix} D_{xx} & D_{xy} \\ -D_{xy} & D_{yy} \end{bmatrix} \begin{bmatrix} E_{1kx} \\ E_{1ky} \end{bmatrix} = 0$ and $D_{zz} E_{1kz} = 0$ (238)

Using (234)-(237), we find from (238) the dispersion relations:

$$\begin{vmatrix} D_{xx} & D_{xy} \\ -D_{xy} & D_{yy} \end{vmatrix} = \begin{vmatrix} 1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2 - \Omega_\alpha^2} & -i \sum_\alpha \frac{\omega_{p\alpha}^2 \Omega_\alpha}{\omega(\omega^2 - \Omega_\alpha^2)} \\ i \sum_\alpha \frac{\omega_{p\alpha}^2 \Omega_\alpha}{\omega(\omega^2 - \Omega_\alpha^2)} & 1 - \frac{k_\perp^2 c^2}{\omega^2} - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2 - \Omega_\alpha^2} \end{vmatrix} = 0 \quad (239)$$

and $$D_{zz} = 1 - \frac{k_\perp^2 c^2}{\omega^2} - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2} = 0 \quad (240)$$

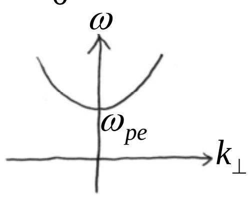
Note: We have assumed $\mathbf{k} = k_\perp \mathbf{e}_x + k_z \mathbf{e}_z$ [(145)]. Thus, with $k_z = 0$, (239) and (240) apply to waves with $\mathbf{k} = k_\perp \mathbf{e}_x$. This explains why D_{xx} and D_{yy} are unequal, although the system is isotropic in x and y



Ordinary mode: (see Nicholson, Sec. 7.9 for a fluid treatment)

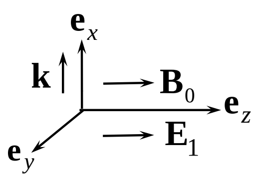
Rewrite $$D_{zz} = 1 - \frac{k_\perp^2 c^2}{\omega^2} - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2} = 0 \quad (240)$$

Since $\omega_{pe} \gg \omega_{pi}$, we may neglect the ion contribution and get



$$\omega^2 - k_\perp^2 c^2 - \omega_{pe}^2 = 0 \quad (241)$$

This is the dispersion relation for the ordinary mode. The ordinary mode is a pure electromagnetic mode, which propagates in a direction perpendicular to $B_0 \mathbf{e}_z$, with the electric field parallel to $B_0 \mathbf{e}_z$. The dispersion relation (241) has the same form as that of electromagnetic waves in an unmagnetized plasma [see (143)] because the electron motion is along $B_0 \mathbf{e}_z$ and hence is unaffected by the external magnetic field.



General properties of modes in $D_{xx}D_{yy} + D_{xy}^2 = 0$:

Rewrite the field equations: $$\begin{bmatrix} D_{xx} & D_{xy} \\ -D_{xy} & D_{yy} \end{bmatrix}\begin{bmatrix} E_{1kx} \\ E_{1ky} \end{bmatrix} = 0 \qquad (238)$$

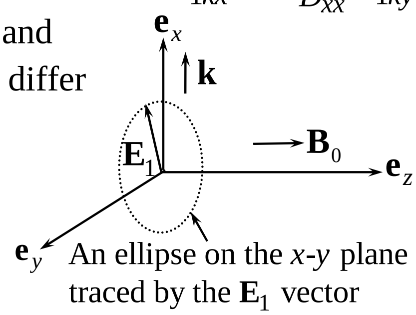
and the dispersion relation: $$D_{xx}D_{yy} + D_{xy}^2 = 0 \qquad (242)$$

(238) gives the following information about the modes in (242):

(1) $\mathbf{E}_1$ $[=E_{1kx}\mathbf{e}_x + E_{1ky}\mathbf{e}_y]$ of these modes lies on the x-y plane.

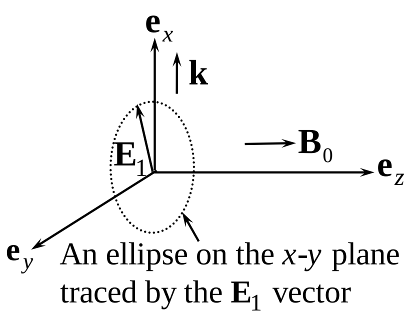
(2) Under (242), either equation in (238) gives $E_{1kx} = -\frac{D_{xy}}{D_{xx}}E_{1ky}$

From (239), we see that D_{xx} is real and D_{xy} is imaginary. Thus, E_{1kx} and E_{1ky} differ by a factor "i", implying E_{1x} and E_{1y} are 90^0 out of phase while having unequal amplitudes ($|E_{1x}| \neq |E_{1y}|$), i.e. $\mathbf{E}_1$ is elliptically polarized.



(3) As shown in the figure, we have

$$\begin{cases} \mathbf{E}_1 = E_{1kx}\mathbf{e}_x + E_{1ky}\mathbf{e}_y \\ \mathbf{k} = k_\perp \mathbf{e}_x \end{cases}$$



Thus, in general, these modes are neither electrostatic ($\mathbf{k} \times \mathbf{E}_1 = 0$) nor electromagnetic ($\mathbf{k} \cdot \mathbf{E}_1 = 0$), except at particular frequencies (such as $\omega \to \infty$) or wave numbers (such as $k_\perp \to \infty$). Consider, for example, the relative amplitude of E_{1kx} and E_{1ky} in the relation:

$$E_{1kx} = -\frac{D_{xy}}{D_{xx}}E_{1ky}$$

If, for some ω or $k_\perp$, we have $D_{xx} \to 0$. Then, $\mathbf{E}_1 \to E_{1kx}\mathbf{e}_x$ and the mode becomes electrostatic. If, $D_{xy} \to 0$ at some ω or $k_\perp$, then $\mathbf{E}_1 \to E_{1ky}\mathbf{e}_y$ and the mode becomes electromagnetic. In either case, $\mathbf{E}_1$ also becomes linearly polarized.

Extraordinary mode: (see Nicholson, Sec. 7.9 for a fluid treatment)

At high frequencies, we may neglect the ion contribution. Then,

(239) can be written
$$\begin{vmatrix} 1-\dfrac{\omega_{pe}^2}{\omega^2-\Omega_e^2} & i\dfrac{\omega_{pe}^2\Omega_e}{\omega(\omega^2-\Omega_e^2)} \\ -i\dfrac{\omega_{pe}^2\Omega_e}{\omega(\omega^2-\Omega_e^2)} & 1-\dfrac{k_\perp^2c^2}{\omega^2}-\dfrac{\omega_{pe}^2}{\omega^2-\Omega_e^2} \end{vmatrix} = 0, \qquad (243)$$

where, as before, $\Omega_e = \dfrac{|e|B_0}{m_e c}$. (243) gives

$$\left(1-\frac{\omega_{pe}^2}{\omega^2-\Omega_e^2}\right)\left(1-\frac{k_\perp^2c^2}{\omega^2}-\frac{\omega_{pe}^2}{\omega^2-\Omega_e^2}\right)-\frac{\omega_{pe}^4\Omega_e^2}{\omega^2(\omega^2-\Omega_e^2)}=0 \qquad (244)$$

After some algebra, (244) can be written

$$k_\perp^2c^2 = \frac{(\omega^2-\omega_{pe}^2)^2-\omega^2\Omega_e^2}{\omega^2-\omega_{pe}^2-\Omega_e^2} = \frac{(\omega^2-\omega_{pe}^2)^2-\omega^2\Omega_e^2}{\omega^2-\omega_{UH}^2}, \quad \left[\begin{matrix}\text{identical to}\\ \text{(7.178)}\end{matrix}\right] \qquad (245)$$

where ω_{UH}^2, called the upper hybrid frequency, is defined as

$$\omega_{UH}^2 = \sqrt{\omega_{pe}^2+\Omega_e^2} \qquad (246)$$

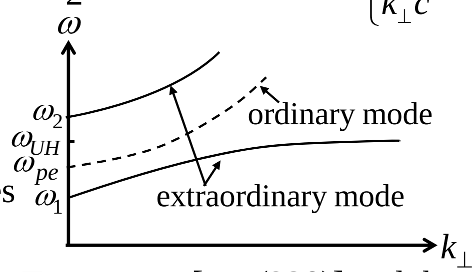
Rewrite
$$k_\perp^2c^2 = \frac{(\omega^2-\omega_{pe}^2)^2-\omega^2\Omega_e^2}{\omega^2-\omega_{UH}^2} \qquad (245)$$

This is the dispersion relation for the extraordinary mode. It has two branches with the following limiting frequencies:

$$\boxed{\Omega_e = |e|B_0/(m_e c)}$$

$$k_\perp = 0 \Rightarrow \omega = \begin{Bmatrix}\omega_1\\ \omega_2\end{Bmatrix} = \sqrt{\omega_{pe}^2+\frac{\Omega_e^2}{4}} \mp \frac{\Omega_e}{2}; \quad k_\perp \to \infty \Rightarrow \omega \to \begin{cases}\omega_{UH}\\ k_\perp c\end{cases}$$

Thus, as shown in the figure, the frequency of the lower branch goes from ω_1 to ω_{UH} and the frequency of the upper branch goes from ω_2 to infinity. Note that at



$\omega = \omega_{UH}$, we have $D_{xx} = 0$. Hence, $\mathbf{E}_1 = E_{1x}\mathbf{e}_x$ [see (238)] and the wave is electrostatic (called upper hybrid resonance). As $\omega \to \infty$, we have $D_{xy} = 0 \Rightarrow \mathbf{E}_1 = E_{1y}\mathbf{e}_y$ and the wave is electromagnetic.

Magnetosonic Wave: (see Nicholson, Sec. 7.12 for a fluid treatment)

Rewrite $$\begin{vmatrix} D_{xx} & D_{xy} \\ D_{yx} & D_{yy} \end{vmatrix} = \begin{vmatrix} 1-\sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2-\Omega_\alpha^2} & -i\sum_\alpha \frac{\omega_{p\alpha}^2\Omega_\alpha}{\omega(\omega^2-\Omega_\alpha^2)} \\ i\sum_\alpha \frac{\omega_{p\alpha}^2\Omega_\alpha}{\omega(\omega^2-\Omega_\alpha^2)} & 1-\frac{k_\perp^2c^2}{\omega^2}-\sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2-\Omega_\alpha^2} \end{vmatrix} = 0 \quad (239)$$

At very low frequencies ($\omega^2 \ll \Omega_i^2$), ions play a major role and we must retain the ion terms in (239). Under the condition: $\omega^2 \ll \Omega_i^2$

$$D_{xx} = 1-\sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2-\Omega_\alpha^2} \approx 1+\sum_\alpha \frac{\omega_{p\alpha}^2}{\Omega_\alpha^2} = 1+\frac{\omega_{pe}^2}{\Omega_e^2}+\frac{\omega_{pi}^2}{\Omega_i^2}$$

$$D_{yy} = 1-\frac{k_\perp^2c^2}{\omega^2}-\sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2-\Omega_\alpha^2} \approx 1-\frac{k_\perp^2c^2}{\omega^2}-\sum_\alpha \frac{\omega_{p\alpha}^2}{\Omega_\alpha^2} \approx 1-\frac{k_\perp^2c^2}{\omega^2}-\frac{\omega_{pi}^2}{\Omega_i^2}$$

$$D_{xy} = -D_{yx} = -i\sum_\alpha \frac{\omega_{p\alpha}^2\Omega_\alpha}{\omega(\omega^2-\Omega_\alpha^2)} \approx i\sum_\alpha \frac{\omega_{p\alpha}^2}{\omega\Omega_\alpha} = \frac{i}{\omega}[-\underbrace{\frac{\omega_{pe}^2}{\Omega_e}}_{=\omega_{pi}^2/\Omega_i}+\frac{\omega_{pi}^2}{\Omega_i}] = 0$$

Thus, (239) gives $D_{xx}D_{yy} = 0$. Because $D_{xx} > 0$, we have

$$D_{yy} = 1-\frac{k_\perp^2c^2}{\omega^2}-\frac{\omega_{pi}^2}{\Omega_i^2} = 0 \quad (247)$$

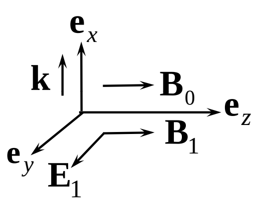
(247) has a form identical to (219) if k_z in (219) is replaced with $k_\perp$. Thus, the solution is simply (221) with k_z changed to $k_\perp$:

$$\omega^2 = \frac{k_\perp^2 V_A^2}{1+V_A^2/c^2} \quad (248)$$

where the Alfven speed V_A is defined in (220) as

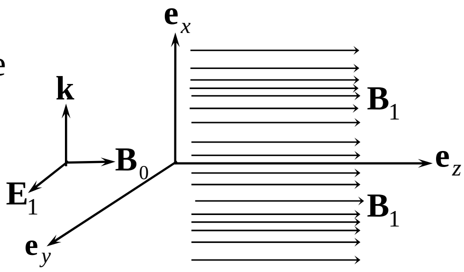
$$V_A = \frac{B_0}{\sqrt{4\pi\rho_i}} \ [\rho_i = n_{i0}m_i] \quad (220)$$

(248) gives the dispersion relation for the <u>magnetosonic wave</u>. Because $D_{yy} = 0$, it has an electric field $\mathbf{E}_1 = E_1\mathbf{e}_y$. Hence, $\mathbf{k}\cdot\mathbf{E}_1 = 0$ and the wave is electromagnetic with $\mathbf{B}_1 = B_1\mathbf{e}_z$.



A physical picture: Like the low-frequency Alfven wave, particle dynamics can be described by 2 types of drift motion. The $E_1\mathbf{e}_y \times B_0\mathbf{e}_z$ drifts move the plasma in the $\pm x$-direction. Since E_1 ($\sim e^{ik_\perp x}$) also varies with x, the drift motion will compress/decompress the plasma, resulting in a density variation along x. On the other hand, the wave magnetic field $B_1\mathbf{e}_z$, when superposed with $B_0\mathbf{e}_z$ will cause a similar density variation of the magnetic field lines (see figure). The field lines are again frozen to the plasma, similar to the Alfven wave. However, the restoring force (thus the oscillation mechanism) is now provided by the magnetic pressure force density: $-\nabla B^2/8\pi$ [(222)].

As in the Alfven wave, there is also a polarization drift current in the y-direction due to the time variation of $E_1\mathbf{e}_y$. This current generates the wave magnetic field $B_1\mathbf{e}_z$, hence the magnetic pressure.



Alternative derivation of the magnetosonic wave dispersion relation:

The polarization drift current is given by

$$\mathbf{J}_p = \frac{\rho_m c^2}{B_0^2}\frac{\partial \mathbf{E}_1}{\partial t} = \frac{-i\omega\rho_m c^2}{B_0^2}\mathbf{E}_{1k} e^{-i\omega t + ik_\perp x}$$

or $$\mathbf{J}_{pk} = \frac{-i\omega\rho_m c^2}{B_0^2}\mathbf{E}_{1k}, \qquad (249)$$

Sub. $\mathbf{J}_{1k} = \mathbf{J}_{pk} = \frac{-i\omega\rho_m c^2}{B_0^2}\mathbf{E}_{1k}$, $\mathbf{E}_{1k} = E_{1ky}\mathbf{e}_y$, and $\mathbf{k} = k_\perp \mathbf{e}_x$ into

the field equation: $$\mathbf{k}\times(\mathbf{k}\times\mathbf{E}_{1k}) + \frac{\omega^2}{c^2}\mathbf{E}_{1k} = -\frac{4\pi i\omega}{c^2}\mathbf{J}_{1k} \qquad (125)$$

we obtain $$\left(k_\perp^2 - \frac{\omega^2}{c^2} - \frac{4\pi\rho_m\omega^2}{B_0^2}\right)E_{1ky}\mathbf{e}_y = 0$$

or $$\omega^2 - k_\perp^2 c^2 + \frac{\omega^2 c^2}{V_A^2} = 0,$$

which gives the same dispersion relation as (248):

Asymptotic behavior of the magnetosonic wave dispersion relation:

The dispersion relation for the magnetosonic wave:

$$\omega^2 = k_\perp^2 V_A^2 /(1+V_A^2/c^2) \qquad (245)$$

is valid under the condition $\omega^2 \ll \Omega_i^2$. It breaks down as $k_\perp \to \infty$. To find the behavior at $k_\perp \to \infty$, we assume $\omega^2 \gg \Omega_i^2$ and $\omega^2 \ll \Omega_e^2$.

Then, $D_{xx} = 1-\sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2-\Omega_\alpha^2} \approx 1+\frac{\omega_{pe}^2}{\Omega_e^2}-\frac{\omega_{pi}^2}{\omega^2}$ and from

$$\begin{vmatrix} D_{xx} & D_{xy} \\ D_{yx} & D_{yy} \end{vmatrix} = \begin{vmatrix} 1-\sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2-\Omega_\alpha^2} & -i\sum_\alpha \frac{\omega_{p\alpha}^2\Omega_\alpha}{\omega(\omega^2-\Omega_\alpha^2)} \\ i\sum_\alpha \frac{\omega_{p\alpha}^2\Omega_\alpha}{\omega(\omega^2-\Omega_\alpha^2)} & 1-\frac{k_\perp^2c^2}{\omega^2}-\sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2-\Omega_\alpha^2} \end{vmatrix} = 0 \qquad (239)$$

we find that, as $k_\perp \to \infty$ but ω remains finite, we must have

$$D_{xx} = 1-\sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2-\Omega_\alpha^2} \approx 1+\frac{\omega_{pe}^2}{\Omega_e^2}-\frac{\omega_{pi}^2}{\omega^2} = 0, \qquad (247)$$

which implies $\mathbf{E}_1 = E_{1x}\mathbf{e}_x$ [see (238)]. Hence, the wave is electrostatic.

Rewrite
$$1+\frac{\omega_{pe}^2}{\Omega_e^2}-\frac{\omega_{pi}^2}{\omega^2} = 0 \qquad (247)$$

(247) gives

$$\omega^2 = \frac{\omega_{pi}^2\Omega_e^2}{\Omega_e^2+\omega_{pe}^2} \underset{\boxed{\text{if } \omega_{pe}^2 \gg \Omega_e^2}}{\approx} \frac{\omega_{pi}^2\Omega_e^2}{\omega_{pe}^2} = \Omega_e\Omega_i = \omega_{LH}^2, \quad [\text{for } k_\perp \to \infty] \qquad (248)$$

where
$$\omega_{LH} = \sqrt{\Omega_e\Omega_i} \qquad (249)$$

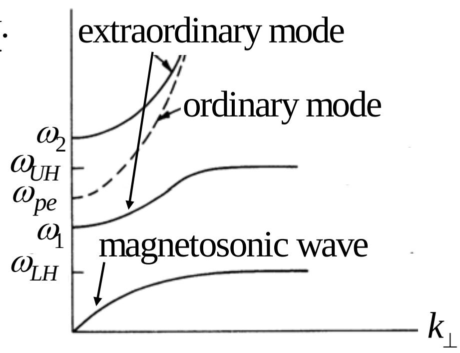
is called the lower hybrid frequency.

This justifies the assumption:

$$\omega^2 \gg \Omega_i^2 \text{ and } \omega^2 \ll \Omega_e^2$$

we made in obtaining (248).

Finally, all the perpendicular modes ($\mathbf{k} \perp \mathbf{B}_0$) discussed so far are summarized in the figure.



Discussion:

(i) We have covered a number of the most familiar modes in a uniform plasma in the framework of the kinetic theory. These modes are treated in Ch. 7 of Nicholson by the fluid theory. However, some other familiar uniform-plasma modes have been left out, for example, the Bernstein modes (Krall & Trivelpiece, Sec. 8.12.3).

(ii) We have only considered waves either along or perpendicular to $B_0\mathbf{e}_z$. In practice, waves can exist at any angle to $B_0\mathbf{e}_z$ (Nicholson, p. 165), with complicated expressions and mixed properties. The general dispersion relation (149) can be the basis for a detailed study of such uncovered uniform-plasma modes.

(iii) There are also modes which are not contained in (149). For example, an inhomogeneous (equilibrium) distribution in density or temperature introduces new modes, such as drift waves (Nicholson, Sec. 7.14; Krall & Trivelpiece, Secs. 8-15 and 8.16). Plasmas in

some devices (e.g. tokamaks) are futher complicated by a complex magnetic field configuration. Such plasmas are usually the subjects of research papers.

(iv) Modes considered in Sec. 6.12 are for a cold plasma. Basic properties and the underlying physics are more clearly exhibited in this limit. However, cold modes are stable because there is no free energy to drive an instability. In the next topic, we will demonstrate show how a mode can become unstable in an anisotropic plasma.

(v) The relativistic Valsov equation can be derived by the same steps as in the derivation of the Vlasov equation in Sec. 6.1. For the case we considered $(\mathbf{E}_0 = 0, \mathbf{B}_0 = B_0\mathbf{e}_z)$, the relativistic factor γ is a constant of the motion in zero-order orbit equations. Hence, the derivation of the relativistic dispersion relation takes exactly the same steps which lead to (171)-(182). In Special Topis I, we will derive the relativistic Vlasov equation and consider a relativistic instability.

A Slow-Wave Instability on the Electron Cyclotron Wave :

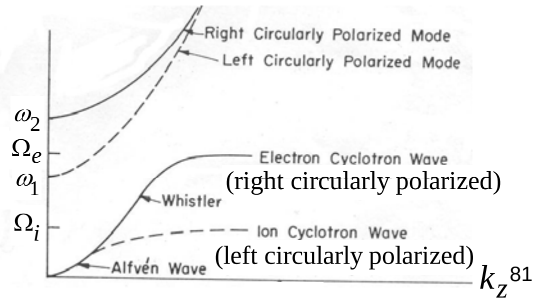
The dispersion relation for "right circularly polarized waves" is

$$\omega^2 - k_z^2c^2 + 2\pi\omega\sum_\alpha \omega_{p\alpha}^2 \int \frac{\frac{\partial \bar f_{\alpha 0}}{\partial v_\perp^2}(1-\frac{k_z v_z}{\omega})+\frac{k_z v_z}{\omega}\frac{\partial \bar f_{\alpha 0}}{\partial v_z^2}}{\omega - k_z v_z + \Omega_\alpha} v_\perp^3 dv_\perp dv_z = 0 \quad (203)$$

At high frequencies, we may neglect the ions. Then, for a cold plasma, (203) reduces to $\omega^2 - k_z^2c^2 - \frac{\omega\omega_{pe}^2}{\omega - \Omega_e} = 0$, $[\Omega_e = \frac{|e|B_0}{m_e c}]$ (214)

which gives the 2 "right circularly polarized" branches in the figure.

Below, we will show that the "electron cyclotron wave" portion of the lower branch can be destabilized by an anisotropy in velocity distribution, resulting in a velocity-space instability.



For an anisotropic plasma, we cannot use (214), but must go back to (203). Again, assume high frequency and neglect the ions. (203) gives

$$\omega^2 - k_z^2c^2 + 2\pi\omega\omega_{pe}^2 \underbrace{\int \frac{\frac{\partial \bar f_{e0}}{\partial v_\perp^2}(1-\frac{k_z v_z}{\omega})+\frac{k_z v_z}{\omega}\frac{\partial \bar f_{e0}}{\partial v_z^2}}{\omega - k_z v_z - \Omega_e} v_\perp^3 dv_\perp dv_z}_{I} = 0 \quad (250)$$

The integral I can be written

$$I = \frac{1}{2}\int \frac{\frac{\partial \bar f_{e0}}{\partial v_\perp}(1-\frac{k_z v_z}{\omega})}{\omega - k_z v_z - \Omega_e} v_\perp^2 dv_\perp dv_z + \frac{1}{2}\int \frac{\frac{k_z}{\omega}\frac{\partial \bar f_{e0}}{\partial v_z}}{\omega - k_z v_z - \Omega_e} v_\perp^3 dv_\perp dv_z$$

$\leftarrow$ $\boxed{\text{integration by parts}}$

$$= -\int \frac{\bar f_{e0}(1-\frac{k_z v_z}{\omega})}{\omega - k_z v_z - \Omega_e} v_\perp dv_\perp dv_z - \frac{1}{2}\int \frac{\frac{k_z^2}{\omega}\bar f_{e0}}{(\omega - k_z v_z - \Omega_e)^2} v_\perp^3 dv_\perp dv_z$$

Now, assume $\bar f_{e0} = \frac{1}{2\pi v_\perp}\delta(v_\perp - v_{\perp 0})\delta(v_z)$, [see (104)] (251)

which represents a uniform distribution of electrons in random-phase gyrational motion, with $v_\perp = v_{\perp 0}$ and $v_z = 0$ for all electrons.

Then, $$I = -\frac{1}{2\pi}\left[\frac{1}{\omega-\Omega_e} + \frac{1}{2\omega}\frac{k_z^2 v_{\perp 0}^2}{(\omega-\Omega_e)^2}\right] \quad (252)$$

Sub. (252) into (250), we obtain the dispersion relation:

$$\omega^2 - k_z^2 c^2 - \omega_{pe}^2\left[\frac{\omega}{\omega-\Omega_e} + \frac{1}{2}\frac{k_z^2 v_{\perp 0}^2}{(\omega-\Omega_e)^2}\right] = 0, \quad (253)$$

which reduces to (214) as $v_{\perp 0} \to 0$.

The two right circularly polarized branches are plotted for $\omega_{pe} = 10\Omega_e$ and $v_{\perp 0} = 0.2c$. The top figure plots ω_r (wave frequency) vs k_z. The upper branch is a fast wave ($\omega_r / k_z > c$), while the lower branch is a slow wave ($\omega_r / k_z < c$). The bottom figure plots ω_i (growth rate) vs k_z. We see that the slow wave is destabilized by the gyrational particles, which feed energy to the wave through cyclotron resonances.