SPECIAL TOPIC I: The Electron Cyclotron Maser

The Relativistic Vlasov Equation:

For a relativistic formalism, we work in the **x-p** space, where **p** is the momentum of a particle. Then, the distribution function is $f(\mathbf{x},\mathbf{p},t)$, implying that $f(\mathbf{x},\mathbf{p},t)d^3xd^3p$ gives the total number of particles in the differential volume d^3xd^3p at point $(\mathbf{x},\mathbf{p})$ and time t.

For the same reason as discussed in Sec. 6.1, in the absence of collisions, $f(\mathbf{x},\mathbf{p},t)$ obeys an equation of continuity of the form:

$$\frac{\partial}{\partial t} f(\mathbf{x},\mathbf{p},t) + \nabla_{\mathbf{x},\mathbf{p}} \cdot [f(\mathbf{x},\mathbf{p},t)(\dot{\mathbf{x}},\dot{\mathbf{p}})] = 0 \qquad (1)$$

In (1), $\nabla_{\mathbf{x},\mathbf{p}}\ [= (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial p_x}, \frac{\partial}{\partial p_y}, \frac{\partial}{\partial p_z})]$ is a 6-dimensional divergence operator, and $(\dot{\mathbf{x}},\dot{\mathbf{p}})\,[= (\dot{x}, \dot{y}, \dot{z}, \dot{p}_x, \dot{p}_y, \dot{p}_z)]$ is a "velocity" vector in the **x-p** space.

Rewrite $\quad \frac{\partial}{\partial t} f(\mathbf{x},\mathbf{p},t) + \nabla_{\mathbf{x},\mathbf{p}} \cdot [f(\mathbf{x},\mathbf{p},t)(\dot{\mathbf{x}},\dot{\mathbf{p}})] = 0 \qquad (1)$

We may derive the relativistic version of the Vlasov equation from (1) by exactly the same method used in Sec. 6.1. The main difference is we now use the relavistic equation of motion:

$$\dot{\mathbf{p}}(= \tfrac{d}{dt}\mathbf{p}) = q(\mathbf{E} + \tfrac{1}{c}\mathbf{v}\times\mathbf{B}) \quad [\text{where } \mathbf{p} = \gamma m\mathbf{v}] \qquad (2)$$

In (1), $\nabla_{\mathbf{x},\mathbf{p}} \cdot [f(\mathbf{x},\mathbf{p},t)(\dot{\mathbf{x}},\dot{\mathbf{p}})] = \nabla_{\mathbf{x}} \cdot (f\dot{\mathbf{x}}) + \nabla_{\mathbf{p}} \cdot (f\dot{\mathbf{p}})$

$\dot{\mathbf{x}} = \mathbf{v}$ →

$$= f\nabla_{\mathbf{x}} \cdot \mathbf{v} + \mathbf{v}\cdot\nabla_{\mathbf{x}} f + f\nabla_{\mathbf{p}} \cdot \dot{\mathbf{p}} + \dot{\mathbf{p}}\cdot\nabla_{\mathbf{p}} f \qquad (3)$$

where $\quad \nabla_{\mathbf{x}} \cdot \mathbf{v} = 0$ [$\because$ **v** is an independent variable] $\qquad (4)$

$$(2) \Rightarrow \nabla_{\mathbf{p}} \cdot \dot{\mathbf{p}} = q\cancel{\nabla_{\mathbf{p}} \cdot \mathbf{E}} + \tfrac{q}{c}\nabla_{\mathbf{p}} \cdot (\mathbf{v}\times\mathbf{B})$$

$\mathbf{p} = \gamma m \mathbf{v}$
γ: relativistic factor
m: rest mass

$$= \tfrac{q}{mc}\nabla_{\mathbf{p}} \cdot (\tfrac{1}{\gamma}\mathbf{p}\times\mathbf{B})$$

$= \mathbf{B}\cdot(\nabla_{\mathbf{p}}\times\mathbf{p}) + \mathbf{p}\cdot(\nabla_{\mathbf{p}}\times\mathbf{B})$
$= 0 + 0 = 0$

$$= \tfrac{q}{mc}[\mathbf{p}\times\mathbf{B}\cdot\nabla_{\mathbf{p}}(\tfrac{1}{\gamma}) + \tfrac{1}{\gamma}\overbrace{\nabla_{\mathbf{p}}\cdot(\mathbf{p}\times\mathbf{B})}]$$

$$= \tfrac{q}{mc}\mathbf{p}\times\mathbf{B}\cdot\nabla_{\mathbf{p}}(\tfrac{1}{\gamma}) \qquad (5)$$

To evaluate $\nabla_{\mathbf{p}}(\frac{1}{\gamma})$, we express γ in terms of p as follows,

$$\gamma^2 m^2 c^2 = \frac{m^2c^2}{1-\frac{v^2}{c^2}} \Rightarrow \gamma^2 m^2 c^2 - \gamma^2 m^2 v^2 = m^2 c^2$$

$p = \gamma m v$

$$\Rightarrow \gamma^2 = 1 + \frac{p^2}{m^2c^2} \Rightarrow \gamma = \left(1 + \frac{p_x^2+p_y^2+p_z^2}{m^2c^2}\right)^{\frac{1}{2}}$$

Thus, $\nabla_{\mathbf{p}}(\frac{1}{\gamma}) = -\frac{1}{\gamma^2}\nabla_{\mathbf{p}}\left(1+\frac{p_x^2+p_y^2+p_z^2}{m^2c^2}\right)^{\frac{1}{2}}$

$$= -\frac{1}{\gamma^2}\frac{1}{2\gamma m^2c^2}[2p_x\mathbf{e}_x + 2p_y\mathbf{e}_y + 2p_z\mathbf{e}_z] = -\frac{\mathbf{p}}{\gamma^3 m^2 c^2}$$

and, from $\nabla_{\mathbf{p}}\cdot\dot{\mathbf{p}} = \frac{q}{mc}\mathbf{p}\times\mathbf{B}\cdot\nabla_{\mathbf{p}}(\frac{1}{\gamma})$ [(5)], we have $\nabla_{\mathbf{p}}\cdot\dot{\mathbf{p}} = 0$ (6)

Sub. (4) and (6) into (3), then sub. the result into (1), we obtain the relativistic Vlasov equation:

$$\frac{d}{dt}f(\mathbf{x},\mathbf{p},t) = \frac{\partial}{\partial t}f + \mathbf{v}\cdot\nabla_{\mathbf{x}}f + \overbrace{q(\mathbf{E}+\frac{1}{c}\mathbf{v}\times\mathbf{B})}^{\dot{\mathbf{p}}}\cdot\nabla_{\mathbf{p}}f = 0, \quad (7)$$

which can be interpreted as "Along a particle's orbit in the **x-p** space, the particle density $f(\mathbf{x},\mathbf{p},t)$ remains unchanged.

Complete Set of Equations: We now have the following set of self-consistent, coupled equations to describe a relativistic plasma:

$$\begin{cases} \frac{\partial}{\partial t}f_\alpha + \mathbf{v}\cdot\nabla f_\alpha + q_\alpha(\mathbf{E}+\frac{1}{c}\mathbf{v}\times\mathbf{B})\cdot\nabla_{\mathbf{p}}f_\alpha = 0, & (7) \\ \nabla\cdot\mathbf{B} = 0 & (8) \\ \nabla\cdot\mathbf{E} = 4\pi\rho & (9) \\ \nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial}{\partial t}\mathbf{B} & (10) \\ \nabla\times\mathbf{B} = \frac{1}{c}\frac{\partial}{\partial t}\mathbf{E} + \frac{4\pi}{c}\mathbf{J} & (11) \end{cases}$$

For simplicity, we shall henthforth denote $\nabla_{\mathbf{x}}$ by ∇.

where
$$\begin{cases} \rho(\mathbf{x},t) = \sum_\alpha q_\alpha \int f_\alpha(\mathbf{x},\mathbf{p},t)d^3p & (12) \\ \mathbf{J}(\mathbf{x},t) = \sum_\alpha q_\alpha \int f_\alpha(\mathbf{x},\mathbf{p},t)\mathbf{v}d^3p & (13) \end{cases}$$

Each particle species, denoted by the subscript "α", is governed by a separate Valsov equation, and q_α carries the sign of the charge.

General Form of Equilibrium Solutions:

The relativistic Vlasov equation can be written

$$\frac{d}{dt} f_\alpha(\mathbf{x},\mathbf{p},t) = \frac{\partial}{\partial t} f_\alpha + \frac{d\mathbf{x}}{dt}\cdot\nabla f_\alpha + \frac{d\mathbf{p}}{dt}\cdot\nabla_{\mathbf{p}} f_\alpha = 0, \qquad (14)$$

where the time differentiation $\frac{d}{dt}$ follows the orbit of a particle. Thus, as in the nonrelativistic case, any function of constants of the motion $C_i = C_i(\mathbf{x},\mathbf{p},t)$, is a solution of relativistic Vlasov equation.

The equilibrium solution (denoted by subscript "0") of interest to us is a steady-state solution formed of constants of the motion with no explicit t-dependence, i.e. $f_{\alpha 0} = f_{\alpha 0}(C_1, C_2 ..)$ with $C_i = C_i(\mathbf{x},\mathbf{p})$.

Examples of Constants of the Motion:

1. If $\mathbf{B}_0 = \mathbf{E}_0 = 0$, γ, p_x, p_y, and p_z are constants of the motion.

2. If $\mathbf{E}_0 = 0$ and $\mathbf{B}_0 = B_0\mathbf{e}_z = const.$, γ, $p_\perp$ and p_z are constants of the motion (note that $\mathbf{p}_\perp$, p_x and p_y are *not* constants of the motion).

3. The relativistic motion of a charged particle (mass m; charge q) in EM fields $\mathbf{A}$ and ϕ is governed by Lagrange's equation: (Goldstein, Poole, & Safko, "Classical Mechanics," 3rd ed., Sec. 7.9)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}, \quad i = 1, 2, 3 \qquad (15)$$

(15) $\Rightarrow \frac{d}{dt}\mathbf{p} = q\mathbf{E} + \frac{q}{c}\mathbf{v}\times\mathbf{B}$. See Heald & Marion "Classical EM Radiation," 3rd ed., Sec. 14.10.

where $\begin{cases} q_i \text{ is a position coordinate.} \\ L = -\frac{1}{\gamma}mc^2 + \frac{q}{c}\mathbf{v}\cdot\mathbf{A} - q\phi \quad [L\text{: relativistic Lagrangian}] \end{cases}$

In cylindrical coordinates, we have $q_i = (r, \theta, z)$, $\dot{q}_i = (\dot{r}, \dot{\theta}, \dot{z})$, $\frac{1}{\gamma} = [1 - (\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2)/c^2]^{1/2}$, and $\mathbf{v}\cdot\mathbf{A} = \dot{r}A_r + r\dot{\theta}A_\theta + \dot{z}A_z$. If $\mathbf{A}$ and ϕ are independent of θ, (15) gives $\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta} = 0$. Hence,

$$\frac{\partial L}{\partial \dot{\theta}} = \gamma m r v_\theta + \frac{q}{c} r A_\theta = const \;\; \text{[canonical angular momentum]} \qquad (16)$$

As a side note, the constant of the motion in (16), $\gamma m r v_\theta + \frac{q}{c} r A_\theta$, is a useful quantity to monitor the accuracy of numerical calculations of relativistic particle motion in, for example, electromagnetic fields.

Examples of Equilibrium Distribution Functions:

As in Ch. 6, we assume that the plasma is immersed in a uniform external magnetic field along the z-axis: $\mathbf{B}_0 = B_0\mathbf{e}_z$, but there is no external electric field ($\mathbf{E}_0 = 0$).

An equilibrium solution $f_{\alpha 0}(\mathbf{p})$ must satisfy the zero-order relativistic Vlasov equation, which for our model is

$$\underbrace{\frac{\partial}{\partial t} f_{\alpha 0}(\mathbf{v})}_{0} + \underbrace{\mathbf{v}\cdot\nabla f_{\alpha 0}(\mathbf{v})}_{0} - q_\alpha(\underbrace{\mathbf{E}_0}_{0} + \tfrac{1}{c}\mathbf{v}\times B_0\mathbf{e}_z)\cdot\nabla_{\mathbf{p}} f_{\alpha 0}(\mathbf{p}) = 0$$

$$\Rightarrow \qquad (\mathbf{v}\times B_0\mathbf{e}_z)\cdot\nabla_{\mathbf{p}} f_{\alpha 0}(\mathbf{p}) = 0 \qquad (17)$$

Thus, any function of the form $f_{\alpha 0}(p_\perp, p_z)$ satisfies (17), provided that the total charge and current densities of all species vanish so that there is no net self field at equilibrium. This in turn makes $p_\perp$ and p_z constants of the motion in the only field present: $B_0\mathbf{e}_z$. In fact, the demonstration in (17) is redundant, since we have already shown that any function of the constants of the motion is an equilibrium solution.

For a weakly relativistic plasma ($kT \ll mc^2$), the Maxwellian distribution function (normalized to n_0) can be approximated by

$$f_0(p) = \frac{n_0}{(2\pi)^{3/2} p_T^3}\exp\left(-\frac{p^2}{2p_T^2}\right), \quad \text{[cf. (102), Ch. 6]} \qquad (17)$$

where $p_T = \sqrt{mkT}$ is the "thermal momentum". The bi-Maxwellian distribution $f_0 = \dfrac{n_0}{(2\pi)^{3/2} p_{T\perp}^2 p_{Tz}}\exp\left(-\dfrac{p_\perp^2}{2p_{T\perp}^2} - \dfrac{p_z^2}{2p_{Tz}^2}\right)$ [cf. (103), Ch. 6] is also a valid equilibrium solution.

We may also have an equilibrium distribution function of the form:

$$f_0(p_\perp, p_z) = \frac{n_0}{2\pi p_\perp}\delta(p_\perp - p_{\perp 0})\delta(p_z), \quad \text{[cf. (104), Ch. 6]} \qquad (18)$$

which represents a uniform distribution (n_0) of particles in random-phase gyrational motion, with $p_\perp = p_{\perp 0}$ and $p_z = 0$ for all particles.

Note: $f_0(p)$ above has the dimension of n_0 / p^3 whereas the non-relativistic $f_0(v)$ has the dimension of n_0 / v^3. Upon the $\int d^3p$ and $\int d^3v$ integrations, respectively, both give results of the same dimension.

First-Order Equations :

Zero-order relativistic particle orbit:

To obtain the first-order solution of (7) by "integrating over the unperturbed orbit" as in Sec. 6.10, we need the zero-order orbit of a particle in $B_0\mathbf{e}_z$. So we begin with a derivation of the relativistic zero-order orbit.

Since γ is a constant of the motion in the static field $B_0\mathbf{e}_z$, we may write the relativistic equation of motion:

$$\frac{d}{dt}\mathbf{p} = \frac{q_\alpha}{c}\mathbf{v}\times B_0\mathbf{e}_z \qquad (2)$$

as

$$\frac{d}{dt}\mathbf{v} = \frac{q_\alpha B_0}{\gamma m_\alpha c}\mathbf{v}\times\mathbf{e}_z = \Omega_\alpha^r \mathbf{v}\times\mathbf{e}_z, \qquad (21)$$

where

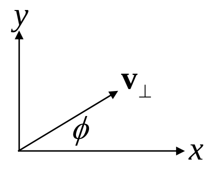
$$\Omega_\alpha^r = \Omega_\alpha / \gamma \qquad (22)$$

and, for later convenience, we have expressed Ω_α^r in terms of the notation for the *rest-mass* cyclotron frequency: $\Omega_\alpha (= \frac{q_\alpha B_0}{m_\alpha c})$.

Rewrite

$$\frac{d}{dt}\mathbf{v} = \Omega_\alpha^r \mathbf{v}\times\mathbf{e}_z \qquad (21)$$

This has the same form as (1.24) of Ch. 1. Thus, under the conditions: $\mathbf{x}'(t'=t)=\mathbf{x}$ and $\mathbf{v}'(t'=t)=\mathbf{v}$, we have



$$\begin{cases} v_x'(t') = v_\perp \cos[\phi - \Omega_\alpha^r (t'-t)] \\ v_y'(t') = v_\perp \sin[\phi - \Omega_\alpha^r (t'-t)] \\ v_z'(t') = v_z \\ x'(t') = x - \frac{v_\perp}{\Omega_\alpha^r}\sin[\phi - \Omega_\alpha^r (t'-t)] + \frac{v_\perp}{\Omega_\alpha^r}\sin\phi \\ y'(t') = y + \frac{v_\perp}{\Omega_\alpha^r}\cos[\phi - \Omega_\alpha^r (t'-t)] - \frac{v_\perp}{\Omega_\alpha^r}\cos\phi \\ z'(t') = v_z (t'-t) + z \end{cases} \qquad (23)$$

(23) and the nonrelativistic orbit equation in (118) of Ch. 6 differ only in the constants $\Omega_\alpha^r \ (= \frac{q_\alpha B_0}{\gamma m_\alpha c})$ and $\Omega_\alpha \ (= \frac{q_\alpha B_0}{m_\alpha c})$. Thus, the relativistic effect has only resulted in the replacement of the rest mass "m_α" with the relativistic mass "γm_α".

Particle dynamics:

We treat the particle dynamics by assuming small perturbations. This allows us to linearize the set of Vlasov/Maxwell equations by

writing
$$\begin{cases} f_\alpha(\mathbf{x},\mathbf{p},t) = f_{\alpha 0}(\mathbf{p}) + f_{\alpha 1}(\mathbf{x},\mathbf{p},t) \\ \mathbf{E}(\mathbf{x},t) = \mathbf{E}_1(\mathbf{x},t) \\ \mathbf{B}(\mathbf{x},t) = B_0\mathbf{e}_z + \mathbf{B}_1(\mathbf{x},t) \\ \rho(\mathbf{x},t) = \rho_1(\mathbf{x},t) \\ \mathbf{J}(\mathbf{x},t) = \mathbf{J}_1(\mathbf{x},t) \end{cases} \quad (24)$$

As before, first-order quantities are denoted by subscript "1".

Sub. the zero-order terms in (24) into the Vlasov/Maxwell equations in (7)-(13) gives the earlier result in (17): $(\mathbf{v}\times B_0\mathbf{e}_z)\cdot\nabla_\mathbf{p} f_{\alpha 0}(\mathbf{p}) = 0$, which is satisfied by any function of the form $f_{\alpha 0}(p_\perp, p_z)$, provided $\rho_0 = \sum_\alpha q_\alpha \int f_{\alpha 0} d^3p = 0$ and $\mathbf{J}_0 = \sum_\alpha q_\alpha \int f_{\alpha 0}\mathbf{v} d^3p = 0$, while the zero-order Maxwell equations give $\mathbf{B}_0 = B_0\mathbf{e}_z$ and $\mathbf{E}_0 = 0$.

Sub. all terms in (24) into the Vlasov/Maxwell equations in (7)-(13) and equating the first-order terms, we obtain the first-order equations:

$$\begin{cases} \frac{\partial}{\partial t} f_{\alpha 1} + \mathbf{v}\cdot\nabla f_{\alpha 1} + \frac{q_\alpha}{c}(\mathbf{v}\times B_0\mathbf{e}_z)\cdot\nabla_\mathbf{p} f_{\alpha 1} \\ \qquad = -q_\alpha(\mathbf{E}_1 + \frac{1}{c}\mathbf{v}\times\mathbf{B}_1)\cdot\nabla_\mathbf{p} f_{\alpha 0} & (25) \\ \nabla\cdot\mathbf{B}_1 = 0 & (26) \\ \nabla\cdot\mathbf{E}_1 = 4\pi\rho_1 & (27) \\ \nabla\times\mathbf{E}_1 = -\frac{1}{c}\frac{\partial}{\partial t}\mathbf{B}_1 & (28) \\ \nabla\times\mathbf{B}_1 = \frac{1}{c}\frac{\partial}{\partial t}\mathbf{E}_1 + \frac{4\pi}{c}\mathbf{J}_1 & (29) \end{cases}$$

where
$$\begin{cases} \rho_1(\mathbf{x},t) = \sum_\alpha q_\alpha \int f_{\alpha 1}(\mathbf{x},\mathbf{v},t) d^3p & (30) \\ \mathbf{J}_1(\mathbf{x},t) = \sum_\alpha q_\alpha \int f_{\alpha 1}(\mathbf{x},\mathbf{v},t)\mathbf{v} d^3p & (31) \end{cases}$$

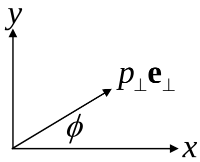
Rewrite (25):

$$\underbrace{\frac{\partial}{\partial t} f_{\alpha 1} + \mathbf{v}\cdot\nabla f_{\alpha 1} + \frac{q_\alpha}{c}(\mathbf{v}\times B_0\mathbf{e}_z)\cdot\nabla_{\mathbf{p}} f_{\alpha 1}}_{\frac{d}{dt} f_{\alpha 1}} = -q_\alpha(\mathbf{E}_1 + \tfrac{1}{c}\mathbf{v}\times\mathbf{B}_1)\cdot\nabla_{\mathbf{p}} f_{\alpha 0},$$

The LHS is a total time derivative along the *zero-order* particle orbit because the acceleration force is $\frac{q_\alpha}{c}\mathbf{v}\times B_0\mathbf{e}_z$. Hence, the RHS of (25) is to be integrated along the zero-order particle orbit, and γ is a constant for the integration. With $\gamma = const$ and

$$f_{\alpha 0} = f_{\alpha 0}(p_\perp, p_z) = f_{\alpha 0}(\gamma m_\alpha v_\perp, \gamma m_\alpha v_z),$$

we may convert the $\nabla_{\mathbf{p}}$ operation into a $\nabla_{\mathbf{v}}$ operation.



$$\nabla_{\mathbf{p}} f_{\alpha 0}(p_\perp, p_z) = \frac{\partial f_{\alpha 0}}{\partial p_\perp}\mathbf{e}_\perp + \frac{1}{p_\perp}\frac{\partial f_{\alpha 0}}{\partial \phi}\mathbf{e}_\phi + \frac{\partial f_{\alpha 0}}{\partial p_z}\mathbf{e}_z$$

$$= \frac{1}{\gamma m_\alpha}\left(\frac{\partial f_{\alpha 0}}{\partial v_\perp}\mathbf{e}_\perp + \frac{1}{v_\perp}\frac{\partial f_{\alpha 0}}{\partial \phi}\mathbf{e}_\phi + \frac{\partial f_{\alpha 0}}{\partial v_z}\mathbf{e}_z\right)$$

$$= \frac{1}{\gamma m_\alpha}\nabla_{\mathbf{v}} f_{\alpha 0}(\gamma m_\alpha v_\perp, \gamma m_\alpha v_z) \qquad (32)$$

Sub. $\nabla_{\mathbf{p}} f_{\alpha 0}(p_\perp, p_z) = \frac{1}{\gamma m_\alpha}\nabla_{\mathbf{v}} f_{\alpha 0}(\gamma m_\alpha v_\perp, \gamma m_\alpha v_z)$ [(32)] into (25):

$$\underbrace{\frac{\partial}{\partial t} f_{\alpha 1} + \mathbf{v}\cdot\nabla f_{\alpha 1} + \frac{q_\alpha}{c}(\mathbf{v}\times B_0\mathbf{e}_z)\cdot\nabla_{\mathbf{p}} f_{\alpha 1}}_{\frac{d}{dt} f_{\alpha 1}} = -q_\alpha(\mathbf{E}_1 + \tfrac{1}{c}\mathbf{v}\times\mathbf{B}_1)\cdot\nabla_{\mathbf{p}} f_{\alpha 0}$$

we obtain

$$\frac{d}{dt} f_{\alpha 1} = -\frac{q_\alpha}{\gamma m_\alpha}(\mathbf{E}_1 + \tfrac{1}{c}\mathbf{v}\times\mathbf{B}_1)\cdot\nabla_{\mathbf{v}} f_{\alpha 0}(\gamma m_\alpha v_\perp, \gamma m_\alpha v_z) \qquad (33)$$

Comparing (33) with its nonrelativistic counterpart [(117) of Ch. 6]:

$$\frac{d}{dt} f_{\alpha 1} = -\frac{q_\alpha}{m_\alpha}(\mathbf{E}_1 + \tfrac{1}{c}\mathbf{v}\times\mathbf{B}_1)\cdot\nabla_{\mathbf{v}} f_{\alpha 0}(v_\perp, v_z)$$

we find that, except for the dimensions of $f_{\alpha 0}$, the 2 equations differ only in the multiplication factor γ on m_α in (33), which is the same difference between relativistic and nonrelativistic orbit equations [see (23)]. Thus, the solution of (33) is simply the nonrelativistic solution [(162) of Ch. 6] with "m_α" replaced by "γm_α" and "$f_{\alpha 0}(v_\perp, v_z)$" replaced by $f_{\alpha 0}(p_\perp, p_z)$. Note that in (162) of Ch. 6, $f_{\alpha 0}$ is contained in the definitions of X, Y, and Z [see (158) of Ch. 6].

Throughout the derivation of the nonrelativistic dispersion relation, there are only two quantities which contain the rest mass m_α:

$$\Omega_\alpha \ (=\frac{q_\alpha B_0}{m_\alpha c}) \text{ and } \omega_{p\alpha}^2 \ (=\frac{4\pi n_{\alpha 0} q_\alpha^2}{m_\alpha})$$

For later convenience, we will retain the above notations for the *rest-mass* cyclotron and plasma frequencies, and write the *relativistic* cyclotron and plasma frequencies as Ω_α / γ and $\omega_{p\alpha}^2 / \gamma$, respectively.

The dimensional difference between $f_0(p_\perp, p_z)$ and $f_{\alpha 0}(v_\perp, v_z)$,

e.g.
$$\begin{cases} f_0(p_\perp, p_z) = \dfrac{n_0}{2\pi p_\perp}\delta(p_\perp - p_{\perp 0})\delta(p_z) \\ f_0(v_\perp, v_z) = \dfrac{n_0}{2\pi v_\perp}\delta(v_\perp - v_{\perp 0})\delta(v_z) \ \ [(104), \text{Ch. } 6] \end{cases} \quad (18)$$

will be rectified when we carry out the $\int d^3 p$ integration to obtain $\mathbf{J}_1$ [see (31)] and subsequently the dispersion relation, because in the nonrelativistic formalsim, we have used the $\int d^3 v$ integration. [see (163) and (180) of Ch. 6].

Relativistic Dispersion Relation for a Plasma in a Uniform External Magnetic Field $\mathbf{B}_0 = B_0 \mathbf{e}_z$:

Based on the discussion above, we have the following recipe for converting the nonrelativistic dispersion relation for a plasma in a uniform external magnetic field $B_0 \mathbf{e}_z$ into a relativistic one.

In any of the equations in Sec. 6.12,

1. Replace "m_α" with "γm_α" or $\Omega_\alpha \ (=\frac{q_\alpha B_0}{m_\alpha c})$ with Ω_α / γ and $\omega_{p\alpha}^2 \ (=\frac{4\pi n_{\alpha 0} q_\alpha^2}{m_\alpha})$ with $\omega_{p\alpha}^2 / \gamma$. (*Note*: if there is an integral, γ must be part of the integrand because γ is a function of $p_\perp$ and p_z.)
2. Replace the integration symbol "$\int_0^\infty v_\perp dv_\perp \int_{-\infty}^\infty dv_z$" with the symbol "$\int_0^\infty p_\perp dp_\perp \int_{-\infty}^\infty dp_z$".
3. Replace "$f_{\alpha 0}(v_\perp, v_z)$" with "$f_{\alpha 0}(p_\perp, p_z)$".
4. All other quantities, such as $v_\perp$ and v_z, remain unchanged. But we may interchange $v_\perp$ with $p_\perp / (\gamma m_\alpha)$, and v_z with $p_z / (\gamma m_\alpha)$.

A specific example: Rewrite the dispersion relation for "right circularly polarized waves" [(203) of Ch. 6]:

$$\omega^2 - k_z^2c^2 + 2\pi\omega\sum_\alpha \omega_{p\alpha}^2 \int \frac{\frac{\partial \bar{f}_{\alpha 0}}{\partial v_\perp^2}(1-\frac{k_z v_z}{\omega})+\frac{k_z v_z}{\omega}\frac{\partial \bar{f}_{\alpha 0}}{\partial v_z^2}}{\omega - k_z v_z + \Omega_\alpha} v_\perp^3 dv_\perp dv_z = 0$$

By steps 1 and 2, we get the relativistiv version of the above dispersion relation:

$$\omega^2 - k_z^2c^2 + 2\pi\omega\sum_\alpha \int \frac{\omega_{p\alpha}^2}{\gamma} \frac{\frac{\partial \bar{f}_{\alpha 0}}{\partial v_\perp^2}(1-\frac{k_z v_z}{\omega})+\frac{k_z v_z}{\omega}\frac{\partial \bar{f}_{\alpha 0}}{\partial v_z^2}}{\omega - k_z v_z + \Omega_\alpha/\gamma} v_\perp^2 p_\perp dp_\perp dp_z = 0 \quad (34)$$

where, by step 3, it is understood that $\bar{f}_{\alpha 0} = \bar{f}_{\alpha 0}(p_\perp, p_z)$.

By step 4, we replace $v_\perp$ with $p_\perp/(\gamma m_\alpha)$, and v_z with $p_z/(\gamma m_\alpha)$

$$\omega^2 - k_z^2c^2 + 2\pi\omega\sum_\alpha \omega_{p\alpha}^2 \int \frac{\frac{\partial \bar{f}_{\alpha 0}}{\partial p_\perp^2}(1-\frac{k_z p_z}{\omega\gamma m_\alpha})+\frac{k_z p_z}{\omega\gamma m_\alpha}\frac{\partial \bar{f}_{\alpha 0}}{\partial p_z^2}}{\gamma\omega - k_z p_z/m_\alpha + \Omega_\alpha} p_\perp^3 dp_\perp dp_z = 0$$

(35)

A Fast-Wave Instability : Rewrite (35) in slightly different form and neglect ions (for high-frequency waves):

$$\omega^2 - k_z^2c^2 + \pi\omega_{pe}^2 \int \frac{\frac{\partial \bar{f}_{e0}}{\partial p_\perp} p_\perp(\omega-\frac{k_z p_z}{\gamma m_e})+\frac{k_z}{\gamma m_e}p_\perp^2\frac{\partial \bar{f}_{e0}}{\partial p_z}}{\gamma\omega - k_z p_z/m_e - \Omega_e} p_\perp dp_\perp dp_z = 0 \quad (36)$$

From $\gamma = (1+\frac{p_\perp^2+p_z^2}{m^2c^2})^{\frac{1}{2}}$ [see derivation between (5) and (6)], we have

$$\frac{\partial\gamma}{\partial p_\perp} = \frac{p_\perp}{\gamma m_e^2c^2};\ \frac{\partial\gamma}{\partial p_z} = \frac{p_z}{\gamma m_e^2c^2};\ \frac{\partial}{\partial p_\perp}\frac{1}{\gamma} = -\frac{p_\perp}{\gamma^3 m_e^2c^2};\ \frac{\partial}{\partial p_\perp}\frac{1}{\gamma} = -\frac{p_\perp}{\gamma^3 m_e^2c^2} \quad (37)$$

Integrating (36) by parts and using (37), we obtain (after some algebraic manipulations)

$$\omega^2 - k_z^2c^2 - \pi\omega_{pe}^2 \int p_\perp dp_\perp dp_z \frac{\bar{f}_{e0}}{\gamma} \cdot\left[\frac{\omega-\frac{k_z p_z}{\gamma m_e}}{\omega-\frac{k_z p_z}{\gamma m_e}-\frac{\Omega_e}{\gamma}} - \frac{p_\perp^2(\omega^2-k_z^2c^2)}{2\gamma^2 m_e^2c^2(\omega-\frac{k_z p_z}{\gamma m_e}-\frac{\Omega_e}{\gamma})^2}\right] = 0 \quad (38)$$

Rewrite $\omega^2 - k_z^2 c^2 - 2\pi\omega_{pe}^2 \int p_\perp dp_\perp dp_z \frac{\overline{f}_{e0}}{\gamma}$

$$\cdot\left[\frac{\omega - \frac{k_z p_z}{\gamma m_e}}{\omega - \frac{k_z p_z}{\gamma m_e} - \frac{\Omega_e}{\gamma}} - \frac{p_\perp^2(\omega^2 - k_z^2 c^2)}{2\gamma^2 m_e^2 c^2 (\omega - \frac{k_z p_z}{\gamma m_e} - \frac{\Omega_e}{\gamma})^2}\right] = 0 \qquad (38)$$

In integrating (36) by parts, we have differentiated γ with respect $p_\perp$ and p_z, with brings out the relativistic effects in the sense that the relativistic mass γm_e depends on $p_\perp$ and p_z. Terms arising from these differentiations can be combined into a single term proportional to ω^2, as appears on the RHS of (38).

We now specialize to $\overline{f}_0(p_\perp, p_z) = \frac{1}{2\pi p_\perp}\delta(p_\perp - p_{\perp 0})\delta(p_z)$ [(18)]. Then, (38) reduces to a simple dispersion relation given by

$$\omega^2 - k_z^2 c^2 - \frac{\omega_{pe}^2}{\gamma_0}\left[\frac{\omega}{\omega - \frac{\Omega_e}{\gamma_0}} + \frac{k_z^2 v_{\perp 0}^2 (1 - \omega^2/k_z^2 c^2)}{2(\omega - \frac{\Omega_e}{\gamma_0})^2}\right] = 0, \qquad (39)$$

where γ_0 and $v_{\perp 0}$ are the equilibrium values of γ and $v_\perp$, respectively.

Rewrite $\omega^2 - k_z^2 c^2 - \frac{\omega_{pe}^2}{\gamma_0}\left[\frac{\omega}{\omega - \frac{\Omega_e}{\gamma_0}} + \frac{k_z^2 v_{\perp 0}^2 (1 - \omega^2/k_z^2 c^2)}{2(\omega - \frac{\Omega_e}{\gamma_0})^2}\right] = 0, \qquad (39)$

and the nonorelativistic version in (253) of Ch. 6:

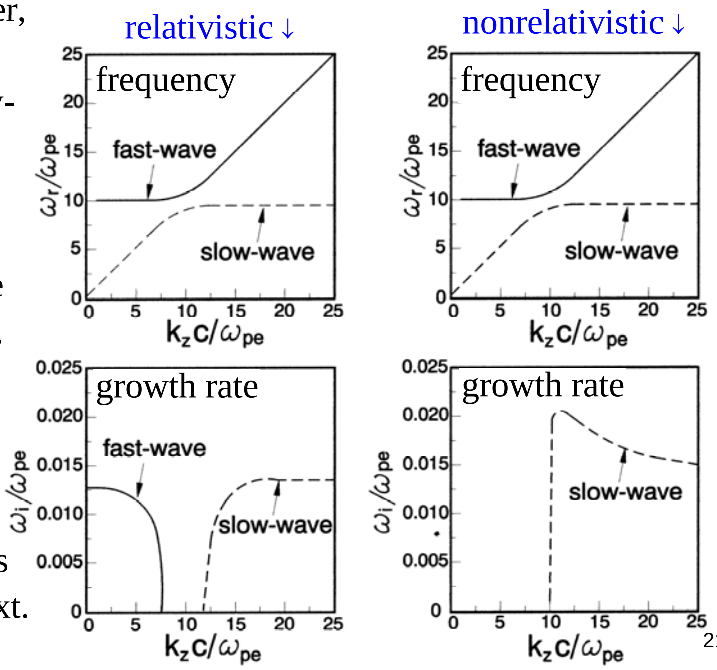
$$\omega^2 - k_z^2 c^2 - \omega_{pe}^2\left[\frac{\omega}{\omega - \Omega_e} + \frac{1}{2}\frac{k_z^2 v_{\perp 0}^2}{(\omega - \Omega_e)^2}\right] = 0,$$

which was also obtained for the right circularly polarized wave and for a similar equilibrium distribution: $\overline{f}_{e0} = \frac{1}{2\pi v_\perp}\delta(v_\perp - v_{\perp 0})\delta(v_z)$

Comparing the two expressions, we find that the plasma frequency and cyclotron frequency are both modified by the relativistic factor γ_0, as expected. In addition, (39) differs from the nonrelativistic version by the presence of the ω^2 term on the RHS. The term has been shown to be of purely relativistic origin.

(39) is plotted on the next page for $\Omega_e / (\gamma_0 \omega_{pe}) = 10$ and $\gamma_0 = 1.02$ $(v_{\perp 0} \approx 0.2c)$, along with the nonrelativistic results shown in Sec. 6.12.

In the relativistic model, both the slow- and fast-wave branches are unstable; however, in the nonrelativistic model, only the slow-wave branch is unstable. Obviously, the relativistic effect is responsible for the fast-wave instability, which is driven by energetic electrons through cyclotron resonances. The physical mechanisms will be discussed next.



Physicsl Interpretation of Fast- and Slow-Wave Instabilities :

Effective cyclotron frequency:

The electron-wave resonance condition is [see the denominator in (34)]

$$\omega - k_z v_z - \frac{\Omega_e}{\gamma} \approx 0, \qquad (40)$$

which implies that the electron sees almost a static wave electric field and hence loses or gains energy for an extended period of time.

Write (40) as

$$\omega - \Omega_{eff} \approx 0, \qquad (41)$$

where

$$\Omega_{eff} \equiv k_z v_z + \frac{\Omega_e}{\gamma} \qquad (42)$$

is defined as the effective cyclotron frequency, which governs the degree of synchronism of each electron with respect to the wave. In (42), a change of γ (hence the relativistic cyclotron frequency Ω_e/γ) affects the synchronism as expected. In addition, since the wave phase varies with z, a change of v_z affects the electron's z-coordinate, hence also the electron's synchronism with the wave.

Effective cyclotron phase space bunching:

Since $\overline{f}_0(p_\perp, p_z) = \frac{1}{2\pi p_\perp}\delta(p_\perp - p_{\perp 0})\delta(p_z)$ is independent of ϕ. It represents a *uniform* (or random) distribution of electrons along any cyclotron orbit. Thus, for every electron which gains energy from the wave (electric field), another electron (180° out of phase) will lose the same amount of energy. This results in zero energy exchange between all electrons and the wave. So, to have a net exchange of energy, the electrons must first be "bunched", which gives rise to an AC current.

It is convenient to visualize the electron bunching by keeping track of each electron's variation in Ω_{eff}. If, for example, v_z and γ of all electrons remain at their initial values 0 and γ_0, respectively, Ω_{eff} will be constant for all electrons. Thus, no bunching occurs. However, v_z and γ will have different variations in the wave fields, causing the electrons to bunch in the effective cyclotron phase space.

Rewrite $\omega - \Omega_{eff} \approx 0 \quad [\Omega_{eff} \equiv k_z v_z + \frac{\Omega_e}{\gamma}]$ (41)

In the presence of a wave, each electron's v_z and γ will vary according to

$$\frac{d}{dt}(\gamma m_e \mathbf{v}) = -e\mathbf{E}_\perp - \frac{e}{c}\mathbf{v}\times(B_0\mathbf{e}_z + \mathbf{B}_\perp), \quad (43)$$

where $\mathbf{E}_\perp$ and $\mathbf{B}_\perp$ are the wave electromagnetic fields, which we denote ny subscript "$\perp$" because they are perpendicular to $\mathbf{e}_z$.

The increment of Ω_{eff} over an infinistesimal time interval Δt is

$$\Delta\Omega_{eff} = k_z \Delta v_z + \Omega_e \Delta(\tfrac{1}{\gamma}) = k_z \Delta v_z - \frac{\Omega_e}{\gamma^2}\Delta\gamma, \quad (44)$$

where Δv_z may be evaluated from the z-component of (43):

$$\frac{d}{dt}(\gamma m_e v_z) = -\frac{e}{c}(\mathbf{v}_\perp \times \mathbf{B}_\perp)\cdot\mathbf{e}_z = -\frac{ek_z}{\omega}\mathbf{v}_\perp \cdot \mathbf{E}_\perp, \quad (45)$$

where, by $\nabla\times\mathbf{E}_\perp = -\frac{1}{c}\frac{\partial}{\partial t}\mathbf{B}_\perp$, we have sub. $\frac{k_z c}{\omega}\mathbf{E}_\perp$ for $\mathbf{B}_\perp$.

Noting that the zero-order v_z is 0, we obtain from (45)

$$\Delta v_z = -\frac{ek_z}{\gamma m_e \omega}(\mathbf{v}_\perp \cdot \mathbf{E}_\perp)\Delta t \quad (46)$$

The change of γ is determined by the work done on the electron:

$$\Delta\gamma = -\frac{e}{m_e c^2}(\mathbf{v}_\perp \cdot \mathbf{E}_\perp)\Delta t \qquad (47)$$

Sub. (47) and $\Delta v_z = -\frac{ek_z}{\gamma m_e \omega}(\mathbf{v}_\perp \cdot \mathbf{E}_\perp)\Delta t$ [(46)] into

$$\Delta\Omega_{eff} = k_z \Delta v_z - \frac{\Omega_e}{\gamma^2}\Delta\gamma \qquad (44)$$

we obtain

$$\Delta\Omega_{eff} = \frac{e}{\gamma m_e \omega}\left(\frac{\Omega_e \omega}{\gamma c^2} - k_z^2\right)(\mathbf{v}_\perp \cdot \mathbf{E}_\perp)\Delta t$$

$$\approx \frac{e}{\gamma m_e \omega}\left(\frac{\omega^2}{c^2} - k_z^2\right)(\mathbf{v}_\perp \cdot \mathbf{E}_\perp)\Delta t \qquad (48)$$

where we have made use of $\omega - k_z v_z - \frac{\Omega_e}{\gamma} \approx 0$ [(40)] and $v_z \approx 0$.

In (48), the terms proportional to ω^2 and k_z^2 are due to $\Delta\gamma$ (a relativistic effect) and Δv_z (a nonrelativistic effect), respectively. Because $\mathbf{v}_\perp \cdot \mathbf{E}_\perp$ (hence $\Delta\Omega_{eff}$) are different for different electrons, bunching occurs. Furthermore, the two terms are joined by a "$-$" sign, indicating the two bunching mechanisms are *competitive.*

Rewrite

$$\Delta\Omega_{eff} \approx \frac{e}{\gamma m_e \omega}\left(\frac{\omega^2}{c^2} - k_z^2\right)(\mathbf{v}_\perp \cdot \mathbf{E}_\perp)\Delta t \qquad (48)$$

Since the two bunching mechanisms are competitive, the non-relativistic model overestimtes the growth rate of the slow-wave instability, which is apparent as we compare the relativistic and non-relativistic slow-wave growth rates in the figure a few pages back.

(48) also shows that the relativistiv mechanism dominates for fast waves ($\omega / k_z > 0$), while the nonrelativistiv mechanism dominates for slow waves ($\omega / k_z < 0$). This explains why there is no growth at the borderline ($\omega / k_z \approx 0$).

The Electron Cyclotron Maser:

The fast-wave instability, known as the electron cyclotron maser, is the basis of a powerful radiation source called the gyrotron, which occupies a unique position in the millimeter and submillimeter regions of the electromagnetic wave spectrum [Ref.: K. R. Chu, "The Electron Cyclotron Maser", Rev. of Modern Phys. 76, 489-540 (2004)]