SPECIAL TOPIC II: Convective and Absolute Instabilities

Ref.: R. J. Briggs, "Electron-Stream Interaction with Plasmas," (The MIT Press, Cambridge, MA), Ch. 2.

Our analysis thus far has restricted the first-order solutions to a single normal mode (e.g. a Fourier component of the form $e^{ik_z z - i\omega t}$), which represents a spatially periodic perturbation of infinite extent. The dispersion relation, in general, yields a complex ω for a range of k_z values. For each k_z, the solution ω tells the temporal behavior of the wave of infinite extent (frequency and growth/damping rates).

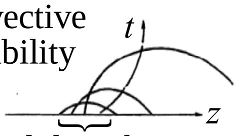
Now, consider a *localized* disturbance, which is a superposition of an infinite number of normal modes with $-\infty < k_z < \infty$. Then, if the plasma is unstable, many normal modes will grow simultaneously. Intuitively, their constructive and destructive interferences may result in a spatial field profile quite different from that of a single mode.

As will be shown below, an instability may grow in time while travelling away from the region of the initial disturbance (of finite duration) so that its amplitude at any *fixed* point along the wave path will grow for a finite length of time, and eventually decay to 0 (left figure). This is called a convective instability (or amplifying wave).

The wave may also grow around the initial disturbance and eventually spread out to every point in space (based on the linear theory, see right figure). This is called an absolute instability.

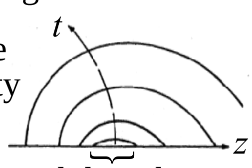
Obviously, the classification of convective/absolute instabilities is frame-dependent, bacause a convectove instability may appear as an absolute instability to an observer moving with the wave.

convective instability



initial disturbance

absolute instability



initial disturbance

Green Function Formalism :

For simplicity, we consider a one-dimensional system in z, i.e. any field component ψ is independent of the transverse coordinates. Then, the Green function $G(z, z', t, t')$ is the response of the system at point z and time t to the excitation of a point source at point $\mathbf{x}'$ and time t'. Assume that the system is unbounded (i.e. no source on the boundary surface), then, $G(z, z', t, t')$ can be written $G(z-z', t-t')$ by reason of symmetry. Assume furthre that a source $s(z', t')$ is turned on at $t=0$ and $s(z', t')$ takes the form: $s(z', t') = g(z')f(t')$ (1)

with $f(t') = 0$ for $t < 0$ (2)

Then, by the principle of linear superposition, we have

$$\psi(z,t) = \int_{-\infty}^{\infty} dz' \int_0^t dt' G(z-z', t-t') g(z') f(t'), \tag{3}$$

where the upper limit of the t'-integration is set at t because the response at t cannot be affected by the source behavior after t.

Before proceeding, we pause for a review of needed theorems.

(1) The <u>convolution theorem for Fourier transform</u> states that the Fourier transform of the <u>convolution</u> of $h_1(x)$ and $h_2(x)$ is given by

$$\int_{-\infty}^{\infty} [\underbrace{\int_{-\infty}^{\infty} h_1(x-\xi) h_2(\xi) d\xi}] e^{-ikx} dx = \phi_1(k)\phi_2(k) \tag{4}$$

called the convolution of $h_1(x)$ and $h_2(x)$

where
$$\begin{cases} \phi_{1,2}(k) = \int_{-\infty}^{\infty} h_{1,2}(x) e^{-ikx} dx & (5) \\ h_{1,2}(x) = \dfrac{1}{2\pi} \int_{-\infty}^{\infty} \phi_{1,2}(k) e^{ikx} dk & (6) \end{cases}$$

Proof: LHS of (4) $= \int_{-\infty}^{\infty} h_2(\xi) d\xi \int_{-\infty}^{\infty} h_1(x-\xi) e^{-ikx} dx$

Let $\eta = x - \xi$ ($\Rightarrow dx = d\eta$)

$$= \int_{-\infty}^{\infty} h_2(\xi) d\xi \int_{-\infty}^{\infty} h_1(\eta) e^{-ik(\xi+\eta)} d\eta$$

$$= \int_{-\infty}^{\infty} h_2(\xi)^{-ik\xi} d\xi \int_{-\infty}^{\infty} h_1(\eta) e^{-ik\eta} d\eta$$

$$= \phi_1(k)\phi_2(k)$$

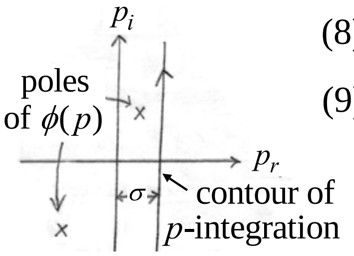
(2) The convolution theorem for Laplace transform states that the Laplace transform of the Laplace convolution of $h_1(t)$ and $h_2(t)$ is given by

$$\int_0^\infty [\underbrace{\int_0^t h_1(t-\tau)h_2(\tau)d\tau}]e^{-pt}dt = \phi_1(p)\phi_2(p) \tag{7}$$

$\boxed{\text{called the Laplace convolution of } h_1(t) \text{ and } h_2(t)}$

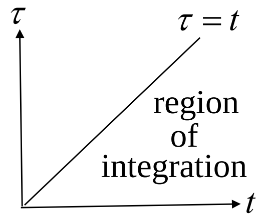
where

$$\begin{cases} \phi_{1,2}(p) = \int_0^\infty h_{1,2}(t)e^{-pt}dt & (8) \\ h_{1,2}(t) = \dfrac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \phi_{1,2}(p)e^{pt}dp & (9) \end{cases}$$



and the contour for the p-integration is shown in the upper figure, where σ is sufficiently large so that all the poles of $\phi(p)$ lie to the left of the p-contour.

Proof: Invert the order of integration in (7) and integrate over the same region in τ-t space.



$$\Rightarrow \quad \text{LHS} = \int_0^\infty h_2(\tau)[\int_\tau^\infty h_1(t-\tau)e^{-pt}dt]d\tau$$

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(let $t = \mu + \tau \Rightarrow dt = d\mu; \; t = 0 \leftrightarrow \mu = 0$)

$$= \int_0^\infty h_2(\tau)[\int_0^\infty h_1(\mu)e^{-p(\mu+\tau)}d\mu]d\tau$$

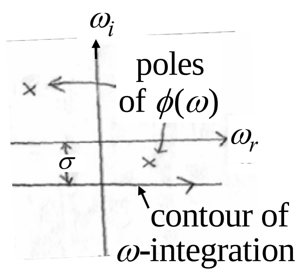
$$= \int_0^\infty h_2(\tau)e^{-p\tau}d\tau\int_0^\infty h_1(\mu)e^{-p\mu}d\mu = \phi_1(p)\phi_2(p)$$

Here, to follow Briggs, we convert the variable p to ω through

$$p = i\omega \tag{10}$$

Then, in terms of ω, the Laplace transform formulae become

$$\begin{cases} \phi(\omega) = \int_0^\infty \phi(t)e^{-i\omega t}dt & (11) \\ \phi(t) = \dfrac{1}{2\pi}\int_{-\infty-i\sigma}^{\infty-i\sigma} \phi(\omega)e^{i\omega t}d\omega & (12) \end{cases}$$



where the contour for the ω-integration is as shown in the figure to the right.

The convolution theorem for Laplace transform can be written in terms of ω as

$$\int_0^\infty [\int_0^t h_1(t-\tau)h_2(\tau)d\tau]e^{-i\omega t}dt = \phi_1(\omega)\phi_2(\omega) \tag{13}$$

Now return to $\psi(z,t)=\int_{-\infty}^{\infty}dz'\int_0^t dt' G(z-z',\, t-t')g(z')f(t')$ (3)

Perform a Laplace transform in t (with $p \to i\omega$) and a Fourier transform in z on (3), i.e. operate (3) with $\int_0^\infty dt\int_{-\infty}^{\infty} dz e^{-i\omega t+ik_z z}$.

The LHS gives

$$\psi(\omega,\, k_z)=\int_0^\infty dt\int_{-\infty}^{\infty} dz\psi(z,t)e^{-i\omega t+ik_z z} \quad \begin{bmatrix} k_z \text{ here is} \\ k \text{ in Briggs.}\end{bmatrix} \qquad (14)$$

The RHS of (3) is a Fourier convolution of G and g, and a Laplace convolution of G and h. Thus, by (7) and (13), the Fourier/ Laplace transform of the RHS of (3) gives $G(\omega,\, k_z)g(k_z)f(\omega)$. Equating it to $\psi(\omega,\, k_z)$ in (14), we obtain

$$\psi(\omega,\, k_z)=G(\omega,\, k_z)g(k_z)f(\omega), \qquad (15)$$

which is (2.12) of Briggs for fields independent of the transverse coordinates. The inverse of $G(\omega,\, k_z)$ [i.e. $G^{-1}(\omega,\, k_z)$] will later be shown to play the role of the dispersion relation of the plasma system.

Rewrite $\quad \psi(\omega,\, k_z)=G(\omega,\, k_z)g(k_z)f(\omega)$ (15)

The inverse Fourier/Laplace transform of (15) gives

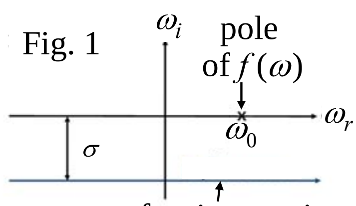
$$\psi(t,z)=\frac{1}{2\pi}\int_{-\infty-i\sigma}^{\infty-i\sigma} d\omega F(\omega,z)f(\omega)e^{i\omega t} \qquad (16)$$

where $\quad F(\omega,z)=\frac{1}{2\pi}\int_{-\infty}^{\infty} dk_z G(\omega,\, k_z)g(k_z)e^{-ik_z z}$ (17)

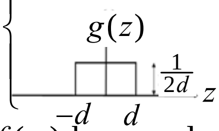
The prescribed ω-contour for (16) is the $\omega_i=-\sigma$ line (Fig. 1). The prescribed k_z-contour for (17) is the real k_z-axis.

We now assume the following source functions:

Fig. 1



$h(t)=e^{i\omega_0 t} \quad \Rightarrow \quad f(\omega)=\int_0^\infty f(t)e^{-i\omega t}dt=\frac{1}{i(\omega-\omega_0)}$ (18)



$\Rightarrow \quad g(k_z)=\int_{-\infty}^{\infty} g(z)e^{ik_z z}dz=\frac{\sin(k_z d)}{k_z d}$ (19)

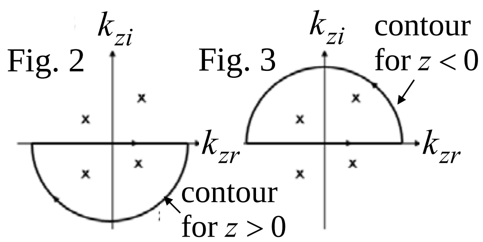
$\Rightarrow f(\omega)$ has a pole at ω_0 on the real ω-axis and $g(k_z)$ has no pole.

Rewrite $$F(\omega,z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dk_z G(\omega,\,k_z)g(k_z)e^{-ik_z z} \qquad (17)$$

where, $g(k_z)=\sin(k_z d)/k_z d$ [(19)]. For simplicity, we assume a point source at $z=0$ (i.e. $d\to 0$). Then, $g(k_z)=1$ and (17) becomes

$$F(\omega,z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dk_z G(\omega,\,k_z)e^{-ik_z z} \qquad (20)$$

(20) can be evaluated by the residue theorem. For $z>0$ and $z<0$, we close the contours on the k_z-plane with different half circles of infinite radius as shown in the figure so that the half circles give vanishing contributions.



Convective Instability in the $z>0$ Region : The result for $z>0$ is

$$F(\omega,z)=\sum_n\frac{-i}{[\frac{\partial}{\partial k_z}G^{-1}(\omega,k_z)]_{k_{zn}(\omega)}}e^{-ik_{zn}(\omega)z} \qquad (21)$$

where k_{zn} are poles in the lower half circle based on $\omega=\omega_r-i\sigma$.

Rewrite $$F(\omega,z)=\sum_n\frac{-i}{[\frac{\partial}{\partial k_z}G^{-1}(\omega,k_z)]_{k_{zn}(\omega)}}e^{-ik_{zn}(\omega)z} \qquad (21)$$

In (21) , since $\mathrm{Im}[k_{zn}]<0$ and $z>0$, $F(\omega,z)$ always decays away from the source. Then, how can there be an unstable solution ? To answer this question, we note that $F(\omega,z)$ is not the end results. The final solution $\psi(t,z)$ is given by (16):

$$\psi(t,z)=\frac{1}{2\pi}\int_{-\infty-i\sigma}^{\infty-i\sigma}d\omega F(\omega,z)f(\omega)e^{i\omega t} \qquad (16)$$

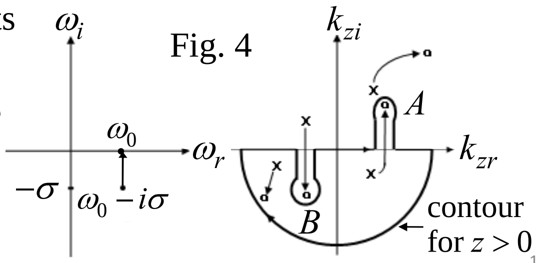
In (16), $\psi(t,z)$ is the superposition of $F(\omega,z)f(\omega)$ ranging from $\omega=-\infty-i\sigma$ to $\omega=\infty-i\sigma$. These superposed signals may still form a growing wave due to constructive interference.

To separate the asymptotic behavior from the transient effects, we will adopt the same technique used for Landau damping; namely, deforming the ω-contour to the upper ω-plane so as to enclose and isolate the effect of the pole at ω_0 (shown later in Fig. 5).

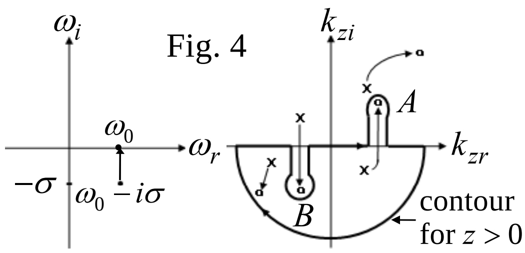
However, $F(\omega, z)$ is defined on the $\omega_i = -\sigma$ line. In order to move the ω-contour to the upper ω-plane, we must analytically continue $F(\omega, z)$ to the region above the $\omega_i = -\sigma$ line. As in the case of Landau damping, analytic continuation of $F(\omega, z)$, denoted by $\tilde{F}(\omega, z)$, is obtained with a deformed k_z-contour:

$$\tilde{F}(\omega, z) = \frac{1}{2\pi}\int_C dk_z G(\omega, k_z) e^{-ik_z z} = \sum_n \frac{-ie^{-ik_{zn}(\omega)z}}{[\frac{\partial}{\partial k_z} G^{-1}(\omega, k_z)]_{k_{zn}(\omega)}} \qquad (22)$$

where the new k_z-contour (denoted by "C") is deformed in such a manner that no pole crosses the k_z-contour, as ω moves upward from the $\omega_i = -\sigma$ line. Pole movements in the k_z-plane (for Fig. 2, $z > 0$) as ω goes from $\omega_0 - i\sigma$ to ω_0 are illustarted in Fig. 4 by mapping the ω path in the ω-plane to k_z paths in the k_z-plane.



Poles that cross the k_{zr}-axis (but not the k_z-contour) as ω goes from $\omega_0 - i\sigma$ to ω_0 deserve special attention. For example, pole A corresponds to an amplying wave which grows expenentially along the $+$z-axis, while pole B corresponds to a wave decaying exponentially along the $+$z-axis. The $z < 0$ region can be similarly considered by using the contour in Fig. 3.



Equation (16), $\psi(t, z) = \frac{1}{2\pi}\int_{-\infty - i\sigma}^{\infty - i\sigma} d\omega F(\omega, z) f(\omega) e^{i\omega t}$, can now be written

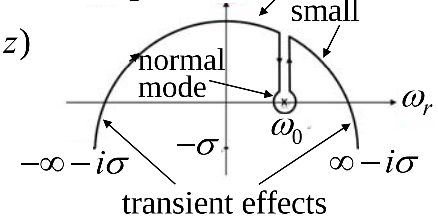
$$\psi(t, z) = \frac{1}{2\pi}\int_{-\infty - i\sigma}^{\infty - i\sigma} d\omega \tilde{F}(\omega, z) f(\omega) e^{i\omega t} \qquad (23)$$

In contrast to $F(\omega, z)$ in (16), $\tilde{F}(\omega, z)$ is analytic in the entire ω-plane so that we can deform the ω-contour in (23).

Rewrite $$\psi(t,z)=\frac{1}{2\pi}\int_{-\infty-i\sigma}^{\infty-i\sigma}d\omega\tilde{F}(\omega,z)f(\omega)e^{i\omega t} \quad (23)$$

From $f(\omega)=\dfrac{1}{i(\omega-\omega_0)}$ [(18)], we find $f(\omega)$ has a pole at $\omega=\omega_0$. If $\tilde{F}(\omega,z)$ has no poles in the $\omega-$plane, then by deforming the ω-contour in (23) as shown in Fig. 5, we obtain

Fig. 5



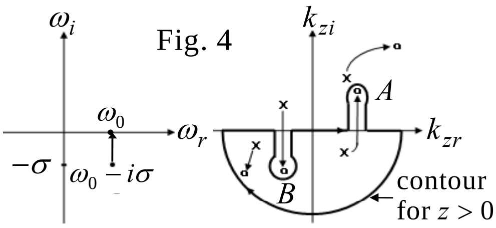
$$\psi(t,z)=\tilde{F}(\omega_0,z)e^{i\omega_0 t}+\text{transient effects}$$

$$=\sum_n \frac{-i}{[\frac{\partial}{\partial k_z}G^{-1}(\omega,k_z)]_{k_{zn}(\omega_0)}}e^{i\omega_0 t-ik_{zn}(\omega_0)z}+\begin{bmatrix}\text{transient}\\ \text{effects}\end{bmatrix} \quad (24)$$

Positions of $k_{zn}(\omega_0)$ in (24) differ from their original positions in Fig.2, which are based on ω values on the original ω-contour (Fig. 1). If $k_{zn}(\omega_0)$ in (24) is in the upper k_z-plane, such as pole A in Fig.4 , we have $\psi\sim e^{\alpha z}$ (α>0), i.e. an amplying wave along the $+z$-axis.

The $\psi\sim e^{\alpha z}$ instablity due to pole A grows only along $+$z, but not in time. Also, it is at the signal frequency ω_0. Hence, it is called a *convective instability,* which amplifies the injected signal and will propagate away from the source when the injected signal is off. Facts leading to such an instability are:

Fig. 4



(i) Pole A is a root of $D(k_z,\omega)=0$ and it crosses the k_{zr}-axis as ω rises from $\omega_0-i\sigma$ to ω_0 (Note that ω_0 is an arbitrarily set frequency);

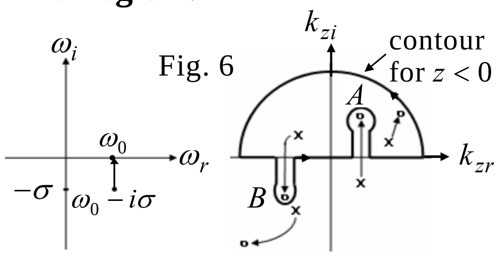
(ii) At the root crossing point on the k_{zr}-axis , ω is still in the lower half of ω-plane (i.e. $\omega_i<0$). Thus, we may state the necessary condition for the existence of a convective instability as (for $\psi\sim e^{i\omega t}$):

"*For some real* k_z , $G^{-1}(\omega,k_z)=0$ *yields a solution* ω *with* $\omega_i<0$." (25)

If we replace the Laplace transform variable p with $-i\omega$ ($\Rightarrow\psi\sim e^{-i\omega t}$), the condition still holds if we replace "$\omega_i<0$" with "$\omega_i>0$".

Convective Instability in the $z<0$ Region :

The $z<0$ region can be similarly considered by using the contour in Fig. 2. Figure 6 shows the mapping of the $\omega_0 - i\sigma$ to ω_0 path in the ω-plane to the pole movements in the k_z-plane. In this case,



$$\tilde{F}(\omega,z)=\frac{1}{2\pi}\int_C dk_z G(\omega,k_z)e^{-ik_z z}=\sum_n \frac{ie^{-ik_{zn}(\omega)z}}{[\frac{\partial}{\partial k_z}G^{-1}(\omega,k_z)]_{k_{zn}(\omega)}} \qquad (26)$$

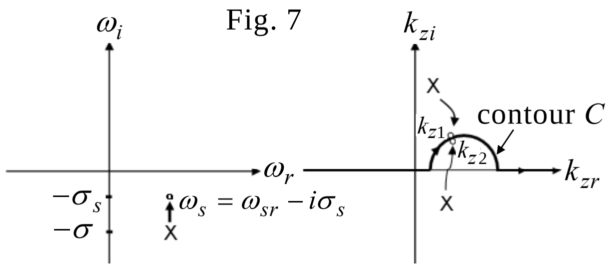
$$\psi(t,z)=\sum_n \frac{i}{[\frac{\partial}{\partial k_z}G^{-1}(\omega,k_z)]_{k_{zn}(\omega_0)}}e^{i\omega_0 t-ik_{zn}(\omega_0)z}+\begin{bmatrix}\text{transient}\\ \text{effects}\end{bmatrix} \qquad (27)$$

Thus, pole B gives a solution $\psi \sim e^{-\alpha z}$ (α>0), i.e. a convective instability which grows expenentially along the $-z$-axis, while pole A corresponds to a wave decaying exponentially along the $-$z-axis.

Absolute Instability :

As discussed earlier, analytic continuation of $F(\omega,z)$ to the region above the $\omega_i=-\sigma$ line is done by deforming the k_z-contour in Fig. 2 (or Fig. 3) in such a way [see Fig. 4 or 6] that no pole crosses the k_z-contour, as ω moves upward from the $\omega_i=-\sigma$ line. In the process, we may runs into difficulty if, for some value of ω (say, $\omega_s=\omega_{sr}-i\sigma$) in the lower half ω-plane, two poles (k_{z1} and k_{z1}) from opposite sides of k_z-contour merge into one (see Fig. 7). This is a difficult case because we have demanded that contour C must pass between 2 poles.

To evaluate $\tilde{F}(\omega,z)$ in (22), we must treat this double root at (ω_s, k_{zs}) by another mothod. We first find out what kind of pole it is.



The condition for pole merging is that $G^{-1}(\omega, k_z)$ has a double root.

Let it be (ω_s, k_{zs}). Then,
$$\begin{cases} G^{-1}(\omega_s, k_{zs}) = 0 & (28) \\ \dfrac{\partial}{\partial k_z} G^{-1}(\omega, k_z)\Big|_{\omega_s, k_{zs}} = 0 & (29) \end{cases}$$

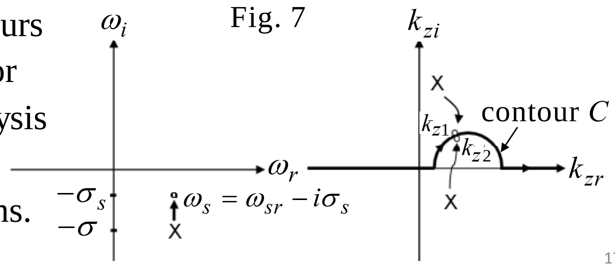
Rewrite

$$\tilde{F}(\omega, z) = \frac{1}{2\pi}\int_C dk_z G(\omega, k_z) e^{-ik_z z} = \sum_n \frac{-ie^{-ik_{zn}(\omega)z}}{[\frac{\partial}{\partial k_z} G^{-1}(\omega, k_z)]_{k_{zn}(\omega)}}, \qquad (22)$$

which treats each pole as a first-order pole. Thus, as two poles merge at (ω_s, k_{zs}), but contour C encloses only one of them, the denominator of (22) vanishes by (29). $\tilde{F}(\omega, z)$ then has a singularity at ω_s.

The same singularity occurs whether k_{zs} is in the upper or lower k_z-plane. So, the analysis below applies to both $z > 0$ [(22)] and $z < 0$ [(26)] regions.

Fig. 7



To determine the nature of the pole at k_{zs}, we perform a two-variable Taylor expansion of $G^{-1}(\omega, k_z)$ about the double root (ω_s, k_{zs}):

$$G^{-1}(\omega, k_z) \approx \underbrace{G^{-1}(\omega_s, k_{zs})}_{0 \text{ by } (28)} + \underbrace{\frac{\partial G^{-1}}{\partial k_z}\Big|_{\omega_s, k_{zs}}}_{0 \text{ by } (29)} (k_z - k_{zs}) + \frac{\partial G^{-1}}{\partial \omega}\Big|_{\omega_s, k_{zs}} (\omega - \omega_s)$$

$$+ \frac{1}{2}\frac{\partial^2 G^{-1}}{\partial k_z^2}\Big|_{\omega_s, k_{zs}} (k_z - k_{zs})^2 = 0, \qquad (30)$$

which gives $\quad (k_z - k_{zs})^2 \approx -2\dfrac{\partial G^{-1}}{\partial \omega}\Big|_s (\omega - \omega_s) \Big/ \dfrac{\partial^2 G^{-1}}{\partial k_z^2}\Big|_s \qquad (31)$

Differentiating (30) with respect to k_z gives

$$\frac{\partial G^{-1}}{\partial k_z} \approx \frac{\partial^2 G^{-1}}{\partial k_z^2}\Big|_s (k_z - k_{zs}) \overset{\boxed{(31)}}{\approx} i[2(\frac{\partial G^{-1}}{\partial \omega})_s (\frac{\partial^2 G^{-1}}{\partial k_z^2})_s]^{\frac{1}{2}} (\omega - \omega_s)^{\frac{1}{2}}, \quad (32)$$

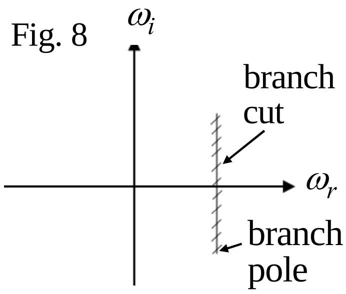
which is correct up to a $\pm$sign of no physical significance.

Sub. (32) into $\tilde{F}(\omega, z) = \sum_n \dfrac{-ie^{-ik_{zn}(\omega)z}}{[\frac{\partial}{\partial k_z} G^{-1}(\omega, k_z)]_{k_{zn}(\omega)}}$ [(22)], we obtain

$$\tilde{F}(\omega,z)\approx\frac{e^{ik_{zs}z}}{[2(\frac{\partial G^{-1}}{\partial\omega})_s(\frac{\partial^2 G^{-1}}{\partial k_z^2})_s]^{\frac{1}{2}}}\frac{1}{(\omega-\omega_s)^{\frac{1}{2}}}\qquad(33)$$

This shows that the singularity in $\tilde{F}(\omega,z)$ is a branch pole in ω-space as shown in Fig. 8. (see Mathews and Walker, Appendix A-1 for the meaning of a branch cut.)

Fig. 8



Sub. of (33) into

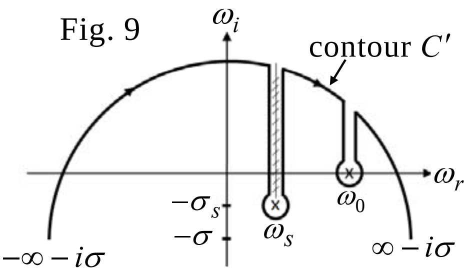
$$\psi(t,z)=\frac{1}{2\pi}\int_{-\infty-i\sigma}^{\infty-i\sigma}d\omega\tilde{F}(\omega,z)f(\omega)e^{i\omega t}\qquad(23)$$

$$\Rightarrow\psi(t,z)\approx\frac{1}{2\pi}\frac{f(\omega_s)}{[2(\frac{\partial G^{-1}}{\partial\omega})_s(\frac{\partial^2 G^{-1}}{\partial k_z^2})_s]^{\frac{1}{2}}}e^{i(\omega_s t-k_{zs}z)}\int_{-\infty-i\sigma}^{\infty-i\sigma}\frac{e^{i(\omega-\omega_s)t}}{(\omega-\omega_s)^{\frac{1}{2}}}d\omega\qquad(34)$$

where we have ignored the pole at ω_0 and set $f(\omega)=f(\omega_s)$ because the dominant contribution to (34) comes from $\omega\approx\omega_s$.

As before, to bring out the asymptotic behavior, we deform the contour for ω-integration as shown in Fig. 9 and denote the deformed contour by C'. Then, (34) becomes

Fig. 9



$$\psi(t,z)\approx\frac{1}{2\pi}\frac{f(\omega_s)}{[2(\frac{\partial G^{-1}}{\partial\omega})_s(\frac{\partial^2 G^{-1}}{\partial k_z^2})_s]^{\frac{1}{2}}}e^{i(\omega_s t-k_{zs}z)}\underbrace{\int_{C'}\frac{e^{i(\omega-\omega_s)t}}{(\omega-\omega_s)^{\frac{1}{2}}}d\omega}_{I}\qquad(35)$$

in which the branch pole gives the asymptotic solution:
[see evaluation of integral I in Appendix A, which gives $I=2(\frac{i\pi}{t})^{1/2}$]:

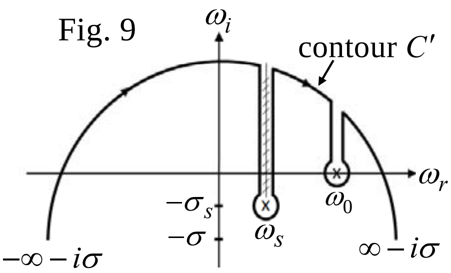
$$\psi(t,z)\approx\frac{f(\omega_s)}{[2\pi i(\frac{\partial G^{-1}}{\partial\omega})_s(\frac{\partial^2 G^{-1}}{\partial k_z^2})_s]^{\frac{1}{2}}}\frac{e^{i(\omega_s t-k_{zs}z)}}{t^{\frac{1}{2}}}\qquad(36)$$

which is correct up to a phase factor of no physical significance. There is no divergence problem at $t=0$ since this is an asymptotic solution.

Rewrite $$\psi(t,z)=\frac{f(\omega_s)}{[2\pi i(\frac{\partial G^{-1}}{\partial\omega})_s(\frac{\partial^2 G^{-1}}{\partial k_z^2})_s]^{\frac{1}{2}}}\frac{e^{i(\omega_s t-k_{zs}z)}}{t^{\frac{1}{2}}}\qquad(36)$$

Discussion:

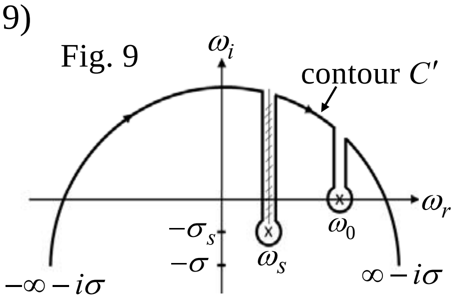
Fig. 9



(i) Since ω_s is in the lower half ω-plane, we have $\omega_{si}<0,$ implying $\psi(z,t)\sim e^{|\omega_{si}|t}$. This indicates that, at a fixed position $z,$ the wave grows exponentially in time. Furthermore, k_{zs} in general has an imaginary part. Thus, the wave also grows expenentially in the $+$ or $-$z-direction. This is called an "absolute instability". In contrast, the convective instability grows exponentially only in the $+$ or $-$z-direction.

(ii) If the double root of $D(k_z,\omega)=0$ occurs in the upper half ω-plane (i.e. $\omega_{si}>0$), the corresponding solution will damp in time and thus does not correspond to an instability.

Fig. 9



(iii) The pole of $f(\omega)$ at ω_0 (Fig. 9) is now of little significance even if it still gives rise to amplifying waves, because the asymptotic behavior is dominated by the branch pole at ω_s. In contrast, the convective instability grows at the frequency ω_0.

(iv) Unlike the convective instability, the frequency of the absolute instability is no longer at the source frequency ω_0. It is instead given by the solution of the two equations for a double root:

$$\begin{cases} G^{-1}(\omega_s,k_{zs})=0 & (28)\\ \dfrac{\partial}{\partial k_z}G^{-1}(\omega,k_z)\Big|_{\omega_s,k_{zs}}=0 & (29)\end{cases}$$

Since ω_s and k_{zs} of the absolute instability are governed by *two* equations, they both have fixed values.

In contrast, the frequency of the convective instability is governed by only *one* equation $[G^{-1}(\omega_s, k_{zs}) = 0]$. Thus, it exists for a range of (ω, k_z) values, and the frequency it grows (ω_0) at is determined *externally* by the source.

(v) The convective instability is often exploited in an amplifier system, in which an external source determines the frequency of the wave while the system amplifies it. The absolute instability is often exploited in an oscillator, which gets started from the noise level and grows rapidly to the saturation level at its intrinsic frequency.

On the basis the analysis, we may summarize the conditions for the existence of an absolute instability as follows:

1. For some value of ω (let it be ω_s) in the lower half ω-plane, $G^{-1}(\omega, k_z) = 0$ has a double root k_{zs}, whcih can be either in the upper or lower half of the k_z-plane. (37)

2. As ω moves downward from ω_s toward the $\omega_i = -\sigma$ line, the double root k_{zs} splits into 2 roots. One remains in the same half of the k_z-plane, while the other moves to the other half of the k_z-plane. (as in Fig. 7 with movements of ω and poles of k_z reversed.) (38)

Note: (i) If we replace the Laplace transform variable p with $-i\omega$ $(\Rightarrow \psi \sim e^{-i\omega t})$, condition 1 holds if we replace "*lower half ω-plane*" with "*upper half ω-plane*", and condition 2 also holds if we replace "*As ω moves downward from ω_s toward the lower ω-plane*" with "*as ω moves upward from ω_s toward the upper ω-plane*".

(ii) If a system is free from the absolute instability, the necessary condition (25) for a convective instability is also a sufficient condition.

Applications of Instability Conditions :

1. The dispersion relation for the EM wave in a cold plasma is

$$G^{-1}(\omega,k_z)[=D(\omega,k_z)]=\omega^2-k_z^2c^2-\omega_{pe}^2=0,\ \ [(143) \text{ of Ch. 6}] \quad (39)$$

which applies to a wave with $E \sim e^{\pm i\omega t \pm ik_z z}$ dependence.

(39) gives $k_z=\pm\frac{1}{c}(\omega^2-\omega_{pe}^2)^{1/2}$ or $k_z=\pm\frac{i}{c}(\omega_{pe}^2-\omega^2)^{1/2}$ (40)

If $\omega<\omega_{pe}$, k_z is purely imaginary, and one of the sign of (40) gives an amplifying wave, but in reality no wave can grow in a cold plasma.

This situation is clarified by the convective instability condition in (25). For a real k_z, (39) does not have a solution ω with an imaginary part. Hence, according to (25), there is no convective instability, i.e. we may only accept the root from (40) with the proper sign, which corresponds to an evanescent wave.

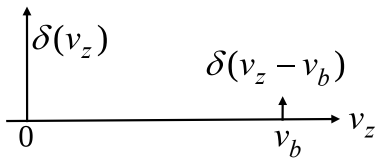
There is no absolute instability in (39) either, because the double root ($k_z=0,\ \omega=\omega_{pe}$) is not in the upper or lower half of the ω-plane.

2. Consider the dispersion relation for the two-stram instability in (83) of Ch. 6 (see figure for the model):

$$G^{-1}(\omega,k_z)[=D(\omega,k_z)]=1-\frac{\omega_{pa}^2}{\omega^2}-\frac{\omega_{pb}^2}{(\omega-k_zv_b)^2}=0, \quad (41)$$

which can be put in the form:

$$G^{-1}(\omega,k_z)=(1-\frac{\omega_{pa}^2}{\omega^2})(1-\frac{k_zv_b}{\omega})^2-\frac{\omega_{pb}^2}{\omega^2}=0$$



$$\Rightarrow \frac{\partial}{\partial k_z}G^{-1}(\omega,k_z)=2(1-\frac{\omega_{pa}^2}{\omega^2})\frac{v_b}{\omega}(1-\frac{k_zv_b}{\omega})=0$$

Apparently, $G^{-1}(\omega,k_z)$ and $\frac{\partial}{\partial k_z}G^{-1}(\omega,k_z)$ have no solution, i.e. there is no double pole. Hence, there is no absolute instability in (41). On the other hand, for a real k_z, there is a complex solution ω in both the upper and lower ω-plane [see (86) of Ch. 6]. Hence, the instability in (81) of Ch. 6 is a convective instability.

3. Consider the dispersion relation for the two-stram instability in (88) of Ch. 6 (see figure for the model):

$$G^{-1}(\omega,k_z)[=D(\omega,k_z)]=1-\frac{\omega_{pb}^2}{(\omega-k_z v_b)^2}-\frac{\omega_{pb}^2}{(\omega+k_z v_b)^2}=0, \qquad (42)$$

which can be put in the form:

$$G^{-1}(\omega,k_z)=(\omega^2-k_z^2v_b^2)^2-\omega_{pb}^2\left[(\omega+k_z v_b)^2+(\omega-k_z v_b)^2\right]$$

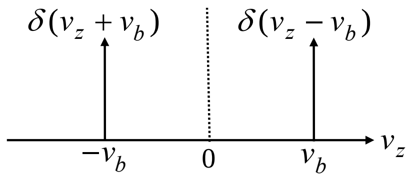
$$=(\omega^2-k_z^2v_b^2)^2-2\omega_{pb}^2(\omega^2+k_z^2v_b^2)=0 \qquad (43)$$

$$\Rightarrow \quad \frac{\partial}{\partial k_z}G^{-1}(\omega,k_z)=2(\omega^2-k_z^2v_b^2)(-2k_z v_b^2)-4\omega_{pb}^2 k_z v_b^2=0$$

$$\Rightarrow \quad \omega^2-k_z^2v_b^2+\omega_{pb}^2=0 \qquad (44)$$

Sub. (44) into (43) gives

$$\omega_{pb}^4-(4k_z^2v_b^2-2\omega_{pb}^2)\omega_{pb}^2=0$$



$$\Rightarrow k_z^2v_b^2=\frac{3}{4}\omega_{pb}^2, \quad \Rightarrow \quad k_{zs}^{\pm}=\pm\frac{\sqrt{3}\omega_{pb}}{2v_b} \qquad (45)$$

Thus, either k_{zs}^{+} or k_{zs}^{-} is a double root of $G^{-1}(\omega,k_z)$.

Sub. $k_{zs}^{\pm}=\pm\frac{\sqrt{3}\omega_{pb}}{2v_b}$ [(45)] for k_z into $\omega^2-k_z^2v_b^2+\omega_{pb}^2=0$ [(44)]

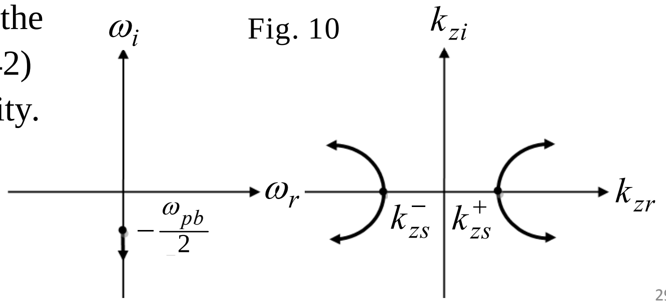
we obtain $\quad \omega_s^2=-\frac{\omega_{pb}^2}{4} \quad \Rightarrow \quad \omega_s^{\pm}=\pm i\frac{\omega_{pb}}{2} \qquad (46)$

Thus, at either $\omega_s^{+}=i\frac{\omega_{pb}}{2}$ or $\omega_s^{-}=-i\frac{\omega_{pb}}{2}$, we have two double roots (k_{zs}^{+} and k_{zs}^{-}) for $G^{-1}(\omega,k_z)=0$.

(42) is derived in Ch. 6 based on $e^{-i\omega t}$ dependence for the wave field, but (42) is independent of the sign of ω, thus applicable to both $e^{-i\omega t}$ and $e^{i\omega t}$ dependence. To be consistent with the convention of the current Special Topic, we adopt the $e^{i\omega t}$ dependence. Then, the double root at $\omega_s^{-}=-i\frac{\omega_{pb}}{2}$ meets condition 1 for the existence of an absolute instability.

To see whether condition 2 is also met by the required pattern of pole movements on the k_z-plane, we move ω from $\omega_s^-(=-i\frac{\omega_{pb}}{2})$ downward along the imaginary axis. Since ω^2 along this path is real, k_z as solved from (43) will always be complex conjugate pairs, i.e. if k_z is a solution of (43) (which has real coefficients), k_z^* must also be a solution. Thus, as ω moves down from ω_s^-, $k_{zs}^\pm$ will each split into 2 complex conjugate roots, which then move into different halves of the k_z-plane, as shown in Fig. 10. Thus, condition 2 is also satisfied. We therefore conclude that the dispersion relation in (42) has an absolute instability.

Fig. 10



Appendix A: Evaluation of Integral I in (35)

The integral I in (35) of main text is $I=\int_{C'}\frac{e^{i(\omega-\omega_s)t}}{(\omega-\omega_s)^{1/2}}d\omega$, (A.1)

where contour C′ is along the two sides (A and B) of the branch cut (Fig. 1).

First, we move the branch pole to the origin of a new coordinate system by defining $y=\omega-\omega_s \Rightarrow I=\int_{C''}\frac{e^{iyt}}{y^{1/2}}d\omega$, (A.2)

where contour C″ is along the relocated branch cut as shown in Fig. 2.

Fig. 1

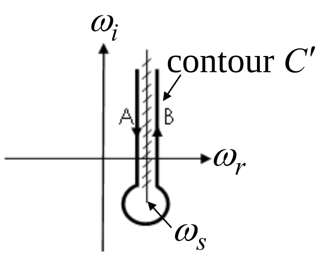
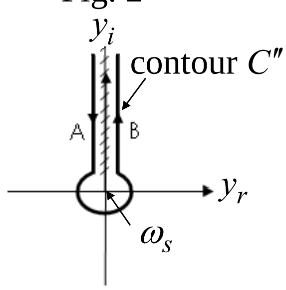


Fig. 2



Rewrite $$I=\int_{C''}\frac{e^{iyt}}{y^{1/2}}dy \qquad \text{(A.2)}$$

Next, let $x=y^{1/2} \Rightarrow dx=dy/(2y^{1/2})$. Then (A.2) can be written

$$I=2\int_{C'''}e^{ix^2t}dx \qquad \text{(A.3)}$$

where contour C''' is shown in Fig. 3, which corresponds to contour C' in the ω-plane and contour C'' in the y-plane. The mapping of contour C'' to contour C''' will be discussed on the next page.

Fig. 1

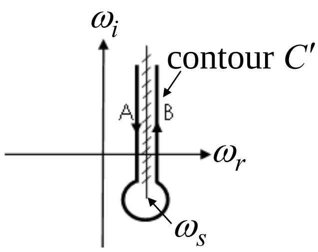


Fig. 2

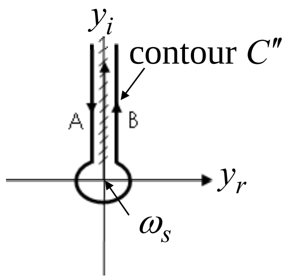
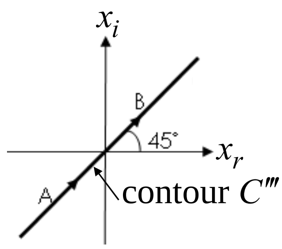


Fig. 3



The downward contour (path A) in the y-plane is given by

$$y=y_0e^{-i\frac{3\pi}{2}} \text{ with } y_0=\infty \text{ to } 0$$

Hence, the corresponding contour in the x-plane is

$$x=y^{1/2}=y_0^{1/2}e^{-i\frac{3\pi}{4}}=y_0^{\frac{1}{2}}e^{-i\pi}e^{i\frac{\pi}{4}}=-y_0^{\frac{1}{2}}e^{i\frac{\pi}{4}} \text{ with } y_0=\infty \text{ to } 0$$

or $$x=\rho e^{i\frac{\pi}{4}} \text{ with } \rho=-\infty \text{ to } 0$$

Note: The values of $y^{1/2}$ on each side of the branch cut in the y-plane differ by a sign (see Mathew & Walker, Appendix A-1).

Fig. 1

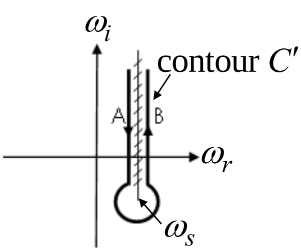


Fig. 2

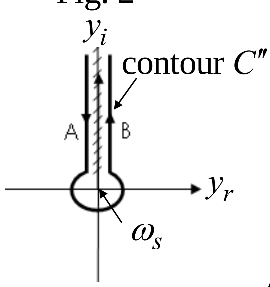
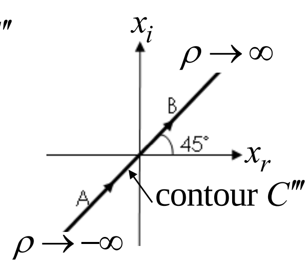


Fig. 3



The upward contour (path B) in the y-plane is given by

$$y = y_0 e^{i\frac{\pi}{2}} \text{ with } y_0 = 0 \text{ to } \infty$$

with the corresponding x-contour given by

$$x = y^{\frac{1}{2}} = y_0^{\frac{1}{2}} e^{i\frac{\pi}{4}} \text{ with } y_0 = 0 \text{ to } \infty$$

or $$x = \rho e^{i\frac{\pi}{4}} \text{ with } \rho = 0 \text{ to } \infty$$

Thus, contour C'' in the $x-$plane (paths A and B) is as shown in Fig. 3 and given by $x = \rho e^{i\frac{\pi}{4}}$ with $\rho = -\infty$ to ∞

Fig. 1

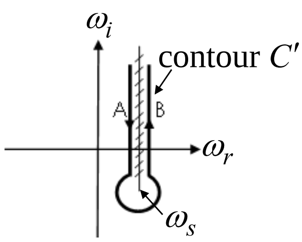


Fig. 2

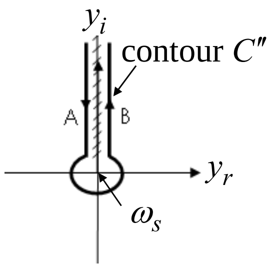
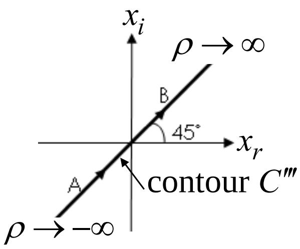


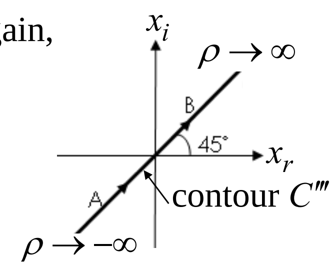
Fig. 3



Now return to the integral:

$$I = 2\int_{C'''} e^{ix^2 t} dx \qquad \text{(A.3)}$$

Fig. 3



Writing $i = e^{i\frac{\pi}{2}}$ and using $x = \rho e^{i\frac{\pi}{4}}$ again,

we have $$\begin{cases} ix^2 = \rho^2 e^{i(\frac{\pi}{2}+\frac{\pi}{2})} = -\rho^2 \\ dx = e^{i\frac{\pi}{4}} d\rho \end{cases}$$

Hence, $$I = \int_{C'''} e^{ix^2 t} dx$$

$$= 2e^{i\frac{\pi}{4}} \int_{-\infty}^{\infty} e^{-\rho^2 t} d\rho = 2e^{i\frac{\pi}{4}} \sqrt{\frac{\pi}{t}}$$

by (18) of Ch. 6

$$= \sqrt{\frac{i\pi}{t}} \qquad \text{(A.4)}$$