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Asymptotic Anti-De Sitter Conditions for Poincaré Gauge Theory*)

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We find conditions on the ten parameters of the Poincaré Gauge Theory of Gravity which are necessary for solutions with an asymptotically anti-de Sitter metric and asymptotically constant Riemann-Cartan curvature or asymptotically constant torsion.

§ 1. Introduction

Motivated by contemporary fundamental theoretical principles, a class of theories with gravity treated as a gauge theory for the Poincaré group has been developed principally by Hayashi and Shirafuji,[1] Hehl[2] and their colleagues. For these Poincaré Gauge Theories (PGT)**) space-time has a Riemann-Cartan geometry, i.e., a metric compatible connection with torsion. The field equations describe how the curvature and torsion are driven by the matter energy-momentum and spin density tensors. The gravitational Lagrangian is assumed to contain terms which are at most quadratic in the curvature and torsion. With normal parity this leads to a ten parameter class of theories. For a wide range of parameters the theory is experimentally viable;[1),2),5),6)] the search for theoretical restrictions on these parameters continues.[7]

One type of condition restricting the parameters is to consider those values which permit solutions with a certain asymptotic behavior. In the past the importance of such conditions and the limitations they imposed on other results was not sufficiently appreciated.

In particular, much initial effort went, quite naturally, into classifying 'massive' modes[5),8)] in the linearized PGT theory. At that time it was not recognized that the flat background assumption is only possible if the parameters satisfy one or more conditions, conditions which render at least one of the modes 'massless'. Only later were these 'massless' modes analysed.[9] A related situation is in the Hamiltonian analysis[10] where again the asymptotically flat conditions require the vanishing of at least one of the 'mass' parameters; consequently, the purely second class 'if constraint' assumption is not valid. Again, only much later, were[11] these degenerate 'first class' constraint cases studied. In both of these instances this oversight led to the relative

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**) We are considering here specifically only what has been referred to as a 'Kibble' type gauge theory, and not, in particular, to the $\overline{\text{Poincaré}}$ gauge theory developed by Kawai.[3] Although this interesting theory is based on a different implementation of the gauge principle, its physical content[4] is quite like the PGT we are considering; so the same conditions may be necessary for asymptotically anti-de Sitter $\overline{\text{P}}$GT solutions.

neglect of the most interesting physical cases. Another instance where the necessary parameter restrictions for the existence of asymptotically flat solutions were overlooked is in a 'proof' of energy positivity for $R+R^2$ theories.[12] Such theories simply do not permit any asymptotically flat solutions.[13] Consequently, the energy expressions which were used are doubtful and thus the 'proof' is not convincing.

The parameter values that allow solutions with asymptotic Newtonian behavior were worked out some time ago.[13] Since then there has been considerable interest in solutions[14] for which the metric asymptotically approaches the *anti-de Sitter* metric. This corresponds to an asymptotically constant (negative) Riemannian curvature. (Only the *negative* case has an asymptotic region.) Asymptotically anti-de Sitter PGT*) solutions are especially interesting physically because they permit total energy-momentum and angular momentum (originally defined only for asymptotically flat spaces) to be defined in terms of the asymptotic geometry.[16),17] If the Riemannian geometry has asymptotically constant curvature the *Riemann-Cartan* geometry could have asymptotically constant curvature *or* torsion. Here we extend the earlier asymptotically flat work,[13] determining the parameter values of the PGT which permit solutions with asymptotically constant curvature or asymptotically constant torsion.

To investigate the *weakest* parameter conditions that permit asymptotically constant curvature we look for *spherically symmetric* deviations from the background. There are several reasons for taking this limited view: (i) Physically, for gravitational fields, higher multipoles should not be possible without a mass monopole; (ii) any non-spherical terms should fall off at least as fast as the mass monopole terms and so they cannot dominate asymptotically; (iii) while we have a good idea of the asymptotics of such terms for asymptotically flat space such is not the case for asymptotically constant curvature spaces; (iv) finally, such terms could only lead to additional restrictions going beyond what we seek: the *weakest* conditions which permit some non-trivial solution.

Although we need concern ourselves here with only spherically symmetric terms the meaning of spherically symmetric should be clarified. While Newtonian gravity is also reflection invariant it is not obvious why we should (or should not) require a more general gravitational theory to have this property. For this reason we investigate both cases, finding the necessary parameter conditions which permit asymptotically $O(3)$ symmetric solutions and those which permit asymptotically $SO(3)$ symmetric solutions.

Our plan is as follows: In § 2 we summarize the Poincaré gauge theory. In § 3 we discuss the background geometry and the rate of asymptotic fall off of the curvature and torsion. Section 4 concerns the asymptotic curvature conditions and § 5 covers the conditions for asymptotically constant torsion. The total energy is expressed in terms of the asymptotic solutions in § 6 thereby obtaining a further restriction which sets the scale of the PGT parameters. Finally we discuss our conclusions in § 7.

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*) The relationship of Poincaré gauge theory and *anti-de Sitter gauge theory* has recently been explored.[15]

§ 2. Poincaré gauge theory

The gauge vector potentials of the Poincaré gauge theory of gravity[1),2)] are the orthonormal coframe ϑ^{α} for translations and the metric-compatible connection one-forms $\omega^{\alpha\beta}=-\omega^{\beta\alpha}=\Gamma_{\mu}{}^{\alpha\beta}\vartheta^{\mu}$ for Lorentz transformations.*) The corresponding field strengths are the *torsion* 2-form $\Theta^{\alpha}:=d\vartheta^{\alpha}+\omega_{\beta}{}^{\alpha}\wedge\vartheta^{\beta}=(1/2)F_{\mu\nu}{}^{\alpha}\vartheta^{\mu}\wedge\vartheta^{\nu}$ and the *curvature* 2-form $\Omega^{\alpha\beta}:=d\omega^{\alpha\beta}+\omega_{\gamma}{}^{\beta}\wedge\omega^{\alpha\gamma}=(1/2)F_{\mu\nu}{}^{\alpha\beta}\vartheta^{\mu}\wedge\vartheta^{\nu}$. The field equations are obtained from the Lagrangian 4-form

$$\mathcal{L}=\mathcal{V}(\vartheta^{\alpha},\ \Theta^{\alpha},\ \Omega^{\alpha\beta})+\mathcal{K}_{\text{source}}\,. \tag{2·1}$$

The gravitational Lagrangian density $\mathcal{V}$ is assumed to be at most quadratic in the curvature and torsion. Consequently the theory has 10 dimensionless parameters controlling the amplitude of the cosmological constant, the scalar curvature, the 3 irreducible quadratic torsion terms and the 6 irreducible quadratic curvature terms (only 5 are effectively independent). The field equation 3-forms are

$$E_{\alpha}:=\frac{\delta\mathcal{L}}{\delta\vartheta^{\alpha}}\equiv -DH_{\alpha}+\frac{\partial\mathcal{V}}{\partial\vartheta^{\alpha}}+\Sigma_{\alpha}=0\,,\quad \text{(FIRST)} \tag{2·2}$$

$$E_{\alpha\beta}:=\frac{\delta\mathcal{L}}{\delta\omega^{\alpha\beta}}\equiv -DH_{\alpha\beta}+\vartheta_{[\beta}\wedge H_{\alpha]}+\tau_{\alpha\beta}=0\,,\quad \text{(SECOND)} \tag{2·3}$$

where

$$H_{\alpha}:=-\frac{\partial\mathcal{V}}{\partial\Theta^{\alpha}}=-\frac{1}{l^{2}}{}^{*}\Big(\sum_{a=1}^{3}A_{a}{}^{(a)}\Theta_{\alpha}\Big)\,, \tag{2·4}$$

$$H_{\alpha\beta}:=-\frac{\partial\mathcal{V}}{\partial\Omega^{\alpha\beta}}=-\frac{A_{0}}{2l^{2}}{}^{*}(\vartheta_{\alpha}\wedge\vartheta_{\beta})-\frac{1}{\kappa}{}^{*}\Big(\sum_{b=1}^{6}B_{b}{}^{(b)}\Omega_{\alpha\beta}\Big) \tag{2·5}$$

are the gravitational field momenta (which contain the weighted irreducible parts, ${}^{(a)}\Theta_{\alpha}$ and ${}^{(b)}\Omega_{\alpha\beta}$ of the torsion and curvature),**)

$$\Sigma_{\alpha}:=\frac{\delta\mathcal{K}}{\delta\vartheta^{\alpha}}\,, \tag{2·6}$$

$$\tau_{\alpha\beta}:=\frac{\delta\mathcal{K}}{\delta\omega^{\alpha\beta}} \tag{2·7}$$

are, respectively, the canonical energy-momentum and spin density of the source and l is the Planck length.

*) We follow the conventions of Hehl and his coworkers,[14),18)] which are convenient for symbolic computer calculations.

**) The relation of the parameters used here to those used in another common convention[1)] is $l^{-2}(A_0, A_1, A_2, A_3)=\pm(-2a, 3a, 6\beta, -2\gamma/3)$, and $\kappa^{-1}(B_1, B_2, B_3, B_4, B_5, B_6)=(3a_2, 4a_3, 4a_1, 2a_5, 2a_4, 24a_6)$, for signature $\pm(-+++)$.

§ 3. Background geometry and asymptotics

The *anti-de Sitter* metric given by

$$g=-\vartheta_0{}^t\otimes\vartheta_0{}^t+\vartheta_0{}^r\otimes\vartheta_0{}^r+\vartheta_0{}^\theta\otimes\vartheta_0{}^\theta+\vartheta_0{}^\phi\otimes\vartheta_0{}^\phi \tag{3·1}$$

with the spherical orthonormal coframe

$$\vartheta_0{}^t=e^{\Phi_0}dt\,,\quad \vartheta_0{}^r=e^{-\Phi_0}dr\,,\quad \vartheta_0{}^\theta=rd\theta\,,\quad \vartheta_0{}^\phi=r\sin\theta d\phi\,, \tag{3·2}$$

where $e^{2\Phi_0}=1+(r/R)^2$ and t, r, θ, ϕ are spherical coordinates, will serve as our background. The underlying *Riemannian* geometry has, of course, vanishing torsion, and the Riemannian curvature 2-form

$$\Omega^{\alpha\beta}_{\text{riem}}=-\frac{1}{R^2}\vartheta_0{}^\beta\wedge\vartheta_0{}^\alpha \tag{3·3}$$

is that of a constant negative curvature space with radius of cuvature R.

There has been much interest in this anti-de Sitter geometry. Its symmetry group has certain similarities to the Poincaré group permitting, in particular, definitions of energy-momentum and angular momentum.[16] The anti-de Sitter geometry is thus well suited to serving as an asymptotic background for dynamic geometry theories since it allows the basic conserved quantities: total energy-momentum and angular momentum of a solution to be defined in terms of its asymptotic geometry. Hence, asymptotically, the gravitational field appears to come from a point particle. We can thus generalize the idea of 'asymptotically Newtonian' gravitational fields. We want to do this in a way that in the limit $1/R\to 0$ we recover the 'asymptotically Newtonian' results.

In the PGT the orbits of structureless test particles are governed *only* by the underlying Riemannian geometry. So we next consider this Riemannian geometry. In the Einstein theory introducing a central mass m into an anti-de Sitter space leads to a unique solution: the Schwarzschild anti-de Sitter metric

$$g=-\vartheta^t\otimes\vartheta^t+\vartheta^r\otimes\vartheta^r+\vartheta^\theta\otimes\vartheta^\theta+\vartheta^\phi\otimes\vartheta^\phi \tag{3·4}$$

with the orthonormal coframe being

$$\vartheta^t=e^{\Phi}dt\,,\quad \vartheta^r=e^{-\Phi}dr\,,\quad \vartheta^\theta=rd\theta\,,\quad \vartheta^\phi=r\sin\theta d\phi\,, \tag{3·5}$$

where $e^{2\Phi}=1-2m/r+(r/R)^2$. We should consider both the regime $m\ll r\ll R$ in which case the geometry is very nearly Schwarzschild (indeed very nearly Newtonian), and the asymptotic regime $m\ll R\ll r$. Within these regimes we make the 'correspondence principle' assumption that the metric (3·4), (3·5) is good also for the PGT at least to order $O(1/r)$ (i.e., falls off faster than $1/r$) for sufficiently small m.

The background Riemann-Cartan geometry has more structure than the background Riemannian geometry. However, at least asymptotically, it should have no less symmetry. Consequently, we *assume* the background Riemann-Cartan curvature is of the form

$$\Omega_0{}^{\alpha\beta}=C_0\vartheta_0{}^\beta\wedge\vartheta_0{}^\alpha \tag{3·6}$$

asymptotically for some constant C_0. There are two obvious candidates: $C_0=-R^{-2}$ and $C_0=0$. It turns out that these are the only possibilities.

We arrive at this conclusion by assuming asymptotically constant $O(3)$ invariant torsion, specifically

$$\Theta^t=\frac{f_0}{R}\vartheta_0{}^t\wedge\vartheta_0{}^r\,, \tag{3·7a}$$

$$\Theta^r=\frac{h_0}{R}\vartheta_0{}^r\wedge\vartheta_0{}^t\,, \tag{3·7b}$$

$$\Theta^\theta=\frac{1}{R}(g_0\vartheta_0{}^r\wedge\vartheta_0{}^\theta-k_0\vartheta_0{}^t\wedge\vartheta_0{}^\theta)\,, \tag{3·7c}$$

$$\Theta^\phi=\frac{1}{R}(g_0\vartheta_0{}^r\wedge\vartheta_0{}^\phi-k_0\vartheta_0{}^t\wedge\vartheta_0{}^\phi)\,, \tag{3·7d}$$

where f_0, g_0, h_0, k_0 are constants, and computing the curvature 2-form:

$$\Omega^{tr}=-R^{-2}(1+f_0)\vartheta_0{}^r\wedge\vartheta_0{}^t\,, \tag{3·8a}$$

$$\Omega^{t\theta}=-R^{-2}(1+f_0)(1-g_0)\vartheta_0{}^\theta\wedge\vartheta_0{}^t+R^{-2}[k_0-h_0(1-g_0)]\vartheta_0{}^\theta\wedge\vartheta_0{}^r\,, \tag{3·8b}$$

$$\Omega^{t\phi}=-R^{-2}(1+f_0)(1-g_0)\vartheta_0{}^\phi\wedge\vartheta_0{}^t+R^{-2}[k_0-h_0(1-g_0)]\vartheta_0{}^\phi\wedge\vartheta_0{}^r\,, \tag{3·8c}$$

$$\Omega^{r\theta}=-R^{-2}(1-g_0-h_0k_0)\vartheta_0{}^\theta\wedge\vartheta_0{}^r+R^{-2}k_0(1+f_0)\vartheta_0{}^\theta\wedge\vartheta_0{}^t\,, \tag{3·8d}$$

$$\Omega^{r\phi}=-R^{-2}(1-g_0-h_0k_0)\vartheta_0{}^\phi\wedge\vartheta_0{}^r+R^{-2}k_0(1+f_0)\vartheta_0{}^\phi\wedge\vartheta_0{}^t\,, \tag{3·8e}$$

$$\Omega^{\theta\phi}=-R^{-2}[(1-g_0)^2-k_0{}^2]\vartheta_0{}^\phi\wedge\vartheta_0{}^\theta\,. \tag{3·8f}$$

The restriction $F_{\phi t}{}^{r\phi}=2R^{-2}k_0(1+f_0)=0$ gives two possibilities, (i) $1+f_0\neq0$ and $k_0=0$ $\Rightarrow C_0=-R^{-2}$, or (ii) $1+f_0=0\Rightarrow C_0=0$.

§ 4. Asymptotic constant curvature conditions

We first consider a background with *constant curvature*, i.e., the torsion vanishes and the *Riemann-Cartan* curvature has the form (3·6) with $C_0=-R^{-2}$. This background solves the PGT field equations (2·2), (2·3) if

$$C_0:=\frac{1}{l^2}\left(A_0/R^2+\frac{\Lambda}{3}\right)=0\,. \tag{4·1}$$

We define *asymptotically constant curvature* to mean that the curvature 2-form asymptotically approaches $-R^{-2}\vartheta^\beta\wedge\vartheta^\alpha$ to order $O(r^{-3})$ (i.e., it may have non-vanishing terms of this order just as in the asymptotically Newtonian case[13]). The principal reason for making this assumption is that then parallel transport around a loop (holonomy) at large r will be governed only by the background anti-de Sitter $1/R^2$ term. In view of the form of the unit radial derivative operator, $e_r=e^\Phi\partial_r$, we infer that the connection (and thus the torsion) should approach the background at rate $O(r^{-2}e^{-\Phi})$ for spherically symmetric terms. This is consistent with our frame assumption (3·5). The factors of $e^{-\Phi}\approx e^{-\Phi_0}$ have been introduced in a way which allows results that are valid also in the asymptotically Newtonian $1/R\to 0$ limit. Note that asymptotically $e^{-\Phi}$ switches from R/r to 1 in this limit.

We first consider the case that the geometry has $O(3)$ symmetry (spherical symmetry *and* reflection invariance) to lowest asymptotic order. Only a few of the torsion tensor components may be non-vanishing to this order with respect to the spherical orthonormal frame (3·5). Hence the torsion 2-forms are assumed to have the form

$$\Theta^t=\frac{m}{r^2e^{\Phi}}f\vartheta^t\wedge\vartheta^r+O(r^{-2}e^{-\Phi_0}), \tag{4·2a}$$

$$\Theta^r=\frac{m}{r^2e^{\Phi}}h\vartheta^r\wedge\vartheta^t+O(r^{-2}e^{-\Phi_0}), \tag{4·2b}$$

$$\Theta^\theta=\frac{m}{r^2e^{\Phi}}(g\vartheta^r\wedge\vartheta^\theta-k\vartheta^t\wedge\vartheta^\theta)+O(r^{-2}e^{-\Phi_0}), \tag{4·2c}$$

$$\Theta^\phi=\frac{m}{r^2e^{\Phi}}(g\vartheta^r\wedge\vartheta^\phi-k\vartheta^t\wedge\vartheta^\phi)+O(r^{-2}e^{-\Phi_0}), \tag{4·2d}$$

where f, g, h and k are constants.

With the above asymptotic forms for the frame and torsion 2-form, we find (using REDUCE[19]) and EXCALC[18])) the components $F_{\alpha\mu}=(*E_\alpha)_\mu$ of the first field equation to lowest asymptotic order

$$F_{tt}=3c_0-\frac{m}{r^3}[c_1f-2(c_1+d_1)g]+O(r^{-3}), \tag{4·3a}$$

$$F_{tr}=-\frac{m}{r^3}[(c_1+d_1-d_0)h+(2c_1+3d_1-d_0)k]+O(r^{-3}), \tag{4·3b}$$

$$F_{rt}=-\frac{m}{r^3}[(3c_1+d_1+d_0)h+(6c_1+3d_1+d_0)k]+O(r^{-3}), \tag{4·3c}$$

$$F_{rr}=-3c_0-\frac{m}{r^3}[(3c_1+2d_1)f-2(3c_1+2d_1)g]+O(r^{-3}), \tag{4·3d}$$

$$F_{\theta\theta}=F_{\phi\phi}=-3c_0-\frac{m}{r^3}[(c_1+d_1)f-(2c_1+d_1)g]+O(r^{-3}); \tag{4·3e}$$

and the components $S^{\alpha\beta}{}_\mu=(*E^{\alpha\beta})_\mu$ of the second field equation to lowest asymptotic order

$$S^{t\phi}{}_\phi=S^{t\theta}{}_\theta=\frac{m}{2r^2e^{\Phi}}[(c_1+d_0)h+(2c_1+d_0)k]+O(r^{-2}e^{-\Phi_0}), \tag{4·4a}$$

$$S^{tr}{}_t=-\frac{m}{2r^2e^{\Phi}}[c_1f-2(c_1+d_1)g]+O(r^{-2}e^{-\Phi_0}), \tag{4·4b}$$

$$S^{tr}{}_r=-\frac{m}{2r^2e^{\Phi}}[(c_1+d_1-d_0)h+(2c_1+3d_1-d_0)k]+O(r^{-2}e^{-\Phi_0}), \tag{4·4c}$$

$$S^{r\phi}{}_{\phi}=S^{r\theta}{}_{\theta}=-\frac{m}{2r^2e^{\Phi}}[(c_1+d_1)f-(2c_1+d_1)g]+O(r^{-2}e^{-\Phi_0})\,. \tag{4·4d}$$

Here we have introduced the convenient parameter combinations*)

$$c_1:=\frac{1}{3l^2}(2A_1+A_2)\,,$$

$$d_0:=\frac{1}{l^2}(A_0-A_1)+\frac{2}{\kappa R^2}(B_6-B_5)\,,$$

$$d_1:=\frac{1}{l^2}(A_0-A_1)+\frac{2}{\kappa R^2}(B_4+B_6)\,. \tag{4·5}$$

Requiring Eqs. (4·3) and (4·4) to vanish gives, in addition to $c_0=0$, the following conditions:

$$\begin{pmatrix} c_1 & -2(c_1+d_1) \\ 3c_1+2d_1 & -2(3c_1+2d_1) \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix}=0\,, \tag{4·6}$$

$$\begin{pmatrix} d_1-2d_0 & 3d_1-2d_0 \\ c_1+d_0 & 2c_1+d_0 \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix}=0\,. \tag{4·7}$$

We relax our requirement of reflection invariance, i.e., we now consider $SO(3)$ symmetry. Then the number of relevant torsion tensor components increases. We now have the following asymptotically to lowest-order non-vanishing torsion components:

$$\Theta^t=(4·2\text{a})-\frac{m}{r^2e^{\Phi}}p\vartheta^\theta\wedge\vartheta^\phi\,, \tag{4·8a}$$

$$\Theta^r=(4·2\text{b})+\frac{m}{r^2e^{\Phi}}q\vartheta^\theta\wedge\vartheta^\phi\,, \tag{4·8b}$$

$$\Theta^\theta=(4·2\text{c})-\frac{m}{r^2e^{\Phi}}(t\vartheta^r\wedge\vartheta^\phi+s\vartheta^t\wedge\vartheta^\phi)\,, \tag{4·8c}$$

$$\Theta^\phi=(4·2\text{d})+\frac{m}{r^2e^{\Phi}}(t\vartheta^r\wedge\vartheta^\theta+s\vartheta^t\wedge\vartheta^\theta)\,, \tag{4·8d}$$

where p, q, s, t are also constants. Without the requirement of reflection invariance, we also have extra lowest-order terms in the field equations:

$$F_{\theta\phi}=-F_{\phi\theta}=\frac{m}{r^3}[c_2q+(2c_2+d_0)t]+O(r^{-3})\,, \tag{4·9a}$$

$$S^{\theta\phi}{}_t=-\frac{m}{2r^2e^{\Phi}}[(c_2+d_2)p+2c_2s]+O(r^{-2}e^{-\Phi_0})\,, \tag{4·10a}$$

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*) For comparison with other results we note that in the other common convention[1)] these combinations are $c_1:=(1/3l^2)(2A_1+A_2)=2(\alpha+\beta)$, $(1/l^2)(A_0-A_1)=-3(\alpha+(2/3)a)$, $(2/\kappa)(B_4+B_6)=4(a_5+12a_6)$, $(2/\kappa)(B_6-B_5)=4(12a_6-a_4)$. All but the last have played important roles in earlier studies.[5),8)~11)]

$$S^{\theta\phi}{}_{r}=-\frac{m}{2r^2e^{\Phi}}[(c_2+d_2)q+(2c_2+d_2-d_0)t]+O(r^{-2}e^{-\Phi_0}), \tag{4·10b}$$

$$S^{t\phi}{}_{\theta}=-S^{t\theta}{}_{\phi}=-\frac{m}{2r^2e^{\Phi}}[c_2p+(2c_2+d_2)s]+O(r^{-2}e^{-\Phi_0}), \tag{4·10c}$$

$$S^{r\phi}{}_{\theta}=-S^{r\theta}{}_{\phi}=\frac{m}{2r^2e^{\Phi}}[c_2q+(2c_2+d_0)t]+O(r^{-2}e^{-\Phi_0}), \tag{4·10d}$$

where*)

$$c_2:=\frac{1}{3l^2}(A_1+2A_3),$$

$$d_2:=\frac{1}{l^2}(A_0-A_1)+\frac{2}{\kappa R^2}(B_2+B_6). \tag{4·11}$$

Requiring the lowest-order equations (4·3), (4·4), (4·9), (4·10) to vanish gives Eqs. (4·6), (4·7) and the following conditions:

$$\begin{pmatrix} c_2 & 2c_2+d_2 \\ c_2+d_2 & 2c_2 \end{pmatrix}\begin{pmatrix} p \\ s \end{pmatrix}=0, \tag{4·12}$$

$$\begin{pmatrix} c_2 & 2c_2+d_0 \\ c_2+d_2 & 2c_2-d_0+d_2 \end{pmatrix}\begin{pmatrix} q \\ t \end{pmatrix}=0. \tag{4·13}$$

In order to have non-trivial asymptotic solutions to Eqs. (4·6), (4·7), (4·12), (4·13) not all of the parameters f, g, h, k, p, q, s, t can vanish. Hence at least one of the

Table I. Asymptotic constant curvature conditions for $O(3)$ symmetry.

	parameter conditions	restriction on the solution amplitudes f, g, h, k
$d_0=0$	$d_1=0=c_1$	no restriction
	$d_1=0\neq c_1$	$f=2g,\quad h=-2k$
	$d_1\neq 0=c_1$	$f=g=0,\quad h=-3k$
	$d_1=-\frac{3}{2}c_1\neq 0$	$f=-g,\quad h=k=0$
	$0\neq d_1\neq -\frac{3}{2}c_1\neq 0$	$f=g=h=k=0$
$d_0\neq 0$	$d_1=-\frac{3}{2}c_1=d_0$	$f=-g,\quad h=k$
	$d_1=c_1=0$	$h=-k$
	$d_1=0\neq c_1$	$f=2g,\quad h=k=0$
	$d_1=-\frac{3}{2}c_1\neq d_0$	$f=-g,\quad h=k=0$
	$d_1\neq -\frac{3}{2}c_1,\quad 0\neq c_1d_1=-2d_0(c_1+d_1)$	$f=g=0=(2c_1+d_1)h+(4c_1+3d_1)k$
	$d_1\neq -\frac{3}{2}c_1,\quad 0\neq c_1d_1\neq -2d_0(c_1+d_1)$	$f=g=h=k=0$

*) For comparison with the results of others we note that in the other common convention[1] these combinations are $c_2:=(1/3l^2)(A_1+2A_3)=\alpha-(4/9)\gamma$, which is familiar from earlier studies,[5),8)~11)] and $(2/\kappa)(B_2+B_6)=8(a_3+6a_6)$.

Table II. Asymptotic constant curvature: additional conditions for $SO(3)$ symmetry.

	parameter conditions	restriction on the solution amplitudes f, g, h, k
$d_0=0$	$d_2=0=c_2$	no restriction
	$d_2=0\neq c_2$	$p=-2s,\quad q=-2t$
	$d_2\neq 0=c_2$	$p=s=0,\quad q=-t$
	$d_2=-3c_2\neq 0$	$p=s,\quad q=t=0$
	$0\neq d_2\neq -3c_2\neq 0$	$p=q=s=t=0$
$d_0\neq 0$	$d_2=-3c_2=d_0$	$p=s;\quad q=t$
	$d_2=c_2=0$	$t=0$
	$d_2=0\neq c_2$	$p=-2s,\quad q=t=0$
	$d_2=-3c_2\neq d_0$	$p=s,\quad q=t=0$
	$d_2\neq -3c_2,\quad 0\neq c_2d_2=-d_0(2c_2+d_2)$	$p=s=0=(2c_2+d_2)q+(4c_2+d_2)t$
	$d_2\neq -3c_2,\quad 0\neq c_2d_2\neq -d_0(2c_2+d_2)$	$p=q=s=t=0$

following determinants must vanish:

$$\det(4\cdot 6)=d_1(3c_1+2d_1)\,, \tag{4·14a}$$

$$\det(4\cdot 7)=-c_1d_1-2d_0(c_1+d_1)\,, \tag{4·14b}$$

$$\det(4\cdot 12)=d_2(3c_2+d_2)\,, \tag{4·15a}$$

$$\det(4\cdot 13)=-d_2c_2-d_0(2c_2+d_2)\,. \tag{4·15b}$$

The conditions on the parameters that allow solutions with constant curvature at large distance from the source with $O(3)$ symmetry are given in Table I. Additional possibilities permitted by $SO(3)$ symmetry are given in Table II.

The limit $1/R^2\to 0$ yields the asymptotically flat case which was studied earlier. In this limit $d_0=d_1=d_2=(A_0-A_1)/l^2$. The conditions reduce to $\Lambda=0$ and the three cases

$$d_1=-\frac{3}{2}c_1\neq 0\,,\quad \text{if}\quad f=-g\,,\quad h=k\,, \tag{4·16A}$$

$$d_1=0\neq c_1\,,\quad \text{if}\quad f=2g\,,\quad h=-2k\,, \tag{4·16B}$$

$$d_1=0=c_1\,,\quad \text{no restriction on } f, g, h, k \tag{4·16C}$$

for $O(3)$ symmetry, and the additional possibilities

$$d_2=-3c_2\neq 0\,,\quad \text{if}\quad p=s\,,\quad q=t\,, \tag{4·17a}$$

$$d_2=0\neq c_2\,,\quad \text{if}\quad p=-2s\,,\quad q=-2t\,, \tag{4·17b}$$

$$d_2=0=c_2\,,\quad \text{no restrictions on } p, s, q, t \tag{4·17c}$$

for $SO(3)$ symmetry. In terms of the original parameters these conditions are

$$A_0-A_1\neq 0=2A_0+A_2\,, \tag{4·18A}$$

$$A_0-A_1=0\neq 2A_0+A_2\,, \tag{4·18B}$$

$$A_0-A_1=0=2A_0+A_2\,, \tag{4·18C}$$

$$A_0-A_1\neq 0=A_0+2A_3\,, \tag{4·19a}$$

$$A_0-A_1=0\neq A_0+2A_3\,, \tag{4·19b}$$

$$A_0-A_1=0=A_0+2A_3\,, \tag{4·19c}$$

respectively, which (aside from a change in notation) are the same as cases A, B, C, a, b, c obtained in the earlier 'asymptotically Newtonian' work.[13] Physically, the combinations A_0-A_1, $2A_0+A_2$ and A_0+2A_3 have been identified as "masses"*) of the PGT linearized modes.[1),5),8)~11)] Note that at least one 'mass' must vanish in order to have asymptotically flat solutions.

§ 5. Asymptotic constant torsion conditions

We now consider a background for which the curvature *vanishes* (i.e., a teleparallel background) but the torsion is non-vanishing. Using the frame (3·2) and the Minkowski connection coefficients $\omega_r{}^\theta=d\theta$, $\omega_r{}^\phi=\sin\theta d\phi$, $w_\theta{}^\phi=\cos\theta d\phi$ we find the background torsion 2-form:

$$\Theta_0{}^t=-(e^{\Phi_0})'\vartheta_0{}^t\wedge\vartheta_0{}^r\,, \tag{5·1a}$$

$$\Theta_0{}^r=0\,, \tag{5·1b}$$

$$\Theta_0{}^\theta=\frac{e^{\Phi_0}-1}{r}\vartheta_0{}^r\wedge\vartheta_0{}^\theta\,, \tag{5·1c}$$

$$\Theta_0{}^\phi=\frac{e^{\Phi_0}-1}{r}\vartheta_0{}^r\wedge\vartheta_0{}^\phi\,. \tag{5·1d}$$

Although the torsion coefficients are only *asymptotically* constant, this geometry can still serve as our background. For this background geometry to solve the PGT field equations (2·2) and (2·3) we must impose the extra conditions

$$2A_0+A_2=0\,, \tag{5·2}$$

$$A_0-A_1=0\,, \tag{5·3}$$

as well as the expected $c_0=0$, Eq. (4·1).

Following the same procedure as in the previous section, we look for the spherical deviations from the background. We first assume that, asymptotically, the torsion has $O(3)$ symmetry to the lowest order. The torsion is thus assumed to have the form

$$\Theta^t=-(e^{\Phi})'\vartheta^t\wedge\vartheta^r+\frac{m}{r^2e^{\Phi}}f\vartheta^t\wedge\vartheta^r+O(r^{-2}e^{-\Phi_0})\,, \tag{5·4a}$$

$$\Theta^r=0+\frac{m}{r^2e^{\Phi}}h\vartheta^r\wedge\vartheta^t+O(r^{-2}e^{-\Phi_0})\,, \tag{5·4b}$$

$$\Theta^\theta=\frac{e^{\Phi}-1}{r}\vartheta^r\wedge\vartheta^\theta+\frac{m}{r^2e^{\Phi}}(g\vartheta^r\wedge\vartheta^\theta-k\vartheta^t\wedge\vartheta^\theta)+O(r^{-2}e^{-\Phi_0})\,, \tag{5·4c}$$

*) In the other common convention[1)] these combinations are $(1/l^2)(A_0-A_1)=-3(\alpha+(2/3)a)$, $(1/l^2)(2A_0+A_2)=6(\beta-(2/3)a)$, $(1/l^2)(A_0+2A_3)=-(4/3)(\gamma+(3/2)a)$.

$$\Theta^{\phi}=\frac{e^{\Phi}-1}{r}\vartheta^{r}\wedge\vartheta^{\phi}+\frac{m}{r^{2}e^{\Phi}}(g\vartheta^{r}\wedge\vartheta^{\phi}-k\vartheta^{t}\wedge\vartheta^{\phi})+O(r^{-2}e^{-\Phi_0}), \tag{5·4d}$$

where f, g, h and k are constants. With the Schwarzschild anti-de Sitter metric given in Eqs. (3·4), (3·5) and the torsion 2-form (5·4), by imposing the conditions (5·2), (5·3) we find that the components of the first field equation vanish to the order $O(r^{-3})$ and the components of the second field equation are

$$S^{t\phi}{}_{\phi}=S^{t\theta}{}_{\theta}=-\frac{m}{r^{2}e^{\Phi}}\left(\frac{1}{\kappa R^{2}}\right)(B_4+B_5)k+O(r^{-2}e^{-\Phi_0}), \tag{5·5a}$$

$$S^{tr}{}_{t}=-\frac{m}{3r^{2}e^{\Phi}}\left(\frac{1}{\kappa R^{2}}\right)[(2B_1+3B_4+B_6)f+2(B_1-B_6)g]+O(r^{-2}e^{-\Phi_0}), \tag{5·5b}$$

$$S^{r\phi}{}_{\phi}=S^{r\theta}{}_{\theta}=-\frac{m}{3r^{2}e^{\Phi}}\left(\frac{1}{\kappa R^{2}}\right)[(B_1-B_6)f+(B_1+3B_4+2B_6)g]+O(r^{-2}e^{-\Phi_0}). \tag{5·5c}$$

Requiring Eqs. (5·5) to vanish gives the following conditions:

$$\begin{pmatrix}2B_1+3B_4+B_6 & 2(B_1-B_6)\\ B_1-B_6 & B_1+3B_4+2B_6\end{pmatrix}\begin{pmatrix}f\\ g\end{pmatrix}=0, \tag{5·6a}$$

$$(B_4+B_5)k=0. \tag{5·6b}$$

We now relax our requirement of reflection invariance, i.e., we consider $SO(3)$ symmetry. We add the extra components given in (4·8) into Eq. (5·4) and get the same extra terms of the field equations—Eqs. (4·9) and (4·10). Requiring the field equations to vanish gives conditions (4·12), (4·13) on p, q, s and t.

The conditions on the parameters that allow solutions with constant torsion at large distance from the source with $O(3)$ symmetry are given in Table III.*) Additional possibilities permitted by $SO(3)$ symmetry are the same as those given in Table II.

Table III. Asymptotic constant torsion conditions for $O(3)$ symmetry.

	parameter conditions	restriction on the solution amplitudes f, g, h, k
$d_0=0$	$B_4+B_6=0=B_1+B_4$	no restriction
	$B_4+B_6=0\neq B_1+B_4$	$f=-g$
	$B_4+B_6\neq 0=B_1+B_4$	$f=2g,\ k=0$
	$B_4+B_6\neq 0\neq B_1+B_4$	$f=g=k=0$
$d_0\neq 0$	$B_4+B_6=0=B_1+B_4$	$k=0$
	$B_4+B_6=0\neq B_1+B_4$	$f=-g,\ k=0$
	$B_4+B_5=0=B_1+B_4$	$f=2g$
	$B_4+B_5=0\neq B_1+B_4$	$f=g=0$
	$(B_4+B_5)(B_4+B_6)\neq 0=B_1+B_4$	$f=2g,\ k=0$
	$(B_4+B_5)(B_4+B_6)(B_1+B_4)\neq 0$	$f=g=k=0$

*) In the other common convention[1] the combinations appearing here are $\kappa^{-1}(B_4+B_6)=2(a_5+12a_6)$, $\kappa^{-1}(B_1+B_4)=(3a_2+2a_5)$, $\kappa^{-1}(B_4+B_5)=2(a_4+a_5)$, all of which have played an important role in earlier studies.[5),8)~11)]

§ 6. Total energy scale

One more *scale* restriction on the PGT parameters can be obtained by matching the strength of the field to the amplitude of the source. Usually such a restriction is obtained by matching the weak field limit in the interior of a source to the Poisson equation of Newtonian gravity. This works well enough for solutions which are asymptotically flat, however, in the more general situation considered here we expect the Poisson equation to hold only to order $R^{-2}\neq 0$, so it is more difficult to get a precise condition on the parameters from the limit to the Poisson equation. Instead we can obtain a scale restriction with no such uncertainty from asymptotic matching to the total energy. For asymptotically anti-de Sitter solutions the total energy is well defined. It depends only upon the asymptotic underlying Riemannian geometry; for the Schwarzschild anti-de Sitter metric (3·4) and (3·5), the total energy is just the mass parameter m. On the other hand, an expression has recently been found[17] which gives the total energy as an asymptotic 2-surface integral in terms of the PGT quantities

$$E=-\frac{l^2}{8\pi}\oint(\partial_t\rfloor\vartheta^\alpha)\Delta H_\alpha+\Delta\omega^{\alpha\beta}\wedge(\partial_t\rfloor H_{\alpha\beta})+(\partial_t\rfloor\omega^{\alpha\beta})\Delta H_{\alpha\beta}\,, \tag{6·1}$$

where Δ applied to any quantity indicates the difference between the asymptotic solution value and the background value and $\rfloor$ indicates the interior product (or contraction) of a vector field with a differential form. By matching the value of this energy expression for our asymptotic solutions with the expected value m for a test mass we can find one further *scale* restriction on the parameters.

For the asymptotically Newtonian limit, $1/R^2\to 0$, we obtain, in general,

$$E=m\left[-A_0-\frac{1}{3}(A_0-A_1)(f+g)+\frac{1}{6}(2A_0+A_2)(f-2g)\right]. \tag{6·2}$$

Setting $E=m$ and using the conditions (4·16), (4·18) gives

$$A_0=-1\,, \tag{6·3}$$

independently of the values of the free torsion parameters.

For the case of asymptotically constant torsion solutions we obtain

$$E=m\left[-A_1-\frac{1}{3}(A_0-A_1)(f+g)+\frac{1}{6}(2A_0+A_2)(f-2g)\right]. \tag{6·4}$$

Using conditions (5·2), (5·3), Table III and $E=m$ then gives

$$A_0=A_1=-1\,, \tag{6·5}$$

independently of the values of the free torsion parameters in the solutions.

The asymptotically constant curvature case is, not surprisingly, more complicated. The general value for the energy expression can be expressed in the alternate forms:

$$E=m\left[-A_0-\frac{1}{3}(A_0-A_1)(f+g)+\frac{1}{6}(2A_0+A_2)(f-2g)\right]$$

$$+\frac{ml^2}{\kappa R^2}[(B_1-B_6)(f+g+2)+(B_4+B_6)(f-g)]\,, \tag{6·6a}$$

$$=m\left[-A_0+\frac{l^2}{\kappa R^2}(B_1-B_6)(f+g+2)\right]$$

$$-\frac{m}{2}(A_0-A_1)(f+g)+\frac{ml^2}{2}[c_1(f-2g)+d_1(f-g)]\,, \tag{6·6b}$$

$$=m\left[-A_0+\frac{l^2}{\kappa R^2}(B_1-B_6)(f+g+2)\right]$$

$$-\frac{m}{2}(A_0-A_1)(f+g)-\frac{ml^2}{4}[c_1(f+g)+(3c_1+2d_1)(f-g)]\,, \tag{6·6c}$$

which are convenient for evaluation. In each of the cases in Table I the last term in Eq. (6·5b) or (6·5c) clearly vanishes so the total energy reduces to

$$E=m\left[-A_0-\frac{1}{2}(A_0-A_1)(f+g)+\frac{l^2}{\kappa R^2}(B_1-B_6)(f+g+2)\right]. \tag{6·7}$$

Note that for asymptotically constant curvature, in contrast to constant torsion or zero curvature, the total energy does depend on the solution parameters f, g except for the special case

$$\frac{1}{2l^2}(A_0-A_1)-\frac{1}{\kappa R^2}(B_1-B_6)=0\,. \tag{6·8}$$

We also note that the total energy depends on the PGT parameter B_1 which does not appear in the asymptotic constant curvature conditions. Perhaps more significantly, we have found that the total energy depends upon the asymptotic value of the curvature R^{-2} *unless* $f+g+2=0$. In this special case $E=-A_1 m$, which leads to the condition $A_1=-1$. (The condition $f+g=-2$ *is* satisfied by the torsion for the known exact PGT anti-de Sitter solution.[14])

Finally, we note that (i) in all cases the solution parameters p, q, s, t, h, k do not make any contribution to the energy expressions; (ii) for anti-de Sitter space without any torsion the total energy is

$$E=m\left[-A_0+\frac{2l^2}{\kappa R^2}(B_1-B_6)\right], \tag{6·9}$$

to be matched with the value m. The values obtained here for the PGT theory should be compared with the corresponding Einstein or ECSK theory results. For these two special PGT theories all parameters vanish except A_0; the matching gives $A_0=-1$.

§ 7. Conclusion

Gravitating systems which have solutions that are well behaved asymptotically are of considerable interest. In particular, if the asymptotic Riemannian background curvature vanishes or is a negative constant (i.e., anti-de Sitter space), total conserved quantities can be defined and the gravitational field appears to have a point source and is thus a natural generalization of an asymptotically Newtonian field.

Here we have considered the Poincaré Gauge Theory (PGT), a promising alternative to Einstein's theory. We have investigated the necessary conditions on the ten PGT parameters which permit solutions with an asymptotically anti-de Sitter metric.

For the Riemann-Cartan geometry of the PGT we have found that there are two possible types of background. Either the Riemann-Cartan curvature has the same constant value asymptotically as the Riemannian curvature or the Riemann-Cartan curvature *vanishes* asymptotically (i.e., the geometry is asymptotically *teleparallel*).

For each case, not surprisingly, the asymptotic solutions to the PGT theory must satisfy a restriction connecting the cosmological constant with the asymptotic value of the Riemannian scalar curvature. Furthermore, the ten PGT parameters must satisfy from one to three additional conditions. In each case the solutions have at least one free constant. All possible spherically symmetric cases have been tabulated in Tables I, II and III for both $SO(3)$ and $O(3)$ symmetry.

By matching the total energy of a test mass, as obtained from an integral over the 2-sphere at infinity, with the known physical value, we have obtained one additional restriction setting the scale of the dimensionless PGT parameters.

Naturally, known exact solutions fit into the possible cases which we have found. However, we have found other cases of parameter values which permit solutions with asymptotically anti-de Sitter behavior for which there is, as yet, no known exact solution. Hence our work reveals possibilities for finding new exact solutions for the PGT.

To our knowledge, aside from our positive energy test results,[7)] our results presented here are the only available set of conditions on the 10 PGT parameters for the important case of solutions with asymptotically anti-de Sitter boundary conditions. Of course other restrictions can and should be found by applying other criteria just as in the asymptotically flat case.[5),8)~11)]

We recall that in the asymptotically flat case overlooking the required asymptotic conditions led to the relative neglect of the physically more interesting possibilities. Asymptotically flat solutions are possible only if one or more parameter combinations vanish. These parameter combinations turn out to be the same as certain combinations which play an important role in the linearized theory where they have a physical interpretation as the "masses" of the linearized modes.[5),8),9)] These same combinations also play an important role in the Hamiltonian analysis[10),11)] of asymptotically flat solutions.

We expect that similar important roles are played by the various parameter combinations which we have identified here as necessary conditions for asymptotically anti-de Sitter PGT solutions.

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