Quantum Geometry and black holes

Chung-Hsien Chou
Institute of Physics, Academia Sinica, Taiwan, ROC

Oct 5, 2006
NTHU HEP seminar

- ✦ Introduction
- ✦ ADM formulation
- ✦ Canonical Quantization
- ✦ From Metric to Connection
- ✦ Quantum Geometry and Black Holes
- ✦ Summary and Conclusion

Prelude

- General relativity [Einstein,1915] is a dynamical theory of spacetime.
- General relativity is a very successful classical field theory.
- Schwarzschild [1916]:

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 + \frac{1}{1 - \frac{2GM}{r}}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

  - $r = 2GM$: horizon; $r = 0$: singularity.

- **Pattern Singularity Theorem**:
  If a spacetime of sufficient differentiability satisfies
  - a condition on the curvature
  - a causality condition
  - and an appropriate initial and/or boundary condition

  then there are null or timelike inextensible incomplete geodesics.

  $\Rightarrow$ Singularities are unavoidable in GR.

- GR can not be complete! It predicts its own breakdown.

✦ Cosmic Censorship Conjecture [Penrose]: "Naked singularities cannot form from gravitational collapse in an asymptotically flat spacetime that is non-singular on some initial spacelike hypersurface."

- Certain types of "trivial" naked singularities must be excluded.
- Initial, cosmological, singularities are excluded.
- There is no proof. This is the major unsolved problem in classical GR.

✦ Why does one want to go beyond general relativity?

✦ One of the big challenges in physics:

how to make general relativity consistent with quantum mechanics?

Black hole thermodynamics

- Hawking (1972): the area of the event horizon of a black hole cannot decrease.
- Bekenstein (1973): associate an entropy to a black hole

$$S_{BH} = kA$$

- Hawking (1975): black hole temperature $T = \frac{1}{8\pi M}$,

$$S_{BH} = \frac{1}{4}A$$

- What are the microscopic degrees of freedom responsible for this entropy?
- What are the higher order corrections to the Benkenstein-Hawking entropy formula?

Sketch of canonical quantization

- Pick a Poisson algebra of classical quantities.
- Represent these quantities as quantum operators acting on a space of quantum states.
- Implement any constraint you may have as a quantum operator equation and solve for the physical states.
- Construct an inner product on physical states.
- Develop a semiclassical approximation and compute expectation values of physical quantities.

Canonical analysis in ADM variable

- Einstein-Hilbert action [in metric variables]

$$I[g_{\mu\nu}] = \frac{1}{16\pi G}\int d^4x\sqrt{-g}(\mathcal{R}-2\Lambda)$$

- ADM Decomposition: introduce a foliation of spacetime $\mathcal{M} = \Sigma \times \mathrm{R}$

  - $g_{\mu\nu} \rightarrow q_{ab}$, N_a : shift function, N: laspe function.
  - $ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -N^2(dx^0)^2 + q_{ab}(dx^a + N^a dx^0)(dx^b + N^b dx^0)$

$$g_{\mu\nu} = \begin{pmatrix} q_{ab}N^aN^b - N^2 & q_{ab}N^a \\ q_{ab}N^b & q_{ab} \end{pmatrix}, g^{\mu\nu} = \begin{pmatrix} -1/N^2 & N^a/N^2 \\ N^b/N^2 & q^{ab} - N^aN^b/N^2 \end{pmatrix}$$

- After performing the Legendra transformation:

$$I[q_{ab}, \pi^{ab}, N_a, N] = \frac{1}{16\pi}\int dt \int_\Sigma d^3x[\pi^{ab}\dot{q}_{ab} - \mathcal{H}]$$

  - $\pi^{ab} = -\frac{\sqrt{q}}{16\pi G}(K^{ab} - Kq^{ab})$: momenta canonically conjugate to q_{ab}, $K_{ab} = \frac{1}{2N}(-\partial_0 q_{ab} + \nabla_a N_b + \nabla_b N_a)$: extrinsic curvature.

$$\mathcal{H}(q_{ab},\pi^{ab},N_a,N) = N^a H_a(q_{ab},\pi^{ab}) + N H(q_{ab},\pi^{ab})$$

- Super-momentum constraint: $H_a(q_{ab},\pi^{ab}) = -\frac{2}{16\pi G}\nabla_b \pi^b{}_a \quad (=0)$
- Super-Hamiltonian constraint:

$$\begin{aligned} H(q_{ab},\pi^{ab}) &= \frac{8\pi G}{\sqrt{q}}(q_{ac}q_{bd} + q_{ad}q_{bc} - q_{ab}q_{cd})\pi^{ab}\pi^{cd} - \frac{\sqrt{q}}{16\pi G}(R(q) - 2\Lambda) \\ &= \frac{\sqrt{q}}{16\pi G}[K^{ab}K_{ab} - K^2 - R(q) + 2\Lambda] \quad (=0) \end{aligned}$$

✦ Degrees of freedom of GR in 4D:

6 pairs (q_{ab},π^{ab}) subject to 4 constraints = 2 FIELD d.o.f.

✦ The Poisson brackets are

$$\begin{aligned} \{\pi^{ab}(x), q_{cd}(y)\} &= 16\pi\delta^a_{(c}\delta^b_{d)}\delta(x,y), \\ \{q_{ab}(x), q_{cd}(y)\} &= \{\pi^{ab}(x), \pi^{cd}(y)\} = 0 \end{aligned}$$

✦ Phase space variables: (q_{ab},π^{cd})

Canonical Quantization of GR

- Does not require background spacetime (background independence)
- Can be used for strong and weak GR fields.
- Conjugate variables:

$$\{q_{ab}(\vec{x}), \pi^{cd}(\vec{y})\}_{P.B.} = \frac{1}{2}(\delta_a{}^c \delta_b{}^d + \delta_b{}^c \delta_a{}^d)\delta^3(\vec{x}-\vec{y})$$

- Canonical Quantization :

$$\{\ ,\ \}_{P.B.} \rightarrow \frac{1}{i\hbar}[\ ,\ ]; \quad q_{ab} \rightarrow \hat{q}_{ab}, \quad \pi^{ab} \rightarrow \hat{\pi}^{ab}$$

- Metric representation: Wavefunction $\Psi[q_{ab}]$
  - $\hat{q}_{ab}\Psi[q_{ab}] = q_{ab}\Psi[q_{ab}]$; $\quad \hat{\pi}^{ab}\Psi[q_{ab}] = \frac{1}{i\hbar}\frac{\delta}{\delta q_{ab}}\Psi[q_{ab}]$
- Constraints (First Class) (Dirac Quantization):

$$\hat{H}_a(\hat{q}_{ab}, \hat{\pi}^{ab})\Psi[q_{ab}] = \hat{H}_a(\hat{q}_{ab}, \frac{1}{i\hbar}\frac{\delta}{\delta q_{ab}})\Psi[q_{ab}] = 0$$

$\Leftrightarrow \Psi[q'_{ab}] = \Psi[q_{ab}]$ if q_{ab} is related to q'_{ab} by a 3-dimensional diffeomorphism

$\Leftrightarrow \Psi[\mathcal{G}]$. 3-geometry $\mathcal{G} \in$ SUPERSPACE:

Space of all 3-geometries (equivalence class of 3-metrics) $q'_{ab} \sim q_{ab}$ iff they are related by 3-dim. general coordinate transformation.

- Constraint Algebras (Classical):

(Definition: $H_a[N^a] \equiv \int_\Sigma N^a(\vec{x}) H_a(\vec{x}) d^3x \quad \Sigma=$ Cauchy surface)

• Dirac Algebra (explicitly with (q_{ab}, π^{ab}) conjugate pair and Einstein's theory)

$$
\begin{aligned}
\{H_a[N^a], H_b[M^b]\}_{P.B.} &= -H_a[(\mathcal{L}_{\vec{N}} M)^a] \\
\{H_a[N^a], H[M]\}_{P.B.} &= -H[(\mathcal{L}_{\vec{N}} M)] \\
\{H[N], H[M]\}_{P.B.} &= -H_a[(q^{ab}(N\partial_b M - M\partial_b N))]
\end{aligned}
$$

- Quantum super-Hamiltonian Constraint: Wheeler-DeWitt Equation

$$
\text{"}[G_{abcd} \frac{\delta}{\delta q_{ab}} \frac{\delta}{\delta q_{cd}} + \sqrt{q}(R(q) - 2\Lambda)]\text{"} \Psi[\mathcal{G}] = 0
$$

Supermetric $G_{abcd} = \frac{8\pi G}{\sqrt{q}}(q_{ac}q_{bd} + q_{ad}q_{bc} - q_{ab}q_{cd})$.

Symbolically,

$$[\frac{\delta^2}{\delta\mathcal{G}^2} + (R(q) - 2\Lambda)]\Psi[\mathcal{G}] = 0$$

✦ Technical issues:

Ordering, Regularization, Anomalies, Explicit Solutions, of Wheeler-DeWitt Equation.

✦ Important conceptual issues: Where/what is physical "time" in Quantum Gravity?

• Note: x^0 is not "time". Theory is reparametrization invariant. H does not generate "time" translation: $\exp\left(\frac{-ix^0 H}{\hbar}\right)\Psi[\mathcal{G}] = \Psi[\mathcal{G}]$.

✦ B. S. DeWitt [Phys. Rev. **160**, 1113 (1967)]:

Supermetric $G^{abcd}\delta q_{ab}\delta q_{cd} = -(\delta\xi)^2 + (\frac{3}{32})\xi^2\bar{G}_{AB}\delta\xi^A\delta\xi^B$ i.e.

$G^{\{ab\}\{cd\}} = diag(-1, \frac{3}{32}\xi^2\bar{G}_{AB})$; $A, B = 1, 2, 3, 4, 5.$

$\bar{G}_{AB}$: positive-definite $\Rightarrow$ supermetric has signature $(-, +, +, +, +, +)$.

" $-$ " direction is associated with "intrinsic time" $\xi = \sqrt{32/3}(\det q)^{1/4}$.

Superspace is hyperbolic.

Super-Hamiltonian constraint has "dynamical" content.

Wheeler-DeWitt Equation:

$$"[-\frac{\delta^2}{\delta\xi^2} + \frac{32}{3\xi^2}\bar{G}^{AB}\frac{\delta}{\delta\xi^A}\frac{\delta}{\delta\xi^B} + \frac{3\xi^2}{32}(R(q) - 2\Lambda)]"\Psi[\mathcal{G}] = 0$$

In simple homogeneous isotropic cosmological models (e.g. of minisuperspace), $\xi \propto [a(t)]^{3/2}$ ($a =$ expansion scale factor).

The triad formulation

- To use a triad (a set of 3 1-forms at each point in Σ)

$$q_{ab} = e^i_a e^j_b \delta_{ij}$$

  - Densitized triad: $E^a_i = \frac{1}{2}\epsilon^{abc}\epsilon_{ijk}e^j_b e^k_c$
  - Additional 3 (Gauss) constraints: $G_i(E^a_j, K^j_a) = \epsilon_{ijk}E^{aj}K^k_a = 0$

- With new variables, the action of GR becomes

$$I[E^a_j, K^j_a, N_a, N, N^j] = \frac{1}{8\pi}\int dt \int_\Sigma d^3x[E^a_i \dot{K}^i_a$$
$$-N^b H_b(E^a_j, K^j_a) - N H(E^a_j, K^j_a) - N^i G_i(E^a_j, K^j_a)]$$

The sympletic structure now becomes

$$\{E^a_j(x), K^i_b(y)\} = 8\pi\delta^a_b\delta^i_j\delta(x,y),$$
$$\{E^a_j(x), E^b_i(y)\} = \{K^j_a(x), K^i_b(y)\} = 0$$

The Ashtekar-Barbero connection variables

- There is a natural $so(3)$-connection (spin-connection Γ^i_a) that defines the notion of covariant derivative compatible with the dreibein

$$\partial_{[a}e^i_{b]} + \epsilon^i{}_{jk}\Gamma^j_{[a}e^k_{b]} = 0$$

  - Ashtekar-Barbero variable: $A^i_a = \Gamma^i_a + \gamma K^i_a$
  - γ : Immirzi parameter, $\gamma \in \mathrm{R} - \{0\}$.

- With the connection variables, the action becomes

$$I[E^a_j, A^j_a, N_a, N, N^j] = \frac{1}{8\pi}\int dt \int_\Sigma d^3x [E^a_i \dot{A}^i_a$$
$$-N^b H_b(E^a_j, A^j_a) - N H(E^a_j, A^j_a) - N^i G_i(E^a_j, A^j_a)]$$

  - $H_b(E^a_j, A^j_a) = E^a_j F^j_{ab} - (1+\gamma^2)K^i_b G_i = 0$
  - $H(E^a_j, A^j_a) = \frac{E^a_i E^b_j}{\sqrt{\det(E)}}(\epsilon^{ij}{}_k F^k_{ab} - 2(1+\gamma^2)K^i_{[a}K^j_{b]}) = 0$
  - $G_i(E^a_j, A^j_a) = D_a E^a_i = 0$

- where $F^i_{ab} = \partial_a A^i_b - \partial_b A^i_a + \epsilon^i{}_{jk} A^j_a A^k_b$ and $D_a E^a_i = \partial_a E^a_i + \epsilon_{ij}{}^k A^j_a E^a_k$

✦ The Poisson bracket of the new variables are

$$\begin{aligned}\{E^a_j(x), A^i_b(y)\} &= 8\pi\gamma\delta^a_b\delta^i_j\delta(x,y), \\ \{E^a_j(x), E^b_i(y)\} &= \{A^j_a(x), A^i_b(y)\} = 0\end{aligned}$$

✦ Phase space variables: (A^i_a, E^b_j)

✦ Series of (Canonical) transformations:

Metric variables: (q_{ab}, π^{ab})

$\longrightarrow (e_{ai}, \pi^{ai})$+ 3 gauge constraints (Gauss' Law)

$\longrightarrow (E^a_i, K^i_a)$+ Gauss' Law

$\longrightarrow (E^a_i, A^{i\,-}_a = \Gamma^i_a - iK^i_a)$+ Gauss' Law (Ashtekar Variable)

$\longrightarrow (E^a_i, A^i_a = \Gamma^i_a + \gamma K^i_a)$+ Gauss' Law (Ashtekar-Barbero Variable)

(related discussion: C.H.C, R.H. Tung, H. L. Yu, PRD **72**, 064016 (2005))

Conceptual Breakthroughs

- Distinction between geometrodynamics and gauge dynamics is bridged. Identify E^a_j as the momentum conjugate to the gauge potential A^i_a;
  $\Rightarrow (E^a_j, A^i_a)$ phase space identical to Yang-Mills Theory.
- Quantum States can be wavefunctions in A-representation $\Psi[A]$, with $E^a_i = (\frac{8\pi G\hbar}{c^3})\frac{\delta}{\delta A^i_a}$. All manipulations done on gauge variables.

Technical Breakthroughs

- Constraints much simpler:
- Exact solution found (e.g. Chern-Simons state, in field theory variables)
- Loop variables: Wilson loops: holonomy elements.
  - Gauss's constraint solved by Ψ[Wilson loops in A] ;
  - $H_a = 0$ solved by Ψ[knot classes of Wilson loops in A].
- Super-Hamiltonian constraint still difficult, but can be made well-defined:
  - Volume V and area $\mathcal{A}$ operators : well-defined operators acting on loop and spin network states and have discrete spectra.
- Derivation of horizon entropy, both for black hole and cosmological horizons.
  - Black hole evaporation via transition from higher $\mathcal{A}$ states to lower $\mathcal{A}$ states.

• Matching Bekenstein-Hawking entropy formula for large black holes

$$k_B \ln N = S_{BH} \approx k_B(\frac{\mathcal{A}}{4l_p^2}); \quad \mathcal{A} \gg l_p^2,$$

including quantum logarithmic correction when $\mathcal{A}$ is small,

$$S_{BH} = k_B(\frac{\mathcal{A}}{4l_p^2}) - \frac{1}{2}k_B \ln \frac{\mathcal{A}}{4l_p^2} + K_0.$$

(related discussion: C.H.C, Y. Ling, C. Soo, H. L. Yu, PLB **637**, 12 (2006))

✦ Resolution of big-bang singularity, curvature bounded and not divergent. (Bojowald)

✦ Addressing black hole singularity (Ashtekar and Bojowald): Minisuperspace (spherical symmetric) investigation.
Classical black hole singularity does not seem to pose difficulties to quantum evolution of wavefunction.

The construction of LQG

- Holonomy:

$$U[A,\gamma](s) = \mathcal{P}\exp\int_\gamma A = \mathcal{P}\exp\int_\gamma ds\dot{\gamma}^a A_a^i(\gamma(s))\tau_i$$

- The key idea of LQG is to choose the loop states as the basis states for quantum gravity

$$\Psi_\alpha(A) = TrU[A,\gamma](s)$$

- The spin network state $\Psi_S(A)$: a cylindrical function f_S associated to spin network S whose graph is Γ

$$\Psi_S(A) = \Psi_{\Gamma,f_S}(A) = f_S(U[A,\gamma_1],...,U[A,\gamma_n])$$

Quantization of area

- Rovelli and Smolin (1994); Ashtekar, Lewandowski et al (1995): given a surface

$$A(\mathcal{S}) = \int_{\mathcal{S}} \sqrt{n_a E_i^a n_b E_i^b} d^2\sigma$$

- The quantum area spectrum is

$$A(\mathcal{S})|S\rangle = 8\pi\gamma \sum_P \sqrt{j_P(j_P+1)}|S\rangle$$

- Why is geometry discrete ?
  - The value of a triad in a given point is conjugate to the connection in the same point but Poisson commute with values of the connection in any other points.
  - The flux operator will only notice intersection points.
  - The eigenvalues of the flux operator: discrete.
  - Triad $\rightarrow$ metric $\rightarrow$ length, area, volume.... Geometry is discrete.

- The result is topological and background independent.
- The spin of the lines of a spin network can be viewed as "quanta of area".

Quantum Geometry and Schwarzschild Singularity

- In connection dynamics the spherical symmetric connection 1-form A_a^i and triad E_i^a are given by:

$$A = c\tau_3 dx + (a\tau_1 + b\tau_2)d\theta + (-b\tau_1 + a\tau_2)\sin\theta d\varphi + \tau_3\cos\theta d\varphi$$
$$E = p_c\tau_3\sin\theta\frac{\partial}{\partial x} + (p_a\tau_1 + p_b\tau_2)\sin\theta\frac{\partial}{\partial\theta} + (-p_b\tau_1 + p_a\tau_2)\sin\theta d\varphi$$

the corresponding co-triad ω:

$$\omega = \omega_c\tau_3 dx + (\omega_a\tau_1 + \omega_b\tau_2)d\theta + (-\omega_b\tau_1 + \omega_a\tau_2)\sin\theta d\varphi$$

where

$$\omega_a = \frac{\sqrt{|\ p_c\ |}p_a}{\sqrt{p_a^2 + p_b^2}}; \quad \omega_b = \frac{\sqrt{|\ p_c\ |}p_b}{\sqrt{p_a^2 + p_b^2}}; \quad \omega_c = \frac{sgnp_c\sqrt{p_a^2 + p_b^2}}{\sqrt{|\ p_c\ |}}$$

$$\gamma K := A - \Gamma = c\tau_3 dx + (a\tau_1 + b\tau_2)d\theta + (-b\tau_1 + a\tau_2)\sin\theta d\varphi$$

- The Super-Hamiltonian constraint now becomes

$$2cp_c(ap_a + bp_b) + (p_a^2 + p_b^2)(a^2 + b^2 + \gamma^2) = 0$$

Due to the symplectic structure of the phase space

$$\Omega = \frac{L_0}{2\gamma G}(2da \wedge dp_a + 2db \wedge dp_b + dc \wedge dp_c)$$

- Changing of variables and performing canonical quantization

$$p_a \equiv x; \quad p_b \equiv y; \quad p_c \equiv z;$$

$$\frac{2a}{2\gamma G} \equiv \frac{\hbar}{i}\frac{\partial}{\partial x} \equiv p_x; \quad \frac{2b}{2\gamma G} \equiv \frac{\hbar}{i}\frac{\partial}{\partial y} \equiv p_y; \quad \frac{c}{2\gamma G} \equiv \frac{\hbar}{i}\frac{\partial}{\partial z} \equiv p_z;$$

$$zp_z(xp_x + yp_y) + (x^2 + y^2)(\frac{p_x^2 + p_y^2}{4}) + \frac{x^2 + y^2}{4G^2} = 0$$

- Let $x = r\cos\theta, y = r\sin\theta$, we have

$$[-z\frac{\partial}{\partial z}r\frac{\partial}{\partial r} - \frac{1}{4}r\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{r^2}{4\hbar^2 G^2}]\Phi(r, z) = 0$$

This is our Wheeler-De Witt equation for the spherical symmetric case.

- One of the solution to this equation is

$$\Phi(r,z) = C\exp[kz^{1/2} + \frac{1}{k}\frac{r^2}{4\hbar^2G^2}z^{-1/2}]$$

where C, k are constants in r, z.

- The solution $\Phi(r,z)$ can be written in the form which satisfies the Hamilton-Jacobi equation where

$$\Phi(r,z) = C\exp[kz^{1/2} + \frac{1}{k}\frac{r^2}{4\hbar^2G^2}z^{-1/2}] \equiv C\exp\frac{iS}{\hbar}$$

$$S = \frac{\hbar}{i}[kz^{1/2} + \frac{1}{k}\frac{r^2}{4\hbar^2G^2}z^{-1/2}]$$

$$p_r = \frac{\partial S}{\partial r} = \frac{\hbar}{i}[\frac{rz^{-1/2}}{2k\hbar^2G^2}]; \quad p_z = \frac{\partial S}{\partial z} = \frac{\hbar}{i}[\frac{kz^{-1/2}}{2} - \frac{r^2z^{-3/2}}{8k\hbar^2G^2}];$$

One can show that it satisfies

$$zp_zrp_r + \frac{1}{4}rp_rrp_r + \frac{r^2}{4G^2} = 0$$

Summary

- Quantum Gravity as Quantum theory of connections $\Psi[A]$.
- Canonical formulation: $4 = (3 + 1)$

  From $(q, \pi) \Rightarrow (E, A)$

  $\Rightarrow$ Loop Quantum Gravity

  $\Rightarrow$ Geometric quantities are discrete.
- Black Hole entropy counting. Higher-order corrections.
- Exact semiclassical solution for the Wheeler-DeWitt Equation for a spherical symmetric Schwarzschild black hole in connection representation is obtained.