

HAGEDORN STRINGS

FENG-LI LIN
師大物理系



OUTLINE

- HIGH TEMPERATURE STRINGS
- STRING/BLACK HOLE CORRESPONDENCE
- HAGEDORN STRINGS ON ADS

A puzzle

- If black hole can form at the LHC, then can field theory explain the Bekenstein-Hawking entropy?
- For field theory, the entropy associated with the high energy states at energy E is proportional to $\log E$.
- However, the black hole entropy is proportional to $E^{(d-1)/(d-2)}$.
- Therefore, there is a mis-match in entropy for the process

$$a + b \longrightarrow \text{virtual particles} \longrightarrow \text{black hole}$$

High Temperature Strings

Peculiarity of string spectrum

- finite set of massless particles, i.e. gauge bosons, graviton

$$A_{\mu}^a, \quad g_{\mu\nu}$$

- exponential degeneracy of states at high energy (Hagedorn Spectrum)

$$e^{\beta_s E} \quad \text{with} \quad \beta_s \sim l_s$$

- Hagedorn entropy

$$S_{string} \sim \beta_s E$$

- At high energy, the string behavior is very different from particle theory

Random Walk of a String

- The above Hagedorn spectrum can be understood as random walk in target space since the energy of a string is proportional to the length of random walk:

$$\omega(\epsilon) \sim V \cdot \frac{1}{\epsilon} \cdot \frac{e^{\beta_s \epsilon}}{W(\epsilon)}$$

- in d non-compact dimensions

$$\omega(\epsilon) \sim V \cdot \frac{e^{\beta_s \epsilon}}{\epsilon^{1+d/2}}$$

- all space are compact

$$\omega(\epsilon) \sim \frac{e^{\beta_s \epsilon}}{\epsilon}$$

Thermal Partition Function of String

- Counting the physical string states, the thermal partition function of a single string is

$$Z_1(\beta) = \int_{F_0} \frac{d^2\tau}{\tau_2} \text{Tr}(e^{-\beta H_{w.s.}})$$

- Or using the path integral formulation

$$\begin{aligned} Z_1(\beta) &= V_{25}\beta \int_{F_0} \frac{d^2\tau}{4\tau_2} (4\pi^2\alpha'\tau_2)^{-13} |\eta(\tau)|^{-48} \sum_{m,w=-\infty}^{\infty} \exp\left(-\frac{\beta^2|m-w\tau|^2}{4\pi\alpha'\tau_2}\right) \\ &= V_{25}\beta \int_R \frac{d^2\tau}{4\tau_2} (4\pi^2\alpha'\tau_2)^{-13} |\eta(\tau)|^{-48} \sum_{r=-\infty}^{\infty} \exp\left(-\frac{\beta^2 r^2}{4\pi\alpha'\tau_2}\right) \end{aligned}$$

Extract Hagedorn Behaviors

- Using free string gas (obeying Maxwell-Boltzmann statistics) approximation, the free energy of string gas is

$$F(\beta) = -\frac{1}{\beta} \frac{Z_1(\beta)}{V_{25}} \quad \text{i.e. } Z_{string\ gas} = e^{Z_1(\beta)}$$

- Using the saddle-point approximation

$$F(\beta) \simeq - \int_0^\infty \frac{d\tau_2}{4\tau_2} (4\pi^2 \alpha' \tau_2)^{-13} \exp(4\pi\tau_2) \exp\left(-\frac{\beta^2 \tau_2}{4\pi\alpha'}\right)$$

- There exists a Hagedorn critical temperature ($\tau_2 \sim \epsilon$)

$$\beta_H = 4\pi\sqrt{\alpha'}$$

Near Hagedorn T

$$e^{4\pi(\beta_H^2 - \beta^2)\tau_2/\beta_H^2} \sim e^{8\pi\beta_H(\beta_H - \beta)\tau_2/\beta_H} \sim e^{(\beta_H - \beta)R}$$

Thermal Winding Scalar

- The Hagedorn phase transition can be understood as the appearance of the thermal tachyonic winding mode as $T > T_H$

$$X^0(\sigma, \tau) = x^0 + \frac{2\pi m\tau}{\beta} + \frac{n\beta\sigma}{\pi} + \dots$$

set $m = 0$, $n = \pm 1$, mass shell condition is

$$M^2 = -8 + \frac{\beta^2}{\pi^2} = -8 + \frac{1}{\pi^2 T^2}.$$

- Or by the observation

$$\epsilon^{-d/2} \sim \int \frac{d^d k}{(2\pi)^d} e^{-\epsilon k^2} \quad \longrightarrow$$

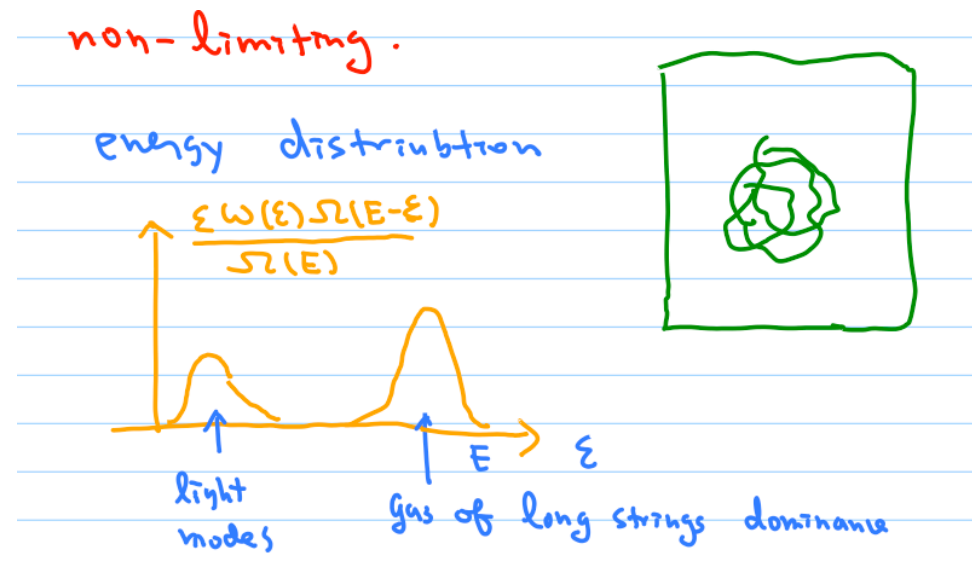
$$\log Z(\beta) \sim V \int_0^\infty \frac{d\epsilon}{\epsilon} \frac{e^{-(\beta - \beta_s)\epsilon}}{\epsilon^{d/2}} \sim -Tr \log[-\nabla^2 + M^2(\beta)]|_{R^d}$$

Hagedorn Thermodynamics I

- From free energy we can evaluate the other thermodynamic quantities. However, the canonical ensemble may break down if it is not equivalent to micro-canonical ensemble. For $d > 2$,

$$\Omega(E) = \int_{L-i\infty}^{L+i\infty} \frac{d\beta}{2\pi i} e^{\beta E} Z(\beta) = CV \frac{e^{\beta_H E + \gamma_0 V}}{E^{(d/2+1)}} (1 + \mathcal{O}(1/E))$$

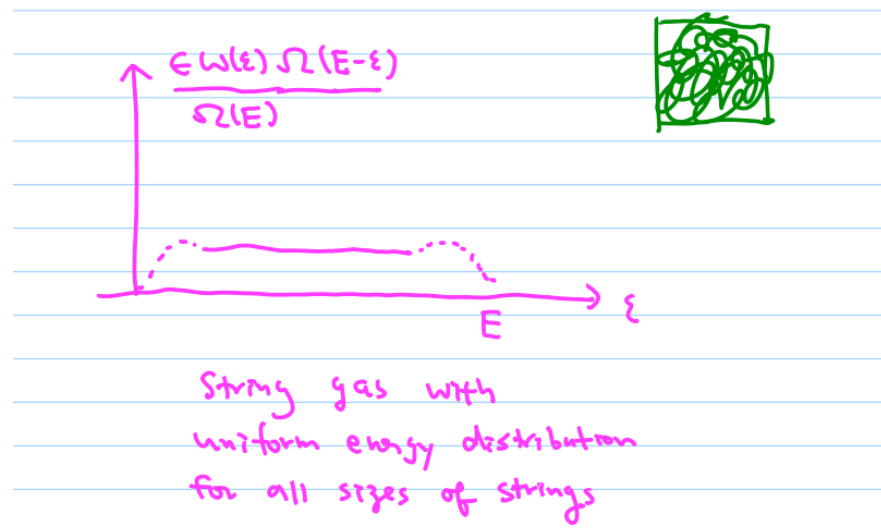
- The specific heat from micro-canonical ensemble density of states is negative and divergent at Hagedorn temperature, so the Hagedorn thermodynamics is not well-defined.



Hagedorn Thermodynamics II

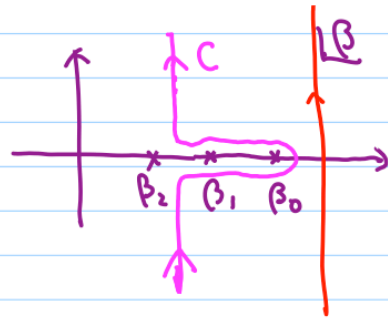
- If all the space are compact, the micro-canonical and canonical ensembles are equivalent but the specific heat from micro-canonical ensemble is divergent for all temperature, therefore, the Hagedorn temperature is limiting.

$$S = \log \Omega = \beta_H E, \quad C_V = -\beta^2 \left(\frac{\partial \beta}{\partial E} \right)_V^{-1} = \infty$$



Hagedorn Thermodynamics III

- We can regularize the Hagedorn thermodynamics by compactifying the large spatial dimensions, this introduces the sub-leading singularities in the partition function,



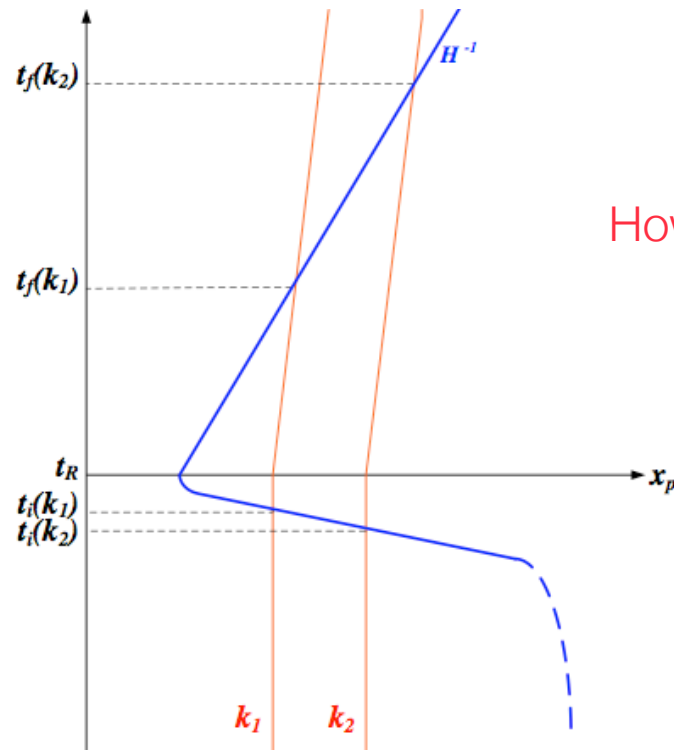
$$\Rightarrow \omega(\varepsilon) = \begin{cases} V \frac{e^{\beta_H \varepsilon}}{\varepsilon^{N/2+1}}, & \varepsilon \ll \frac{R^2}{\ell_s^3} \\ \frac{e^{\beta_H \varepsilon}}{\varepsilon}, & \varepsilon \gg \frac{R^2}{\ell_s^3} \end{cases}$$

$$C_V^{\text{nc}} = -\frac{d+2}{2} \left(\frac{T_H}{T - T_H} \right)^2, \quad \longrightarrow \quad C_V^{\text{closed}} \approx \frac{R^2 / \ell_s^3}{T(1 - T/T_H)},$$

Hagedorn Cosmology

- In Brandenberger et al, they proposed a model using Hagedorn thermal fluctuation to produce CMB spectrum

$$\begin{aligned}
 C^{\mu}_{\nu}{}^{\sigma}{}_{\lambda} &= \langle \delta T^{\mu}_{\nu} \delta T^{\sigma}_{\lambda} \rangle = \langle T^{\mu}_{\nu} T^{\sigma}_{\lambda} \rangle - \langle T^{\mu}_{\nu} \rangle \langle T^{\sigma}_{\lambda} \rangle \\
 &= 2 \frac{G^{\mu\alpha}}{\sqrt{-G}} \frac{\partial}{\partial G^{\alpha\nu}} \left(\frac{G^{\sigma\delta}}{\sqrt{-G}} \frac{\partial \ln Z}{\partial G^{\delta\lambda}} \right) + 2 \frac{G^{\sigma\alpha}}{\sqrt{-G}} \frac{\partial}{\partial G^{\alpha\lambda}} \left(\frac{G^{\mu\delta}}{\sqrt{-G}} \frac{\partial \ln Z}{\partial G^{\delta\nu}} \right),
 \end{aligned}$$



However, the power spectrum is not scale-invariant!

String/Black Hole Correspondence

Bekenstein-Hawking v.s. Hagedorn

- The Bekenstein-Hawking entropy for AdS_{d+1} -charged Schwarzschild black hole ($d > 2$)

$$S_{BH} = \frac{d-2}{d-1} \frac{1 + \frac{d}{d-2} \frac{r_+^2}{l_{AdS}^2} - \mu^2}{1 + \frac{r_+^2}{l_{AdS}^2} + \mu^2} \beta_{BH} M, \quad \beta_{BH} = \frac{4\pi l_{AdS}^2 r_+}{(d-2)(1-\mu^2)l_{AdS}^2 + dr_+^2}, \quad \mu = \frac{w_{d+1} Q}{2r_+^{d-2}}$$

- BTZ

$$S_{BH} = 2 \frac{1 - (l_{AdS}\mu)^2}{1 + (l_{AdS}\mu)^2} \beta_{BH} M, \quad \beta_{BH} = \frac{2\pi l_{AdS}^2}{r_+} \frac{1}{1 - (l_{AdS}\mu)^2}, \quad \mu = 4G_N J / r_+^2,$$

- Hagedorn entropy

$$S_{string} = \beta_s(k) M,$$

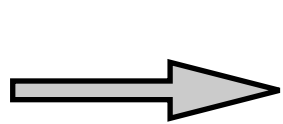
Horowitz-Polchinski

- It suggests that there is a correspondence between Hagedorn strings and black hole when

$$\beta_s = \beta_{BH}$$

- This leads Horowitz and Polchinski (first by Susskind) to conjecture that a black hole will turn into a single string when the size of the horizon is of order of string length, e.g. d=3 (true for other d's)

$$M_{BH}^2 \sim \frac{r_0^2}{G^2} \sim \frac{\alpha'}{G^2} \underset{\substack{\uparrow \\ \text{c.p.}}}{\sim} M_s^2 \sim \frac{N}{\alpha'}$$

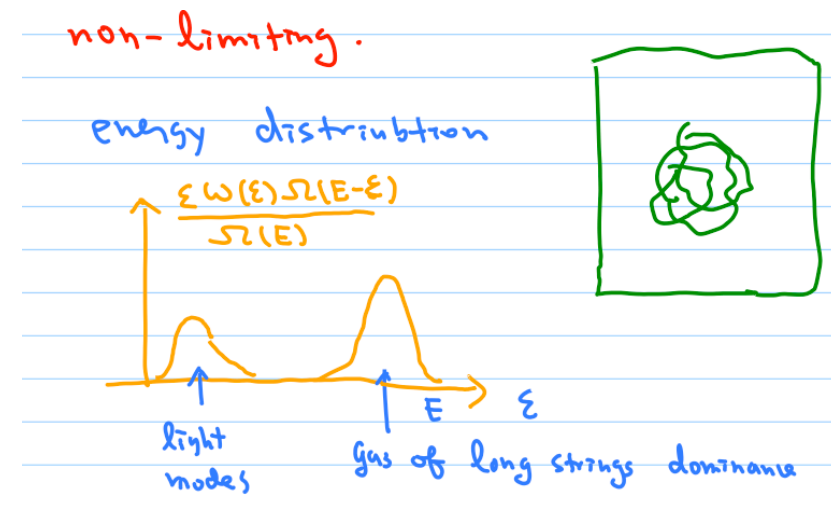


$$S_{BH} \sim \frac{r_0^2}{G} \sim \frac{\alpha'}{G} \sim \sqrt{N} \sim S_s$$

Hagedorn long string

- At the correspondence point, the black hole temperature is of order of string scale, so the strings are in the Hagedorn regime. However, as in the earlier discussions we see that the long string configurations dominate at Hagedorn regime. This is also consistent with the random walk picture, i.e. diffusion of a thermal scalar.
- At zero string coupling, the typical size of a Hagedorn string (diffusion length)

$$l \sim N^{1/4} l_s$$



Self interaction

- If we start with a long string, we may expect the string to condense into a black hole. However, this is not the case if the string coupling is weak as required by the Hoop conjecture.
- It is interesting to see if there is a critical string coupling so that the long string will collapse by self interaction and turn into a black hole as the size of the string ball is of order of the Schwarzschild radius.
- The correspondence principle implies

$$\frac{\alpha'}{G} \sim \sqrt{N}, \quad G \sim g^2 \alpha' \quad \longrightarrow \quad g_c \sim N^{-1/4}$$

- However, some preliminary study for a thermal scalar coupled to gravity yields

$$g_0 \sim N^{(d-6)/8} \quad \longrightarrow \quad \ell \sim \frac{l_s}{g^2 N^{1/2}} \quad (d=3)$$

Hagedorn strings on AdS

Hagedorn Strings and Correspondence Principle in AdS(3).

Feng-Li Lin, Toshihiro Matsuo, Dan Tomino (Taiwan, Natl. Normal U.) . Jun 2007. 28pp.

e-Print: arXiv:0705.4514 [hep-th]

Thermal AdS(3), BTZ and competing winding modes condensation.

Micha Berkooz, Zohar Komargodski, Dori Reichmann (Weizmann Inst.) . WIS-06-07-JUN-DPP, Jun 2007. 40pp.

e-Print: arXiv:0706.0610 [hep-th]

Winding tachyons in BTZ.

Mukund Rangamani, Simon F. Ross (Durham U. & Durham U., Dept. of Math.) . DCPT-07-21, Jun 2007. 37pp.

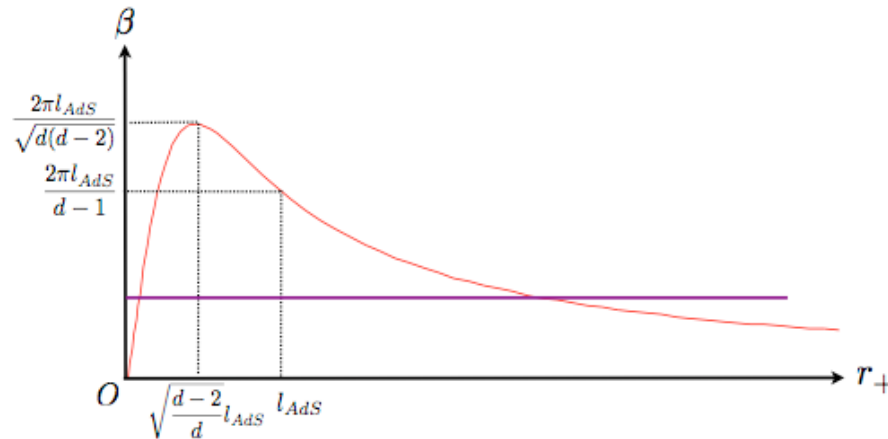
e-Print: arXiv:0706.0663 [hep-th]

Hawking-Page v.s. Hagedorn

- AdS-Schwarzschild metric

$$ds_{d+1}^2 = f(r)d^2\tau + f^{-1}(r)d^2r + r^2 d\Omega_{d-1}^2$$

$$f(r) = 1 + r^2/l_{AdS}^2 - w_{d+1}M/r^{d-2}, \quad w_{d+1} = 16\pi G_{d+1}/(d-1)\Omega_{d-1}$$



- There is Hawking-Page transition from thermal AdS to AdS-black-hole, which could be different Hagedorn transition.

BTZ black hole

- We need to superheat to reach Hagedorn regime without collapsing into a black hole, i.e.

$$T_{HP} \sim 1/l_{AdS}, \quad T_s \sim 1/l_s$$

- The nature of string/black hole transition is not clear in AdS space.

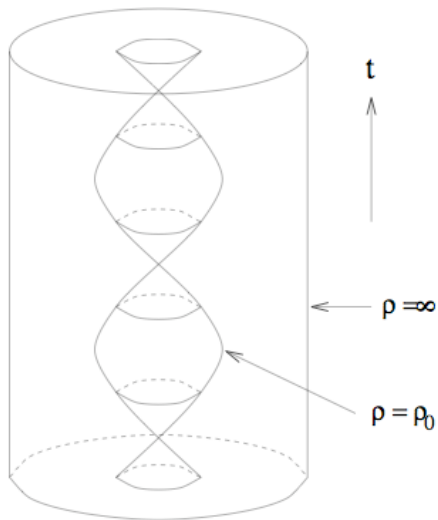
$$k = (l_{AdS}/l_s)^2$$

- For d=2 case, it is called BTZ black hole. In this case, there is no “small black hole” phase.
- The asymptotic torus of thermal AdS(3) and BTZ are related by the SL(2,R) transformation. The BTZ black hole is topological in nature.

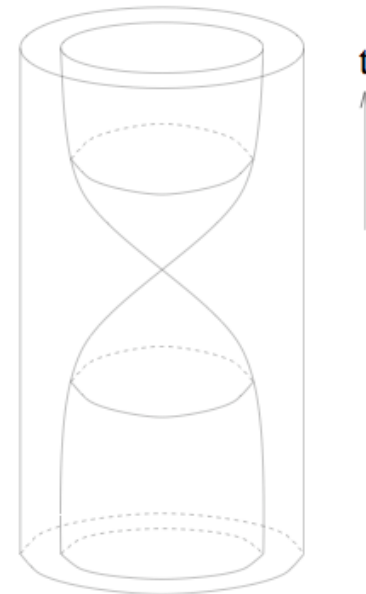
AdS(3) Strings

- String theory on AdS(3) is exactly solvable. This is not the case for higher dimensional AdS space.
- The spectrum is the unitary representations of the $SL(2, \mathbb{R})$ (isometry of AdS(3)) algebra

short strings (discrete)



long strings (continuous)



Maldacena-Ooguri-Son

- The AdS(3) thermal string partition function is

$$\begin{aligned} Z(\beta) &= \int_{F_0} \frac{d^2\tau}{4\tau_2} \mathcal{Z}_{gh} \mathcal{Z}_{int} \mathcal{Z}_{AdS} \\ &= V_{int} \int_{F_0} \frac{d^2\tau}{4\tau_2} \frac{\beta(1-2/k)^{1/2} |\eta(\tau)|^{4-2c_{int}}}{(4\pi^2\alpha'\tau_2)^{(c_{int}+1)/2}} \sum_{n,m=-\infty}^{\infty} \frac{e^{-\beta^2|m-n\tau|^2/4\pi\alpha'\tau_2 + 2\pi(\text{Im}U_{n,m})^2/\tau_2}}{|\vartheta_1(\tau, U_{n,m})|^2}, \end{aligned}$$

- We have assumed the internal space to be trivial, and the central charges are

$$c_{SL(2,R)} = 3 + \frac{6}{k-2}, \quad c_{int} = 26 - c_{SL(2,R)}$$

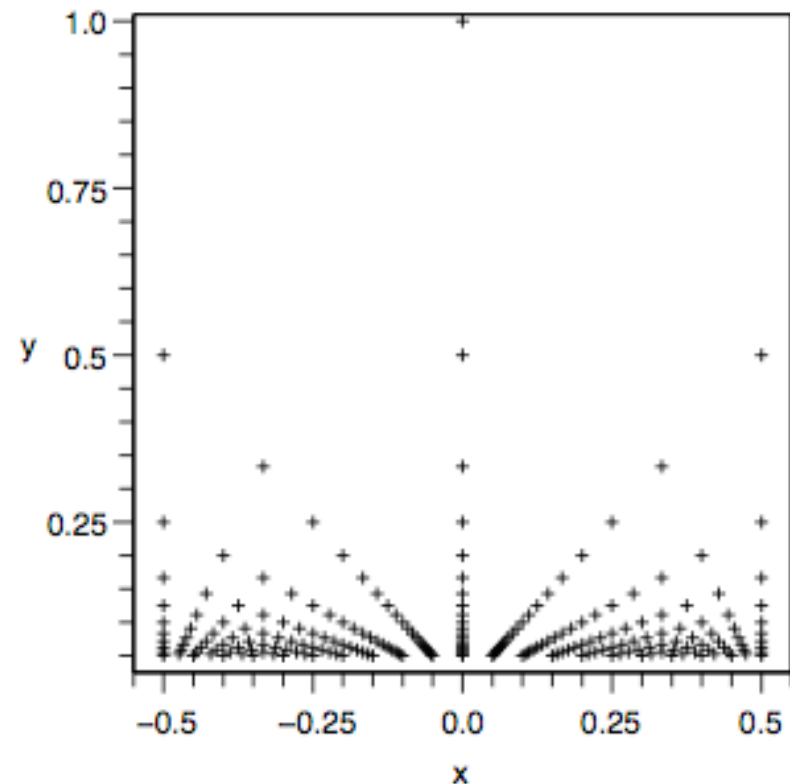
- For internal CFT to be unitary, we need

$$k \geq k_0 = 2 + \frac{6}{23}$$

pole structures & saddle-point

- The Theta function has complicated pole structure, which makes the saddle point approximation more subtle than in the flat space.
- These poles are IR divergence, we need to introduce the IR cutoff before doing saddle-point approximation, i.e.,

$$\int_0^\infty d\tau_2 \cdots = \lim_{\epsilon \rightarrow 0} \sum_{w=0}^{\infty} \int \frac{\beta}{2\pi\sqrt{k}} \frac{1}{(w+\epsilon)} \frac{\beta}{2\pi\sqrt{k}} \frac{1}{(w+1-\epsilon)} d\tau_2 \cdots .$$



Extract Hagedorn behavior I

- We perform the saddle-point approximation for each sub-strip separately, and we find that the sub-strip near the τ_2 -axis dominates.
- We perform the saddle-point approximation of τ_1 -integral with fixed τ_2 , and find out the dominant saddle-point at $\tau_1 = 0$. The resulting partition function is

$$z_0(\beta) \simeq \frac{V_{int}\beta(1-2/k)^{1/2}}{8(4\pi^2)^{(c_{int}+1)/2}} \lim_{\epsilon \rightarrow 0} \sum_{w=w_{min}}^{\infty} \int_{\frac{\beta}{2\pi\sqrt{k}(w+1-\epsilon)}}^{\frac{\beta}{2\pi\sqrt{k}(w+\epsilon)}} d\tau_2 \tau_2^{(c_{int}-3)/2} \sqrt{\frac{2\pi}{G''(0; \tau_2)}} \\ \times \left| \sin\left(\frac{\beta}{2\sqrt{k}\tau_2}\right) \right|^{-2} \exp\left[-\frac{\beta^2 - \beta_{AdS}^2}{4\pi\tau_2}\right],$$

$$G''(0; \tau_2) = 1 - c_{int} + \frac{\pi}{3\tau_2} \left(c_{int} + 1 - \frac{3\beta^2}{k\pi^2} \right) - \frac{\beta}{2\sqrt{k}\tau_2} \left[4 \cot\left(\frac{\beta}{2\sqrt{k}\tau_2}\right) - \frac{\beta/\sqrt{k}\tau_2}{\sin^2(\beta/2\sqrt{k}\tau_2)} \right]$$

$$\beta_{AdS} := \sqrt{\frac{2\pi^2(c_{int}+1)}{3}} = 4\pi \sqrt{\frac{k-9/4}{k-2}}.$$

Extract Hagedorn behavior II

- We can extract the IR divergence of the AdS volume, and the partition function becomes

$$z_0(\beta) = \frac{|\ln \epsilon| V_{int} (k-2)^{1/2}}{2\sqrt{\pi}} \left(\frac{\beta}{8\pi^3 \sqrt{k}} \right)^{\frac{c_{int}+1}{2}} \sum_{w=w_{min}}^{\infty} w^{-\frac{c_{int}+3}{2}} \exp \left[-\frac{\beta^2 - \beta_{AdS}^2}{2\beta} \sqrt{k} w \right] + O(\epsilon^0).$$

- Near the Hagedorn temperature, we can extract the long string spectrum by introducing the momentum integration

$$z_0(\beta) \sim \int_{-\infty}^{\infty} dp \sum_w w^{-(c_{int}+4)/2} e^{-(\beta - \beta_{AdS})E}$$

$$E = \left(w + \frac{p^2}{wk} \right) \sqrt{k}.$$

Extract Hagedorn behavior III

- Taking the continuum limit for the w-sum, the partition function can be put in a suggestive form

$$z_0(\beta) = |\ln \epsilon| \frac{V_{int}(k-2)^{1/2}}{2\sqrt{\pi}} \left(\frac{\beta_{AdS}(\beta - \beta_{AdS})}{8\pi^3} \right)^{\frac{c_{int}+1}{2}} \Gamma \left(-\frac{c_{int}+1}{2}, (\beta - \beta_{AdS})\sqrt{k}w_{min} \right).$$

- The form for flat space Hagedorn string is

$$\beta F_{flat} = -CV_{int}(\beta - \beta_H)^{d/2} \Gamma \left(-\frac{d}{2}, (\beta - \beta_H)m_0 \right)$$

- We see that they have the same form except that d is replaced by 1+c_{int} where "1" corresponds to the radial direction. Therefore, the Hagedorn thermodynamics is similar to the flat space one.

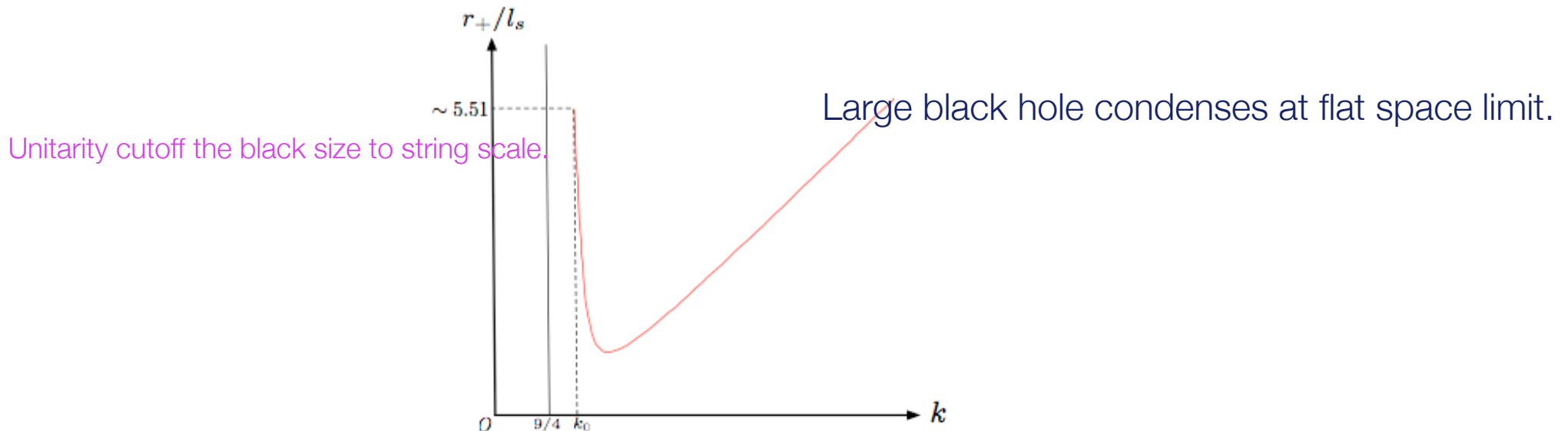
Correspondence principle on AdS(3)

- Our Hagedorn temperature in AdS(3) is monotonically decreasing as k grows, and note that $k > k_0 = 2.26 > 9/4$ so that the maximal Hagedorn temperature is finite about $0.388 l_s^{-1}$

$$\beta_{AdS} := \sqrt{\frac{2\pi^2(c_{int} + 1)}{3}} = 4\pi \sqrt{\frac{k - 9/4}{k - 2}}.$$

- At the correspondence point

$$\beta_{AdS} = \beta_{BH} := \frac{2\pi l_{AdS}^2}{r_+} \quad \text{it yields}$$



Strings on BTZ

- By performing the $SL(2, Z)$ duality on the thermal $AdS(3)$ partition function and swap winding modes associated with the thermal and angular circles, we can obtain the string partition function on BTZ.

$$z_{BTZ}(\beta_{BTZ}) \sim \int d\tau_2 \cdots \exp\left(-\frac{(4\pi^2 k / \beta_{BTZ})^2 - \beta_{AdS}^2}{4\pi\tau_2}\right).$$

- This implies that the low temperature BTZ black hole (stringy size) is unstable due to the appearance of the tachyonic angular winding modes since it diverges as

$$\beta_{BTZ} \geq \frac{4\pi^2 k}{\beta_{AdS}} \equiv \beta_{BTZ}^{Hag} \quad \longrightarrow \quad r_+^{(min)} = \frac{\beta_{AdS}}{2\pi}.$$

- However, z_{BTZ} is pathological since it has wrong Boltzmann weight.

Conclusion

- In this talk, we have discussed the various aspects of the Hagedorn strings, especially on the implication to the strings/black hole correspondence.
- In the AdS(3) case, we have analyzed the string partition function, and the results illuminate the effect of the stringy effect on the Hagedorn thermodynamics.
- Some subtleties cannot be captured by the thermal partition function such as (1) self-interaction (2) strings on BTZ (3) $k=3$ transition for fermionic AdS(3) string at which $T_c^{\text{BTZ}}=T_{\text{AdS}}$.
- Hagedorn strings may have implication at earlier Universe.