Quantum Mechanics (II): Homework 1
Due: March 11, 2013

Ex. 1 Magnetic translation operator
Consider an electron in a uniform magnetic field $\mathbf{H}$. Except for the magnetic field, the electron is otherwise free.

(a) 5 Verify that the vector potential $\mathbf{A}(\mathbf{r})$ can be expressed as $\frac{e}{2}\mathbf{H} \times \mathbf{r}$ (this is the so-called symmetry gauge.).

(b) 5 Let the wavefunction of the electron be $\psi(\mathbf{r})$ which satisfies the Schrödinger equation $\hat{H}_0(\mathbf{r})\psi(\mathbf{r}) = \varepsilon\psi(\mathbf{r})$. Under a translation $\mathbf{r} \rightarrow \mathbf{r} + \mathbf{a}$, one expects that the system is translationally invariant. Show that however, in the symmetry gauge, if $\mathbf{a} \perp \mathbf{H}$, $\hat{H}_0(\mathbf{r})$ is not invariant, i.e., $\hat{H}_0(\mathbf{r} + \mathbf{a}) \neq \hat{H}_0(\mathbf{r})$. In fact, one finds that

$$\hat{H}_0(\mathbf{r} + \mathbf{a}) = \exp \left( i \frac{e}{2\hbar c} \mathbf{r} \cdot \mathbf{H} \times \mathbf{a} \right) \hat{H}_0(\mathbf{r}) \exp \left( -i \frac{e}{2\hbar c} \mathbf{r} \cdot \mathbf{H} \times \mathbf{a} \right).$$

(c) 10 If we define $\hat{T}_a\psi(\mathbf{r})$ as the translated state of $\psi(\mathbf{r})$ and require that $\hat{T}_a\psi(\mathbf{r})$ satisfies the same Schrödinger equation as $\psi(\mathbf{r})$ does. Show that

$$\hat{T}_a\psi(\mathbf{r}) = \psi(\mathbf{r} + \mathbf{a}) \exp \left( -i \frac{e}{2\hbar c} \mathbf{r} \cdot \mathbf{H} \times \mathbf{a} \right).$$

In other words,

$$\hat{T}_a = \exp \left( -i \frac{e}{2\hbar c} \mathbf{r} \cdot \mathbf{H} \times \mathbf{a} \right) \hat{T}_a^\dagger,$$

where $\hat{T}_a$ is the ordinary translation operator. Thus, the probability density $|\hat{T}_a\psi(\mathbf{r})|^2 = |\psi(\mathbf{r} + \mathbf{a})|^2$ is getting translated.

(d) 10 Show that in general two magnetic translations operators, $\hat{T}_a$ and $\hat{T}_b$, do not commute and they satisfy

$$\hat{T}_a\hat{T}_b = \hat{T}_b\hat{T}_a \exp \left( i \frac{e}{\hbar c} \mathbf{H} \cdot \mathbf{a} \times \mathbf{b} \right).$$

(e) 10 We now put this electron in a two dimensional lattice with the primitive translation vectors $\mathbf{c}$ and $\mathbf{d}$, and assume that $\mathbf{H} = \mathbf{H}(\mathbf{r})$. The Hamiltonian is

$$\hat{H}(\mathbf{r}) = \frac{1}{2m} \left( \frac{\hbar}{i} \nabla - \frac{e}{c} \mathbf{A} \right)^2 + V(\mathbf{r}),$$

where $V(\mathbf{r})$ is periodic: $V(\mathbf{r} + m\mathbf{c} + n\mathbf{d}) = V(\mathbf{r})$ for any integers $m$ and $n$. Show that $[\hat{T}_a(\mathbf{r}), \hat{H}(\mathbf{r})] = 0$ and $[\hat{T}_a(\mathbf{r}), \hat{H}(\mathbf{r})] = 0$.

(f) 10 Show that when the magnetic field is such that the flux through the primitive cell is $p/q\hbar c/|e|$ ($p$ and $q$ are two integers, which are relative prime, i.e., which have no common factors, and $\hbar c/|e|$ is the so-called flux quantum), $[\hat{T}_{ne}(\mathbf{r}), \hat{T}_{mqd}(\mathbf{r})] = 0$ for any integers $n$ and $m$. Hence the commutation relations between $\hat{T}_{ne}(\mathbf{r}), \hat{T}_{mqd}(\mathbf{r})$, and $\hat{H}(\mathbf{r})$ are the same as those of $\hat{T}_{c}(\mathbf{r}), \hat{T}_{d}(\mathbf{r})$, and $\hat{H}(\mathbf{r})$ where $\hat{H}(\mathbf{r})$ is the Hamiltonian with zero magnetic field. The magnetic field thus introduces new periodicity into the system of the electron.

Ex 2 10 (i) Suppose that $\psi$ is known to be a simultaneous eigenstate of two Hermitian operators $\hat{A}$ and $\hat{B}$ which anticommute,

$$\hat{A}\hat{B} + \hat{B}\hat{A} = 0.$$

What can you say about the eigenvalues of $\hat{A}$ and $\hat{B}$ for state $\psi$? Illustrate your point using the parity operator and the momentum operator.

(ii) An example of anticommuting operators is the so-called chiral Hamiltonian. A Hamiltonian $H$ is chiral if one can find an operator $\Sigma$ that anticommutes with $H$, i.e., $[H, \Sigma] = 0$. What can you say about the energy eigenvalues?

Ex. 3 Scale transformation and the squeezed state Consider a scale transformation $\hat{S}(\varepsilon)$ which is defined to change the length scale in the following way

$$\langle x|\hat{S}(\varepsilon)|\psi \rangle \equiv \psi(xe^{-\varepsilon}),$$

that is, $x$ is scaled down by a factor $\exp(-\varepsilon)$.

(a) 5 The scale transformation $\hat{S}(\varepsilon)$ thus defined does not preserve the normalization of $\psi$. Show that if we define
$\hat{S}(\varepsilon) \equiv \hat{S}(\varepsilon) \exp(-\varepsilon/2)$, $\hat{S}(\varepsilon)$ preserves the normalization of $\psi$. The resulting operator $\hat{S}(\varepsilon)$ is the usual definition for the scale transformation.

(b) 10 In terms of creation and annihilation operators, show that

$$\hat{S}(\varepsilon) = \exp\left[\varepsilon/2(a^\dagger a - a^2)\right].$$

(c) 10 Show that $\hat{S}^\dagger(\varepsilon) \hat{x} \hat{S}(\varepsilon) = e^{\varepsilon \hat{x}}$ and $\hat{S}^\dagger(\varepsilon) \hat{p} \hat{S}(\varepsilon) = e^{-\varepsilon} \hat{p}$. As a result the value of the product of uncertainties $\Delta x \Delta p$ is not changed. One can thus use $\hat{S}(\varepsilon)$ to squeeze the $\Delta x$ of any state, while not changing $\Delta x \Delta p$!

(d) bonus problem Suppose the state of a harmonic oscillator (mass is $m$ and its frequency is $\omega$) at $t = 0$ is a squeezed coherent state, i.e.,

$$|\psi(t = 0)\rangle = \hat{S}(\varepsilon) |\lambda\rangle,$$

where $|\lambda\rangle$ is a normalized coherent state with eigenvalue $\lambda$. Find $\Delta x(t)$ and $\Delta p(t)$.

**Ex. 4 Galilean transformation**

(a) 5 Consider the Galilean transformation $r \to r - \upsilon t$ on a the state of a system, show that this transformation can be realized by the operator

$$\hat{G}(\upsilon, t) = \exp\left[i\upsilon \hat{p} \cdot (m\hat{r} - \hat{p}t)\right].$$

(b) 5 For a system of N particles, what would be the corresponding operator for the whole system?

The followings are bonus problems (+1):

(c) 15 To get a feeling what $\hat{G}(\upsilon, t)$ does, consider two frames in one dimension: one with coordinate $(x, t)$ is not moving, the other with coordinate $(x', t')$ co-moves with the particle toward $+x$ with the velocity $\upsilon$. In other words, the particle is at rest in the prime frame. Find the wavefunctions in both frames and show that they are related exactly by $\hat{G}(\upsilon, t)$. From the above, conclude that in general, consider two coordinates, $(\mathbf{r}, t)$ and $(\mathbf{r}', t')$, that describe the same space-time point in different frames, the wavefunctions in the two frames are related as follows

$$\Psi'(\mathbf{r}', t') = \exp\left[-i\frac{\hbar}{\upsilon} (m\upsilon \cdot \mathbf{r} - \frac{m\upsilon^2 t}{2})\right] \Psi(\mathbf{r}, t),$$

where $\mathbf{r}' = \mathbf{r} - \upsilon t$ and $t' = t$. Note that any quantities with prime represent the quantities in the moving frame.

(d) 5 Argue that the Galilean invariance for Quantum Mechanics is expressed by the following condition

$$\hat{G}^\dagger(\upsilon, t) \hat{U}'(t, t_0) \hat{G}(\upsilon, t_0) = \hat{U}(t, t_0) \times \text{a phase factor},$$

where $\hat{U}(t, t_0)$ is the evolution operator.

(e) 10 Show that the above condition reduces to the form

$$\hat{G}^\dagger(\upsilon, t) \left[ih \frac{\partial}{\partial t} - \hat{H}\right] \hat{G}(\upsilon, t) = ih \frac{\partial}{\partial t} - \hat{H} + f,$$

where $f$ is an arbitrary real function of time. Verify that this condition is indeed satisfied by the Schrödinger Hamiltonian. Find $f$ in this case.

**Ex. 5 10** Consider a rotation with respect to $\hat{\mathbf{n}}$ through an angle $\Phi$. Suppose $\hat{\mathbf{n}} = (n_x, n_y, n_z)$, find the rotation matrix in terms of $n_x, n_y, n_z$,and $\Phi$.

**Ex. 6 10** Consider a rotation specified by the Euler angles $(\pi/4, \pi/4, \pi/4)$. What are the corresponding rotation axis $\hat{\mathbf{n}}$ and $\Phi$ in the notations of Ex. 5? Use any mathematical software (Mathematica, Maple, ...) to help you solve this problem.

**Ex. 7 10** Show that if we restrict the 3-dimensional rotations to rotate only those lattice points in crystals, the trace of the rotational matrix must be an integer. From this result, show that two dimensional crystals can not have 5-fold rotational symmetry.