Useful formulas
\[ \nabla V = \frac{1}{r} \frac{\partial V}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi} \hat{\phi} \quad \text{and} \quad \nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 V_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (r \sin \theta V_{\theta}) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} V_{\phi} \]

1. \( (8\%, 12\%) \) \( \mathbf{v} = r^2 \cos \theta \hat{r} + r^2 \cos \phi \hat{\theta} - r^2 \cos \theta \sin \phi \hat{\phi} \)

(a) Compute \( \nabla \cdot \mathbf{v} \).

(b) Check the divergence theorem using the volume shown in the figure (one octant of the sphere of radius \( R \)).

[Hint: Make sure you include the entire surface.]

2. \( (10\%, 10\%) \) Suppose the potential at the surface of a hollow hemisphere is specified, as shown in the figure, where \( V_1(a, \theta) = 0 \), \( V_2(b, \theta) = V_0(2 \cos \theta - 5 \cos \theta \sin^2 \theta) \), \( V_3(r, \pi/2) = 0 \). \( V_0 \) is a constant.

(a) Show the general solution in the region \( b \leq r \leq a \) and determine the potential in the region \( b \leq r \leq a \), using the boundary conditions.

(b) When \( V_2(b, \theta) = V_0 \sin \theta \) and \( V_1(a, \theta) = V_3(r, \pi/2) = 0 \), how do you solve this problem? Please explain as detailed as possible.

[Hint: \( P_0(x) = 1 \), \( P_1(x) = x \), \( P_2(x) = (3x^2 - 1)/2 \), and \( P_3(x) = (5x^3 - 3x)/2 \).]
3. (7%, 7%, 6%) The potential of some configuration is given by the expression \( V(\mathbf{r}) = A e^{-\lambda r} / r \), where \( A \) and \( \lambda \) are constants.
   (a) Find the energy density (energy per unit volume).
   (b) Find the charge density \( \rho(\mathbf{r}) \).
   (c) Find the total charge \( Q \) (do it two different ways) and verify the divergence theorem.

4. (7%, 7%, 6%) A uniform line charge \( \lambda \) is placed on an infinite straight wire, a distance \( d \) above a grounded conducting plane.
   (a) Find the potential \( V \) in the region above the plane.
   (b) Find the surface charge density \( \sigma \) induced on the conducting plane.
   (c) Find the force on the wire per unit length.
   [Hint: Use the method of images.]

5. (8%, 6%, 6%) Consider a hollowed charged sphere with radius \( R \) and uniform charge density \( \rho \) as shown in the figure. The inner radius of the spherical cavity is \( R/2 \).
   (a) If the observer is very far from the charged sphere, find the multiple expansion of the potential \( V \) in power of \( 1/r \)
   (b) Find the dipole moment \( \mathbf{p} \).
   (c) Find the electric field \( \mathbf{E} \) up to the dipole term.
   [Note: Specify a vector with both magnitude and direction.]
1.
(a)
\[ \mathbf{v} = r^2 \cos \theta \mathbf{\hat{r}} + r^2 \cos \phi \mathbf{\hat{\phi}} - r^2 \cos \theta \sin \phi \mathbf{\hat{\theta}} \]

\[ \nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta v_{\theta}) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} v_{\phi} \]

\[ = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \cos \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta r^2 \cos \phi) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} (-r^2 \cos \theta \sin \phi) \]

\[ = 4r \cos \theta + r \frac{\partial}{\partial \phi} (-r^2 \cos \theta \sin \phi) - r \frac{\partial}{\partial \phi} \cos \phi \]

\[ = 4r \cos \theta \]

The divergence theorem \[ \int \nabla \cdot \mathbf{v} \, dV = \oint_S \mathbf{v} \cdot d\mathbf{a} \]

\[ \oint_S \mathbf{v} \cdot d\mathbf{a} = xy\text{-plane} + yz\text{-plane} + zx\text{-plane} + \text{curved surface} \]

xy-plane: \[ d\mathbf{a} = -r \, d\theta \, d\phi, \quad \mathbf{v} \cdot d\mathbf{a} = (r^2 \cos \theta \sin \phi) \, r \, d\theta \, d\phi = 0, \]

yz-plane: \[ d\mathbf{a} = r \, d\theta \, d\phi, \quad \mathbf{v} \cdot d\mathbf{a} = -(r^2 \cos \theta \sin \phi) \, r \, d\theta \, d\phi = -r^3 \cos \theta \, d\theta \, d\phi = -\frac{1}{4} R^4 \]

zx-plane: \[ d\mathbf{a} = r \, d\theta \, d\phi, \quad \mathbf{v} \cdot d\mathbf{a} = (r^2 \cos \phi) \, r \, d\phi = r^3 \cos \phi \, d\phi = \frac{1}{4} R^4 \]

curved surface: \[ d\mathbf{a} = R^2 \sin \theta \, d\theta \, d\phi \, \mathbf{\hat{r}}, \quad r = R, \quad \mathbf{v} \cdot d\mathbf{a} = (R^2 \cos \theta) R^2 \sin \theta \, d\theta \, d\phi = \frac{R^4}{2} \sin 2\theta \, d\phi = \frac{\pi R^4}{4} \]

\[ \oint_S \mathbf{v} \cdot d\mathbf{a} = 0 - \frac{1}{4} R^4 + \frac{1}{4} R^4 + \frac{\pi R^4}{4} = \frac{\pi R^4}{4} = \int_r \nabla \cdot \mathbf{v} \, dV \]

(b)

2.
(a)

\[ \begin{cases} 
(i) \quad V_i(a, \theta) = 0 \\
(ii) \quad V_j(b, \theta) = V_0 (2 \cos \theta - 5 \cos \theta \sin^2 \theta) = V_0 (5 \cos^3 \theta - 3 \cos \theta) = 2V_0 P_i \\
(iii) \quad V_3(r, \theta = \pi/2) = 0 
\end{cases} \]

General solution \[ V(r, \theta) = \sum_{i=0}^{\infty} (A_i r^i + B_i r^{-(i+1)}) P_i (\cos \theta) \]
B.C. (i) \( V(a, \theta) = \sum_{i=0}^{n} (A_i a^i + B_i a^{-(i+1)}) P_i(\cos \theta) = 0 \Rightarrow B_i = -A_i a^{2i+1} \)

B.C. (ii) \( V(b, \theta) = \sum_{i=0}^{n} (A_i b^i + B_i b^{-(i+1)}) P_i(\cos \theta) = 2V_0 P_3(\cos \theta) \)

Comparing the coefficient \( \Rightarrow A_i b^3 + B_i b^{-4} = 2V_0, \quad A_i = B_i = 0 \) for \( \ell = 0,1,2,4,5,... \)

B.C. (iii) \( V(r, \theta = \frac{\pi}{2}) = (A_3 r^3 + B_3 r^{-4}) P_3(0) = 0 \Rightarrow A_3 = B_3 = 0 \) except \( \ell = 3, \)

\[ A_3 = \frac{2V_0 b^4}{b^3 - a^3} \quad \text{and} \quad B_3 = -\frac{2V_0 b^4 a^7}{b^3 - a^3} \]

\[ V(r, \theta) = \left( \frac{2V_0}{b^3 - a^3} b^4 r^3 - \frac{2V_0}{b^3 - a^3} b^4 a^7 r^{-4} \right) \left( \frac{5\cos^3 \theta - 3\cos \theta}{2} \right) \]

(b) Boundary condition

\[
\begin{align*}
(i) \quad & V_1(a, \theta) = 0 \\
(ii) \quad & V_2(b, \theta) = V_0 \sin \theta \\
(iii) \quad & V_2(r, \theta = \frac{\pi}{2}) = 0
\end{align*}
\]

General solution \( V(r, \theta) = \sum_{i=0}^{n} (A_i r^i + B_i r^{-i-1}) P_i(\cos \theta) \)

B.C. (i) \( \sum_{i=0}^{n} (A_i a^i + B_i a^{-(i+1)}) P_i(\cos \theta) = 0 \Rightarrow B_i = -A_i a^{2i+1} \)

B.C. (iii) \( \sum_{i=0}^{n} (A_i r^i + B_i r^{-i-1}) P_i(0) = 0 \quad \Rightarrow \ell = 1,3,5,... \) only odd terms survive

B.C. (ii) \( \sum_{i=0}^{n} A_i (b^i - a^{2i+1}) P_i(\cos \theta) = V_0 \sin \theta \)

\[ \int_{-1}^{1} P_i(x) P_i(x) \, dx = \int_{0}^{\pi} P_i(\cos \theta) P_i(\cos \theta) \sin \theta \, d\theta = \begin{cases} 0 & \text{if } \ell' \neq \ell \\ \frac{2}{2\ell + 1}, & \text{if } \ell' = \ell \end{cases} \]

\[ \sum_{i=0}^{n} A_i \left( b^i - a^{2i+1} \right) \int_{0}^{\pi} P_i(\cos \theta) P_i(\cos \theta) \sin \theta \, d\theta = \int_{0}^{\pi} V_0 \sin \theta P_i(\cos \theta) \sin \theta \, d\theta \]

\[ A_\ell = (\frac{b^i}{b^{2i+1} - a^{2i+1}}) \frac{2\ell + 1}{2} \int_{0}^{\pi} V_0 \sin \theta P_i(\cos \theta) \sin \theta \, d\theta \]

But \( A_\ell = 0 \) for \( \ell = 1,3,5,... \) It does not make sense. Why?

Add an artificial boundary condition \( V_2(b, \theta) = \begin{cases} V_0 \sin \theta & \text{for } 0 \leq \theta \leq \frac{\pi}{2} \\ -V_0 \sin \theta & \text{for } \frac{\pi}{2} \leq \theta \leq \pi \end{cases} \)

or \( V_2(b, \theta) = \begin{cases} V_0 \sin \theta & \text{for } 0 \leq \theta \leq \frac{\pi}{2} \\ 0 & \text{for } \frac{\pi}{2} \leq \theta \leq \pi \end{cases} \)
3. (a)

\[ \mathbf{E} = -\nabla V = -A \frac{\partial}{\partial r} \left( \frac{e^{-\lambda r}}{r} \right) \hat{r} = -A \left\{ -\lambda r e^{-\lambda r} - e^{-\lambda r} \right\} \hat{r} = A \left( \frac{\lambda r + 1) e^{-\lambda r}}{r^2} \right) \hat{r} \]

Energy density

\[ \frac{\varepsilon_0}{2} E^2 = \frac{\varepsilon_0}{2} A^2 \left( \frac{\lambda r + 1) e^{-\lambda r}}{r^2} \right) \]

(b)

\[ \rho = \varepsilon_0 \nabla \cdot \mathbf{E} = \varepsilon_0 A (\nabla \cdot \frac{(\lambda r + 1) e^{-\lambda r}}{r^2}) = \varepsilon_0 A (\lambda r + 1) e^{-\lambda r} \left( \nabla \cdot \frac{\hat{r}}{r^2} \right) + \varepsilon_0 A \frac{\hat{r}}{r^2} \cdot \nabla ((\lambda r + 1) e^{-\lambda r}) \]

\[ (\nabla \cdot \frac{\hat{r}}{r^2}) = 4\pi \delta^3 (\mathbf{r}) \text{ and } (\lambda r + 1) e^{-\lambda r} \delta^3 (\mathbf{r}) = \delta^3 (\mathbf{r}) \]

\[ \frac{\hat{r}}{r^2} \cdot \nabla ((\lambda r + 1) e^{-\lambda r}) = \left[ \frac{\hat{r}}{r^2} \cdot \frac{\partial}{\partial r} \right] ((\lambda r + 1) e^{-\lambda r}) = \left[ \frac{1}{r^2} \frac{\partial}{\partial r} \right] ((\lambda r + 1) e^{-\lambda r}) = -\frac{\lambda^2}{r} e^{-\lambda r} \]

\[ \rho = \varepsilon_0 A \left[ 4\pi \delta^3 (\mathbf{r}) - \frac{\lambda^2}{r} e^{-\lambda r} \right] \]

(c)

\[ Q = \int \rho d\tau = \int \varepsilon_0 A [4\pi \delta^3 (r) - \frac{\lambda^2}{r} e^{-\lambda r}] d\tau = 4\pi \varepsilon_0 A [1 + \int_{r=0}^{\infty} \frac{\lambda^2}{r} e^{-\lambda r} r^2 dr] \]

\[ \int_{r=0}^{\infty} \frac{\lambda^2}{r} e^{-\lambda r} r^2 dr = \int_{r=0}^{\infty} e^{-\lambda r} \lambda^2 r^2 dr = -\int_{r=0}^{\infty} \lambda r e^{-\lambda r} = -\int_{x=0}^{\infty} x e^{-x} = -1 \Rightarrow Q = \int \rho d\tau = 0 \]

Use Gauss's law, the charge enclosed in a sphere of radius \( R \)

\[ Q_R = \oint_S \mathbf{E} \cdot d\mathbf{a} = 4\pi \varepsilon_0 A (\lambda R + 1) e^{-\lambda R} \quad \Rightarrow \quad \text{The total charge } Q_{R \to \infty} = 4\pi \varepsilon_0 A (\lambda R + 1) e^{-\lambda R} \bigg|_{R=\infty} = 0 \]

4. (a)

Assume the image line charge of \(-\lambda\) is placed at a distance \( d \) below the plane.

Using the Gauss's law, the electric field outside a line charge \( \lambda \) is \( \mathbf{E} = -\frac{\lambda}{2\pi \varepsilon_0 r} \hat{r} \).

So \( V = \int_{r_0}^{r} \mathbf{E} \cdot d\mathbf{l} = \frac{\lambda}{2\pi \varepsilon_0} \ln \frac{r_0}{r} = V(r) - V_{ref}(r_0) \)

\[ V = V_+ + V_- = \frac{\lambda}{2\pi \varepsilon_0} \left[ \ln \frac{r_0}{\sqrt{(x-d)^2 + y^2}} - \ln \frac{r_0}{\sqrt{(x+d)^2 + y^2}} \right] = \frac{\lambda}{4\pi \varepsilon_0} \left( \ln \frac{(x+d)^2 + y^2}{(x-d)^2 + y^2} \right) \]

(b)
\[ \sigma = \varepsilon_0 E \cdot \hat{n} = \varepsilon_0 E_x = -\frac{\partial}{\partial x} \frac{\lambda}{4\pi} \left\{ \ln \left( \frac{(x+d)^2 + y^2}{(x-d)^2 + y^2} \right) \right\}_{x=0} = -\frac{\lambda}{4\pi} \left\{ \frac{2(x+d)}{(x+d)^2 + y^2} - \frac{2(x-d)}{(x-d)^2 + y^2} \right\}_{x=0} \]

\[ = -\frac{\lambda}{4\pi} \frac{4d}{d^2 + y^2} = -\frac{\lambda}{\pi} \frac{d}{d^2 + y^2} \]

Simple check: \[ \lambda' = \int_{-\infty}^{\infty} \sigma dy = \int_{-\infty}^{\infty} -\frac{\lambda}{\pi} \frac{d}{d^2 + y^2} dy \]

Let \( y = d \tan \theta, \ dy = d \sec^2 \theta d\theta \)

\[ \lambda' = -\frac{\lambda}{\pi} \int_{-\pi/2}^{\pi/2} d^2 \sec^2 \theta \frac{d}{d^2 \sec^2 \theta} d\theta = -\lambda \]

(c) \[
\begin{align*}
dF &= Edq = E \lambda d\ell \\
\frac{dF}{d\ell} &= E\lambda = \frac{\lambda}{2\pi\varepsilon_0 (2d)} = \frac{\lambda^2}{4\pi\varepsilon_0 d}
\end{align*}
\]

5.

(a) Consider this problem as two charge spheres, one with charge density \( \rho \) the other with opposite charge density \( -\rho \).

\[
V_{\text{big}} = \frac{1}{4\pi\varepsilon_0 r} \left( \rho \frac{4\pi}{3} R^3 \right) \quad \text{and} \quad V_{\text{small}} = \frac{1}{4\pi\varepsilon_0} \left[ -\rho \frac{4\pi}{3} \left( \frac{R}{2} \right)^3 \right]
\]

\[
\frac{1}{|r - \frac{1}{2} R|} = \frac{1}{r} \left( 1 + \frac{1}{2r} \right) \cos \theta + \ldots
\]

Using the principle of superposition, we find,

\[
V = \frac{1}{4\pi\varepsilon_0 r} \left( \rho \frac{4\pi}{3} R^3 \right) - \frac{1}{4\pi\varepsilon_0 r} \left( \rho \frac{4\pi}{3} \left( \frac{R}{2} \right)^3 \right) \left( 1 + \frac{1}{2r} \right) \cos \theta + \ldots
\]

\[
= \frac{1}{4\pi\varepsilon_0 r} \frac{7}{8} \left( \rho \frac{4\pi}{3} R^3 \right) - \frac{1}{4\pi\varepsilon_0 r} \frac{1}{2r} \left( \rho \frac{4\pi}{3} \left( \frac{R}{2} \right)^3 \right) \cos \theta + \ldots, \quad \text{let} \ Q = \rho \frac{4\pi}{3} R^3
\]

\[
= \frac{1}{4\pi\varepsilon_0 r} \frac{7Q}{8} - \frac{1}{4\pi\varepsilon_0 r^2} \frac{Q R}{8} \cos \theta + \ldots
\]

(b) \[
Q = \rho \frac{4\pi}{3} R^3
\]

\[
V = \frac{1}{4\pi\varepsilon_0 r} \frac{7Q}{8} - \frac{1}{4\pi\varepsilon_0 r^2} \frac{Q R}{8} \cos \theta + \ldots
\]

The first term is the monopole term and the second term is the dipole term.

So the dipole moment \( p = -\frac{QR}{16} \hat{z} \)

(c)
\[ V = \frac{1}{4\pi\varepsilon_0 r} \left( \frac{7Q}{8} - \frac{1}{4\pi\varepsilon_0 r^2} (\frac{Q R}{8} \cos \theta + \ldots) \right) \]

\[ \mathbf{E} = -\nabla V = -\frac{\partial V}{\partial r} \hat{r} - \frac{1}{r} \frac{\partial V}{\partial \theta} \hat{\theta} - \frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi} \hat{\phi} \]

\[ = \left( \frac{1}{4\pi\varepsilon_0 r^2} \left( \frac{7Q}{8} - \frac{2p}{4\pi\varepsilon_0 r^3} \cos \theta \right) \right) \hat{r} - \frac{p}{4\pi\varepsilon_0 r^3} \sin \theta \hat{\theta} \]