Chapter 32 Inductance and Magnetic Materials

The appearance of an induced emf in a circuit associated with changes in its own magnetic field is called **self-induction**. The corresponding property is called **self-inductance**. A circuit element, such as a coil, that is designed specifically to have self-inductance is called an **inductor**.

Can we find an induced emf due to its own magnetic field changes? Yes!

The appearance of an induced emf in one circuit due to changes in the magnetic field produced by a nearby circuit is called **mutual induction**. The response of the circuit is characterized by their **mutual inductance**.

32.1 Inductance

**Close:** As the flux through the coil changes, there is an induced emf that opposes this change. The self-induced emf tries to prevent the rise in the current. As a result, the current does not reach its final value instantly, but instead rises gradually as in right figure.

**Open:** When the switch is opened, the flux rapidly decreases. This time the self-induced emf tries to maintain the flux.

When the current in the windings of an electromagnet is shut off, the self-induced emf can be large enough to produce a spark across the switch contacts.
32.1 Inductance (II)

Since the self-induction and mutual induction occur simultaneously, both contribute to the flux and to the induced emf in each coil.

The flux through coil 1 is the sum of two terms:

\[ N_1 \Phi_1 = N_1 (\Phi_{11} + \Phi_{12}) \]

The net emf induced in coil 1 due to changes in \( I_1 \) and \( I_2 \) is

\[ V_{\text{emf}} = -N_1 \frac{d}{dt} (\Phi_{11} + \Phi_{12}) \]

Self-Inductance

It is convenient to express the induced emf in terms of a current rather than the magnetic flux through it.

The magnetic flux is directly proportional to the current flowing through it.

\[ N_1 \Phi_{11} = L_1 I_1 \]

where \( L_1 \) is a constant of proportionality called the self-inductance of coil 1. The SI unit of self-inductance is the henry (H). The self-inductance of a circuit depends on its size and its shape.

The self-induced emf in coil 1 due to changes in \( I_1 \) takes the form

\[ V_{\text{emf,11}} = -L_1 \frac{dI_1}{dt} \]
Mutual Inductance

The magnetic flux produced by coil 2 is proportional to $I_2$. Thus the flux contributed from coil 2 may be written as

$$N_1 \Phi_{12} = M I_2$$

where the constant of proportionality is $M$ is called the mutual inductance of the two coils. The SI unit of mutual inductance is the henry (H). The mutual inductance of the two coils depends on its separation and orientation.

The emf induced in coil 1 due to changes in $I_2$ takes the form

$$V_{\text{emf}_{12}} = -M \frac{dI_2}{dt}$$

Are the mutual inductances $M_{12}$ and $M_{21}$ the same? Why?

Example 32.1

A long solenoid of length $L$ and cross-sectional area $A$ has $N$ turns. Find its self-inductance. Assume that the field is uniform throughout the solenoid.

Solution:

$$\Phi = BA = \mu_0 \frac{N}{L} IA$$

$$N\Phi = LI$$

$$L = \mu_0 n^2 AL$$

Notice that the self-inductance is proportional to the square of the number density.

This result may be compared to the capacitance of two parallel plates ($C=\varepsilon_0 A/d$), which depends on the geometry of the capacitor.
Example 32.2
A coaxial cable consists of an inner wire of radius $a$ that carries a current $I$ upward, and an outer cylindrical conductor of radius $b$ that carries the same current downward. Find the self-inductance of a coaxial cable of length $L$. Ignore the magnetic flux within the inner wire.

Solution:

$$B = \frac{\mu_0 I}{2\pi x}, \quad d\Phi = B dA = \frac{\mu_0 I}{2\pi} \ell dx$$

$$\Phi = \int_a^b \frac{\mu_0 I}{2\pi} \ell dx = \frac{\mu_0 I \ell}{2\pi} \ln \frac{b}{a} = LI$$

$$L = \frac{\mu_0 \ell}{2\pi} \ln \frac{b}{a}$$

Hint 1: The direction of the magnetic field.

Hint 2: What happens when considers the inner flux?

Example 32.3
A circular coil with a cross-sectional area of 4 cm$^2$ has 10 turns. It is placed at the center of a long solenoid that has 15 turns/cm and a cross-sectional area of 10 cm$^2$, as shown below. The axis of the coil coincides with the axis of the solenoid. What is their mutual inductance?

Solution:

$$\Phi_{12} = B_2 A_1 = \mu_0 n_2 I_2 A_1$$

$$M = \frac{N_1 \Phi_{12}}{I_2} = \mu_0 n_2 N_1 A_1$$

$$= (4\pi \times 10^{-7})(1500)(10)(0.000)$$

$$= 7.54 \text{ } \mu\text{F}$$

Notice that although $M_{12}=M_{21}$, it would have been much difficult to find $\Phi_{21}$ because the field due to the coil is quite nonuniform.
32.2 LR Circuits
How does the current rise and fall as a function of time in a circuit containing an inductor and a resistor in series?

**Rise**

\[ V_{\text{emf}} - IR - L \frac{dI}{dt} = 0 \]

Let \( I = I_0 e^{-\alpha t} + \beta \Rightarrow \frac{dI}{dt} = -\alpha I_0 e^{-\alpha t} \)

\[
\begin{align*}
\alpha & = \frac{R}{L} \\
0 & : V_{\text{emf}} - R \beta = 0 \Rightarrow \beta = \frac{V_{\text{emf}}}{R} \\
t = 0 & : I_0 = -\beta = -\frac{V_{\text{emf}}}{R}
\end{align*}
\]

\[ I = \frac{V_{\text{emf}}}{R} \left( 1 - e^{-\frac{R}{L} t} \right) \]

The quantity \( \tau = L/R \) is called the **time constant**.

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32.2 LR Circuits

**Decay**

\[ -IR - L \frac{dI}{dt} = 0 \]

Let \( I = I_0 e^{-\alpha t} \Rightarrow \frac{dI}{dt} = -\alpha I_0 e^{-\alpha t} \)

\[
\begin{align*}
\alpha & = \frac{R}{L} \\
t = 0 & : I_0 = \frac{V_{\text{emf}}}{R}
\end{align*}
\]

\[ I = \frac{V_{\text{emf}}}{R} e^{-\frac{R}{L} t} = \frac{V_{\text{emf}}}{R} e^{-\frac{t}{\tau}} \]

The quantity \( \tau = L/R \) is called the **time constant**.
31.3 Energy Stored in an Inductor

The battery that establishes the current in an inductor has to do work against the opposing induced emf. The energy supplied by the battery is stored in the inductor.

In Kirchhoff’s loop rule, we obtain

\[ V_{\text{emf}} = iR + L \frac{di}{dt} \]

\[ iV_{\text{emf}} = i^2R + Li \frac{di}{dt} \]

\[ iV_{\text{emf}} = i^2R + \frac{dU_L}{dt}, \text{ where } U_L = \frac{1}{2}LI^2 \]

power supplied by the battery  
power dissipated in the resistor  
energy change rate in the inductor

Energy Density of the Magnetic Field

We have expressed the total energy stored in the inductor in terms of the current and we know the magnetic field is proportional to the current. Can we express the total magnetic energy in terms of the B-field? Yes.

Let’s consider the case of solenoid.

\[ L = \mu_0 n^2 A\ell \]

\[ U_L = \frac{1}{2}LI^2 = \frac{1}{2\mu_0} (\mu_0 nI)^2 A\ell = \frac{B^2}{2\mu_0} A\ell \]

\[ u_B = \frac{B^2}{2\mu_0} \text{ (The energy density of a magnetic field in free space) } \]

Although this relation has been obtained from a special case, the expression is valid for any magnetic field.
Example 31.4
A 50-mH inductor is in series with a 10-Ω resistor and a battery with an emf of 25 V. At t=0 the switch is closed. Find: (a) the time constant of the circuit; (b) how long it takes the current to rise to 90% of its final value; (c) the rate at which energy is stored in the inductor; (d) the power dissipated in the resistor.

Solution: (a) The time constant is $\tau = L/R = 5$ ms

(b) $0.9I_0 = I_0(1 - e^{-t/\tau}) \Rightarrow t = -\tau \ln 0.1 = 11.5$ ms

(c) $\frac{dU_L}{dt} = LI \frac{dI}{dt}$, $I = \frac{\text{emf}}{R}(1 - e^{-t/\tau})$

$$\frac{dU_L}{dt} = R\left(\frac{\text{emf}}{R}\right)^2 \left(e^{-t/\tau} - e^{-2t/\tau}\right)$$

(d) $P_R = I^2R = R\left(\frac{\text{emf}}{R}\right)^2 (1 - 2e^{-t/\tau} + e^{-2t/\tau})$

Example 31.5
The breakdown electric field strength of air is $3\times10^6$ V/m. A very large magnetic field strength is 20 T. compare the energy densities of the field.

Solution: $U_E = \frac{1}{2} \varepsilon_0 E^2 = (0.5)(8.85\times10^{-12})(3\times10^6)^2$

$$= 40 \text{ J/m}^3$$

$U_B = \frac{1}{2\mu_0} B^2 = \frac{20^2}{2\times4\pi\times10^{-7}}$

$$= 3.2\times10^8 \text{ J/m}^3$$

Magnetic fields are an effective means of storing energy without breakdown of the air. However, it is difficult to produce such large fields over large regions.
Example 31.6

Use the expression for the energy density of the magnetic field to calculate the self-inductance of a toroid with a rectangular cross section.

Solution:

\[ B = \frac{\mu_0 NI}{2\pi r} \]

\[ dU_B = \frac{B^2}{2\mu_0} dV = \frac{B^2}{2\mu_0} h (2\pi r dr) = \frac{\mu_0 h (NI)^2}{4\pi r} dr \]

\[ U_B = \int_a^b \frac{\mu_0 h (NI)^2}{4\pi r} dr = \frac{\mu_0 h N^2 I^2}{4\pi} \ln\left(\frac{b}{a}\right) = \frac{1}{2} LI^2 \]

\[ L = \frac{\mu_0 N^2 h}{2\pi} \ln\left(\frac{b}{a}\right) \]

Can we use the concept of magnetic flux to derive the self-inductance?

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31.4 LC Oscillations

The ability of an inductor and a capacitor to store energy leads to the important phenomena of electrical oscillations.
31.4 LC Oscillations (II)

Consider the situation shown in Fig. 32.11. A circuit consists of an inductor and a capacitor in series. According to Kirchhoff’s loop rule,
\[
\frac{Q}{C} - L \frac{dI}{dt} = 0
\]

\[I = -\frac{dQ}{dt}\] (note there is a minus sign)

\[
\frac{Q}{C} + L \frac{d^2Q}{dt^2} = 0
\]

\[Q = Q_0 \sin(\omega_0 t + \phi)\]

where \(\omega_0 = \frac{1}{\sqrt{LC}}\) (natural angular frequency),

\(Q_0\) and \(\phi\) can be determined from initial condition.

31.4 LC Oscillations (III)

The current and charge relation
\[
Q = Q_0 \sin(\omega_0 t + \phi)
\]

\[I = -I_0 \omega_0 \cos(\omega_0 t + \phi)\]

The energy stored in the system is
\[
U_E = \frac{1}{2C} Q^2 = \frac{1}{2C} Q_0^2 \sin^2(\omega_0 t + \phi)
\]

\[
U_B = \frac{1}{2} LI^2 = \frac{1}{2} LQ_0^2 \omega_0^2 \cos^2(\omega_0 t + \phi)
\]

\[= \frac{1}{2C} Q_0^2 \cos^2(\omega_0 t + \phi)\]

\[U = U_E + U_B = \frac{1}{2C} Q_0^2\] (constant)
The Analogies Between $LC$ Oscillations and the Mechanical Oscillations

\[ \frac{d^2 Q}{dt^2} + \frac{1}{LC} Q = 0 \quad \frac{d^2 x}{dt^2} + \frac{k}{m} x = 0 \quad \frac{d^2 \theta}{dt^2} + \frac{g}{\ell} \theta = 0 \]

<table>
<thead>
<tr>
<th>LC Osc.</th>
<th>Spring</th>
<th>Pendulum.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q$</td>
<td>$x$</td>
<td>$m$</td>
</tr>
<tr>
<td>$\frac{1}{2} m v^2$</td>
<td>$k$</td>
<td>$\frac{1}{2} k x^2$</td>
</tr>
<tr>
<td>$F$</td>
<td>$P = F v$</td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{2} L I^2$</td>
<td>$\frac{1}{C}$</td>
<td>$\frac{1}{2} \frac{Q^2}{C}$</td>
</tr>
<tr>
<td>$V$</td>
<td>$P = V I$</td>
<td></td>
</tr>
</tbody>
</table>

表 32.1 機械與電振盪之間相關的數

Example 32.7

In an $LC$ circuit, as in Fig. 32.10, $L=40$ mH, $C=20$ uF, and the maximum potential difference across the capacitor is 80 V. Find: (a) the maximum charge on $C$; (b) the angular frequency of the oscillation; (c) the maximum current; (d) the total energy.

Solution:

(a) $Q_0 = CV = (20 \times 10^{-6})(80) = 2.4 \text{ mC}$

(b) $\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{40 \times 10^{-3} \times 20 \times 10^{-6}}} = 1120 \text{ rad/s}$

(c) $I_0 = Q_0 \omega_0 = 1.79 \text{ A}$

(d) $U = \frac{1}{2C} Q_0^2 = 6.4 \times 10^{-2} \text{ J}$
31.5 Damped $LC$ Oscillations

Consider the situation shown in Fig. 32.14. A circuit consists of an inductor, a capacitor, and a resistor in series. According to Kirchhoff’s loop rule,

$$\frac{Q}{C} - RI - L \frac{dI}{dt} = 0 \quad \text{since} \quad I = - \frac{dQ}{dt}$$

$$\frac{d^2Q}{dt^2} + \frac{R}{L} \frac{dQ}{dt} + \frac{Q}{LC} = 0 \quad \Rightarrow Q = Q_0 e^{-\frac{Rt}{L}} \cos(\omega t + \phi)$$

where $\omega' = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$ (damped angular frequency),

$Q_0$ and $\phi$ are parameters to be detremined.

31.5 Damped $LC$ Oscillations (II)

The behavior of the system depends on the relative value of $\omega_0$ and $R/2L$.

under damping \hspace{1cm} critical and over damping
32.6 Magnetic Properties of Matter

When a material is placed in an external magnetic field $B_0$, the resultant field within the material is different from $B_0$. The field due to the material itself, $B_M$, is directly proportional to $B_0$:

$$B_M = \chi_m B_0 \quad (\chi_m : \text{magnetic susceptibility})$$

The total field within the material is

$$B = B_0 + B_M = (1 + \chi_m)B_0$$

$$= \kappa_m B_0 \quad (\kappa_m : \text{relative permeability})$$

The magnetic properties of matter can be classified according to their magnetic susceptibility.

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32.6 Magnetic Properties of Matter (II)

*Ferromagnetic* ($\chi_m=10^3$~$10^5$): Fe, Ni, Co, Gd, and Dy.
Depends on temperature

*Paramagnetic* ($\chi_m=10^{-5}$): Al, Cr, K, Mg, Mn, and Na.
Depends on temperature

*Diamagnetic* ($\chi_m=-10^{-5}$): Cu, Bi, C, Ag, Au, Pb, and Zn.
Not depends on temperature
32.6 Magnetic Properties of Matter (III)

When the surrounding temperature is greater than the **Curie temperature**, the *ferromagnetic* material becomes **paramagnetic**.

32.6 Magnetic Properties of Matter (IV)

When the induced magnetic field “lags” behind the change in applied field $B_0$, this lagging effect is called **magnetic hysteresis**.
Exercises and Problems

Ch.32:
Ex. 13, 16, 24, 29
Prob. 3, 4, 5, 9, 10